NUMERICAL METHODS AND PROGRAMMING **2024/2025**

Basic Techniques for Numerical Inteegration **(PROBLEMS 6)**

1.— Calculate analytically the integral $\int_1^2 \ln(x)dx$. Evaluate numerically the above integral using the Newton-Cotes quadratures of 3 and 4 integration points. Calculate the error involved in each case and compare it with the error bounds of the quadratures. Compare the efficiency of both formulas.

Solution 1.a The analytical integral calculation is:

$$I = \int_1^2 \ln(x)dx = (x\ln(x) - x)\Big|_1^2 = 2\ln(2) - 1 = 0.386294361$$

Solución 1.b Newton-Cotes with 3 and 4 integration points

- Newton-Cotes with 3 integration points, $\Rightarrow n = 2 \ \Rightarrow h = \dfrac{2-1}{2} = \dfrac{1}{2}$

$$x_0 = 1, \qquad x_1 = 1.5, \qquad x_2 = 2$$

$$I_2 = \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2)) = 0.385834602$$

- Newton-Cotes with 4 integration points, $\Rightarrow n = 3 \ \Rightarrow h = \dfrac{2-1}{3} = \dfrac{1}{3}$

$$x_0 = 1, \qquad x_1 = 4/3, \qquad x_2 = 5/3, \qquad x_3 = 2$$

$$I_3 = \frac{3h}{8}(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)) = 0.386083784$$

Solution 1.c We can calculate the error committed in each case by comparing with the analytical solution.

$$E_2 = I - I_2 = 0.000459759 \ (\approx 0.12\%)$$

$$E_3 = I - I_3 = 0.000210577 \ (\approx 0.05\%)$$

We can compare the errors obtained with the corresponding error values of each quadrature:

- Newton-Cotes with 3 integration points, upper error bound:

$$\left.\begin{array}{c} E_2 = -\dfrac{f^{(4}(\mu)h^5}{90} \\ f^{(4}(\mu) = -6/\mu^4 \end{array}\right\} \Rightarrow \quad \left.\begin{array}{c} E_2 = \dfrac{6}{\mu^4}\dfrac{(1/2)^5}{90} \\ \mu \in [1,2] \end{array}\right\} |E_2| \leq \frac{6}{1^4}\frac{(1/2)^5}{90} = 0.002083 \ (\approx 0.54\%)$$

- Newton-Cotes with 4 integration points, upper error bound:

$$\left.\begin{array}{c} E_3 = -3\dfrac{f^{(4}(\mu)h^5}{80} \\ f^{(4}(\mu) = -6/\mu^4 \end{array}\right\} \Rightarrow \quad \left.\begin{array}{c} E_3 = 3\dfrac{6}{\mu^4}\dfrac{(1/3)^5}{80} \\ \mu \in [1,2] \end{array}\right\} |E_3| \leq \frac{3.6}{1^4}\frac{(1/3)^5}{80} = 0.000925926 \ (\approx 0.24\%)$$

Both formulas work reasonably well. Logically, the quadrature with 4 integration points obtains a better numerical value of the integral. The errors obtained in each case agree with the error rates of both quadratures.

2.— Repeat the previous problem for the integral $\int_0^1 \ln(x)dx$. Critically analyze the problems posed in this case.

Solution 2.a In this case we are dealing with an improper integral, since $\lim_{x\to 0} \ln(x) = -\infty$. The analytical calculus of the integral:

$$I = \int_0^1 \ln(x)dx = \lim_{a\to 0^+} \int_a^1 \ln(x)dx = \lim_{a\to 0^+} (x\ln(x) - x)\Big|_a^1 = \lim_{a\to 0^+} (-1 + a - a\ln(a)) = -1$$

Solution 2.b To numerically evaluate the integral, in this case, we have to use open Newton-Cotes quadratures.

- Open Newton-Cotes with 3 integration points, $\Rightarrow n = 2 \ \Rightarrow h = \dfrac{1-0}{4} = \dfrac{1}{4}$

$$x_0 = 1/4, \qquad x_1 = 1/2, \qquad x_2 = 3/4$$

$$I_2 = \frac{4h}{3}(2f(x_0) - f(x_1) + 2f(x_2)) = -0.884935229$$

- Open Newton-Cotes with 4 integration points, $\Rightarrow n = 3 \ \Rightarrow h = \dfrac{1-0}{5} = \dfrac{1}{5}$

$$x_0 = 1/5, \qquad x_1 = 2/5, \qquad x_2 = 3/5, \qquad x_3 = 4/5$$

$$I_3 = \frac{5h}{24}(11f(x_0) + f(x_1) + f(x_2) + 11f(x_3)) = -0.899396352$$

Solution 2.c We can calculate the error committed in each case by comparing with the analytical solution.

$$E_2 = I - I_2 = -0.115064771 \ (\approx 12\%)$$

$$E_3 = I - I_3 = -0.100603648 \ (\approx 10\%)$$

We can compare the errors obtained with the corresponding error values of each quadrature:

- Open Newton-Cotes with 3 integration points, error bound:

$$\left.\begin{array}{r} E_2 = 28\dfrac{f^{(4}(\mu)h^5}{90} \\ f^{(4}(\mu) = -6/\mu^4 \end{array}\right\} \Rightarrow \quad \left.\begin{array}{r} E_2 = -28\dfrac{6}{\mu^4}\dfrac{(1/4)^5}{90} \\ \mu \in [0,1] \end{array}\right\} \Rightarrow |E_2| \text{ can not be bounded}$$

- Open Newton-Cotes with 4 integration points, error bound:

$$\left.\begin{array}{r} E_3 = 95\dfrac{f^{(4}(\mu)h^5}{144} \\ f^{(4}(\mu) = -6/\mu^4 \end{array}\right\} \Rightarrow \quad \left.\begin{array}{r} E_3 = -95\dfrac{6}{\mu^4}\dfrac{(1/5)^5}{144} \\ \mu \in [1,2] \end{array}\right\} \Rightarrow |E_3| \text{ can not be bounded}$$

Solution 2.d The formulas perform relatively poorly. Although the results between the two are comparable. Errors, in this case, cannot be bounded.

3.— Obtain better approximations to the values of the integrals of problems 1 and 2 by combining the results obtained with the 3-point and 4-point quadratures in each case. Study to what extent the hypotheses made in the combination of the quadratures are reliable and analyze the results obtained.

Solution 3. For the case of exercise 1, we can combine the formulas used since they have the same order of error. Thus, assuming $f^{(4}(\mu_2) \approx f^{(4}(\mu_3)$, we know that:

$$I = I_2 + E_2 = I_3 + E_3$$

$$\frac{E_2}{E_3} \approx \frac{80\ 3^5}{3\ 90\ 2^5} = 2.25$$

Therefore

$$I_2 + 2.25E_3 \approx I_3 + E_3 \Rightarrow E_3 \approx \frac{I_3 - I_2}{1.25} \Longrightarrow I \approx \hat{I} = I_3 + \frac{I_3 - I_2}{1.25} = 0.386283130$$

The error in this case:

$$E = I - \hat{I} = 0.000011232\ (\approx 0.003\%)$$

The error is reduced by a factor of 20-30.

For the case of exercise 2, we can again combine the formulas used since they have the same order of error. Thus, assuming $f^{(4}(\mu_2) \approx f^{(4}(\mu_3)$, we know that:

$$I = I_2 + E_2 = I_3 + E_3 \Rightarrow \frac{E_2}{E_3} \approx \frac{28\ 3^5\ 144}{95\ 4^5\ 90} = 1.439144737$$

Therefore

$$I_2 + 1.439144737E_3 \approx I_3 + E_3 \Rightarrow E_3 \approx \frac{I_3 - I_2}{0.439144737} \Longrightarrow I \approx \hat{I} = I_3 + \frac{I_3 - I_2}{0.439144737} = -0.932326550$$

The error in this case:

$$E = I - \hat{I} = -0.067673450\ (\approx 6.77\%)$$

In this case the errors are only slightly reduced.

4.— In order to analyze numerically the integral of problem 1 more accurately two alternatives are proposed:

a) Using a Newton-Cotes formula with 7 points.

b) Using the Composite Simpson formula with 7 integration points.

Estimate the error bounds in each case and compare them. Which alternative seems more convenient? Evaluate the integral using both formulas and compare (with each other and with their respective bounds) the errors made.

Solution 4.a Closed Newton-Cotes $m = 6$

$$I = I_6^N + E_6^N; \quad h = \frac{2-1}{6} = \frac{1}{6};$$

$$x_0 = 1, \quad x_1 = 7/6, \quad x_2 = 4/3, \quad x_3 = 3/2, \quad x_4 = 5/3, \quad x_5 = 11/6, \quad x_6 = 2$$

$$I_6^N = \frac{h}{140}\left(41f_0 + 216f_1 + 27f_2 + 272f_3 + 27f_4 + 216f_5 + 41f_6\right) = 0.386294205$$

$$E_6^N = I - I_6^N = 0.000000156 \ (\approx 0.00004\%)$$

$$\left.\begin{array}{r} E_6^N = -9\,\dfrac{f^{(8}(\mu)h^9}{1400} \\ f^{(8}(\mu) = -5040/\mu^8 \end{array}\right\} \Rightarrow \left.\begin{array}{r} E_6^N = 9\,\dfrac{5040}{\mu^8}\dfrac{(1/6)^9}{1400} \\ \mu \in [1,2] \end{array}\right\} \Rightarrow \left\{\begin{array}{r} |E_6^N| \leq 9\,\dfrac{5040}{2^8}\dfrac{(1/6)^9}{1400} = 0.000003215 \\ (\approx 0.00083\%) \end{array}\right.$$

Solution 4.b Composite Simpson formula $p = 6$

$$I = I_6^S + E_6^S; \quad h = \frac{2-1}{6} = \frac{1}{6};$$

$$x_0 = 1, \quad x_1 = 7/6, \quad x_2 = 4/3, \quad x_3 = 3/2, \quad x_4 = 5/3, \quad x_5 = 11/6, \quad x_6 = 2$$

$$I_6^S = \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + f_6\right) = 0.386287163$$

$$E_6^S = I - I_6^S = 0.000007198 \ (\approx 0.00186\%)$$

$$\left.\begin{array}{r} E_6^S = -\left(\dfrac{p}{2}\right)\dfrac{f^{(4}(\mu)h^5}{90} \\ f^{(4}(\mu) = -6/\mu^4 \end{array}\right\} \Rightarrow \left.\begin{array}{r} E_6^S = \left(\dfrac{p}{2}\right)\dfrac{6}{\mu^4}\dfrac{(1/6)^5}{90} \\ \mu \in [1,2] \end{array}\right\} \Rightarrow \left\{\begin{array}{r} |E_6^S| \leq 3\,\dfrac{6\ (1/6)^5}{90} = 0.000025720 \\ (\approx 0.0067\%) \end{array}\right.$$

The results are good in both cases.

5.— To make a FORTRAN subroutine to calculate the integral of a function $f(x)$ over an interval $[a, b]$ by means of an adaptive Simpson's composite quadrature. The number of subintervals will be doubled at each iteration until the integration error is less than a predetermined value. The integration error will be estimated by Richardson extrapolation.

Solution 5. The program would look like:

```
c____________________________________________MainProgram_______________
c
c Calculation of an integral using Simpson's Composite Formula (Adaptive)
c

      implicit real*8 (a-h,o-z)
      logical cnvgc
      external fprueba        !name of the subintegrating function

      a=-4.d+00               !lower integration bound
      b= 4.d+00               !upper integration bound

      n_ini=1                 !We start with 1 subinterval (1+1  points)
      nitr_max=10             !maximum number of iterations

      epsl=1.d-20             !maximum admissible absolute error of the integral
      rltv=1.d-12             !maximum permissible relative error of the integral

      call IntFCSA(a,b,fprueba,n_ini,nitr_max,epsl,rltv,n,vint,cnvgc)

      write(6,*) ' n     = ',n      !number of sub-intervals used,
      write(6,*) ' vint  = ',vint   !numerical approx. of the value of the integral
      write(6,*) ' cnvgc = ',cnvgc  !the errors are (T) or are not (F) admissible

      end

c____________________________________________fprueba(x)=1/(1+x*x)___________
c     Subintegrand function that we use as an example

      real*8 function fprueba(x)
      implicit real*8 (a-h,o-z)

      fprueba=1.d+00/(1+x*x)

      return
      end

c____________________________________________IntFCSA________________________
!            /b
! Calculus of | f(x)dx  using the Composite Simpson Formula (Adaptive)
!            /a
!
! Otros datos:
!
!     n_ini     = initial number of subintervals <=> 2*n_ini+1 points
!     niter_max = maximum number of iterations
!     epsl      = maximum absolute error admissible for the value of the integral
!     rltv      = maximum relative error admissible for the value of the integral
!
```

```
! Resultados:
!
!     n         = number of subintervals that have been used <=> 2*n+1 points
!     vint      = numerical approach of the value of the integral
!     cnvgc     = logical variable that indicates if the final errors
!                 are (.TRUE.) or they are not (.FALSE.) admissible
!
! Notas:
!
!   * At each iteration the number of subintervals is doubled.
!   * To estimate the error of the numerical approximation in each iteration
! and to calculate the final value of the integral, a
! RICHARDSON EXTRAPOLATION is performed.
!------------------------------------------------------------------------------

      Subroutine IntFCSA(a,b,f,n_ini,nitr_max,epsl,rltv,n,vint,cnvgc)
      implicit real*8 (a-h,o-z)
      logical cnvgc

c     Simpson's Composite Formula with n_ini subintervals
        nitr=0
        n=n_ini
        h=(b-a)/dfloat(2*n)  !distance between points
        t1=f(a)+f(b)         !terms to be multiplied times 1
        t4=0.d+00            !terms to be multiplied times 4
        do i=1,n
          x=a+dfloat(2*i-1)*h
          t4=t4+f(x)
        enddo
        t2=0.d+00            !terms to be multiplied times 2
        do i=1,n-1
          x=a+dfloat(2*i)*h
          t2=t2+f(x)
        enddo
        vint1=(t1+4.d+00*t4+2.d+00*t2)*h/3.d+00  !Composite Simpson formula
        vint=vint1

c     Adaptive Composite Simpson formula
      cnvgc=.false.
      do while(.not.cnvgc.and.nitr.lt.nitr_max)
        nitr=nitr+1
        n=n*2                !We duplicate the number of subintervals
        h=h/2.d+00           !reduce the distance between points to the half
        t2=t2+t4             !terms to be multiplied now times 2
        t4=0.d+00
        do i=1,n             !terms to be multiplied now times 4
          x=a+dfloat(2*i-1)*h
          t4=t4+f(x)
        enddo
```

```
    vint2=(t1+4.d+00*t4+2.d+00*t2)*h/3.d+00 !Composite Simpson formula
    e2=(vint2-vint1)/15.d+00
    vint=vint2+e2                           !Richardson Extrapolation
    e2mx=abs(vint)*rltv
    if (epsl.gt.e2mx) e2mx=epsl             !Admissible error bound
    cnvgc=(abs(e2).le.e2mx)
    vint1=vint2
  enddo

  return
  end
```

6.— For a road that is being designed, cross-sectional profiles have been obtained every 5 meters. In each profile, the areas of cut and embankment have been measured with a planimeter and are shown in the attached table. We wish to calculate the volume of earthworks in the cut and fill that would be required in the section between kilometer points 1730 and 1810 of the road. Calculate the volumes by means of the following techniques: **a)** the Trapezoid rule using only the data of the kilometer points that are multiples of 10, **b)** the composite trapezoid rule using all the available data, **c)** Richardson extrapolation between the previously obtained values, and **d)** Simpson's composite rule using all available data. Compare and comment on the results obtained with the different methods.

Kilometer Point (m)	Cut Area (m^2)	Fill Area (m^2)
1730	2.51	0.05
1735	1.32	0.61
1740	1.12	0.82
1745	0.85	0.95
1750	0.63	1.21
1755	0.05	1.35
1760	0.00	1.56
1765	0.00	2.58
1770	0.00	2.41
1775	0.25	2.21
1780	0.56	1.90
1785	0.85	1.50
1790	0.94	0.85
1795	1.57	0.34
1800	1.83	0.11
1805	2.61	0.00
1810	2.57	0.20

Solution 6.a The compound trapezoid rule using only data from kilometer points that are multiples of 10 would be:

$$h = \frac{1810-1730}{8} = 10 \qquad \begin{cases} D_8^T = \dfrac{h}{2}(2.51 + 2\ (1.12 + \ldots + 1.83) + 2.57) & = & 76.20\ \text{m}^3 \\ T_8^T = \dfrac{h}{2}(0.05 + 2\ (0.82 + \ldots + 0.11) + 0.20) & = & 89.85\ \text{m}^3 \end{cases}$$

Solution 6.b The composite trapezoidal rule using all available data would be:

$$h = \frac{1810-1730}{16} = 5 \qquad \begin{cases} D_{16}^T = \dfrac{h}{2}(2.51 + 2\ (1.32 + \ldots + 2.61) + 2.57) & = & 75.60\ \text{m}^3 \\ T_{16}^T = \dfrac{h}{2}(0.05 + 2\ (0.61 + \ldots + 0.00) + 0.20) & = & 92.625\ \text{m}^3 \end{cases}$$

Solution 6.c An extrapolation of Richardson between those obtained previously would be:

$$\left. \begin{array}{l} D = D_8^T + E_8^T \\ D = D_{16}^T + E_{16}^T \end{array} \right\} \quad \left. \begin{array}{l} E_8^T = \dfrac{(b-a)^3}{8^2\ 12} f''(\xi_1) \\ E_{16}^T = \dfrac{(b-a)^3}{16^2\ 12} f''(\xi_2) \end{array} \right\} \Rightarrow \begin{cases} \text{Hypothesis: } f''(\xi_1) = f''(\xi_2) \\ \dfrac{E_8^T}{E_{16}^T} \approx \dfrac{16^2}{8^2} = 4 \end{cases}$$

$$D = D_8^T + E_8^T = D_{16}^T + E_{16}^T \qquad \Rightarrow \qquad E_{16}^T \approx \frac{D_{16}^T - D_8^T}{3}$$

$$D^R = D_{16}^T + \frac{D_{16}^T - D_8^T}{3} \approx 75.40\ \text{m}^3$$

$$\left. \begin{array}{l} T = T_8^T + \widehat{E}_8^T \\ T = T_{16}^T + \widehat{E}_{16}^T \end{array} \right\} \quad \left. \begin{array}{l} \widehat{E}_8^T = \dfrac{(b-a)^3}{8^2\ 12} f''(\widehat{\xi}_1) \\ \widehat{E}_{16}^T = \dfrac{(b-a)^3}{16^2\ 12} f''(\widehat{\xi}_2) \end{array} \right\} \Rightarrow \begin{cases} \text{Hypothesis: } f''(\widehat{\xi}_1) = f''(\widehat{\xi}_2) \\ \dfrac{\widehat{E}_8^T}{\widehat{E}_{16}^T} \approx \dfrac{16^2}{8^2} = 4 \end{cases}$$

$$T = T_8^T + \widehat{E}_8^T = T_{16}^T + \widehat{E}_{16}^T \qquad \Rightarrow \qquad \widehat{E}_{16}^T \approx \frac{T_{16}^T - T_8^T}{3}$$

$$T^R = T_{16}^T + \frac{T_{16}^T - T_8^T}{3} \approx 93.55\ \text{m}^3$$

Solution 6.d Composite Simpson Rule with all the available data would be:

$$h = \frac{1810-1730}{16} = 5$$

$$D_{16}^S = \frac{h}{3}(2.51 + 4\ (1.32 + 0.85 + \ldots + 2.61) + 2\ (1.12 + 0.63 + \ldots + 1.83) + 2.57) = 75.40\ \text{m}^3$$

$$T_{16}^S = \frac{h}{3}(0.05 + 4\ (0.61 + 0.95 + \ldots + 0.00) + 2\ (0.82 + 1.21 + \ldots + 0.11) + 0.20) = 93.55\ \text{m}^3$$

The results obtained with the Composite Simpson rule and by Richardson extrapolation are the same.

In practice, the Composite Trapezium Rule with all the data is sufficient.

7.— The following fourth order ODE is to be solved by means of a One-Step Method,

$$u'''' = 3(u')^2 + \frac{9}{2}u^3, \qquad x \in [0,1],$$

with the boundary conditions

$$u(1) = 4, \qquad u'(1) = 8, \qquad u''(1) = 24, \qquad u'''(1) = 96.$$

Se pide:

a) Propose and develop completely the application of Euler's method and the Improved Euler's method. Explicit how is the order of the ODE reduced and how are the calculations performed in each case.

b) Using both methods, obtain the values of u, u', u'' and u''' at $x = 0.8$ moving from $x = 1$ in two steps.

c) Analyze the results obtained, comparing them and with those of the analytic solution $u(x) = (1 - x/2)^{-2}$.

Solution 7. First we convert the 4th order ODE into a 1st order ODE system:

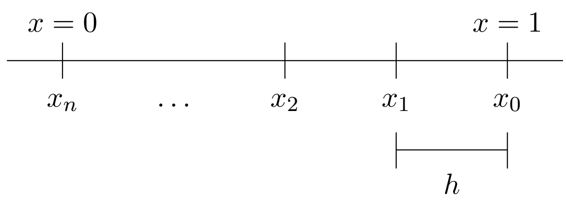
$$\vec{u} = \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \right\} = \left\{ \begin{array}{c} u \\ u' \\ u'' \\ u''' \end{array} \right\} \Rightarrow \vec{u}' = \left\{ \begin{array}{c} u_1' \\ u_2' \\ u_3' \\ u_4' \end{array} \right\} = \left\{ \begin{array}{c} u' \\ u'' \\ u''' \\ u'''' \end{array} \right\} = \left\{ \begin{array}{c} u_2 \\ u_3 \\ u_4 \\ 3\ u_2^2 + \frac{9}{2}\ u_1^3 \end{array} \right\}$$

Then,

$$\left\{ \begin{array}{ll} \dfrac{d\vec{u}}{dx} = \vec{\varphi}(x,\vec{u}) \quad ; \quad \vec{u} = \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \right\} & \vec{\varphi}(x,\vec{u}) = \left\{ \begin{array}{c} u_2 \\ u_3 \\ u_4 \\ 3\ u_2^2 + \frac{9}{2}\ u_1^3 \end{array} \right\} \\ \vec{u}(1) = \vec{u}_0 \quad ; \quad \vec{u}_0 = \left\{ \begin{array}{c} 4 \\ 8 \\ 24 \\ 96 \end{array} \right\} & \end{array} \right.$$

Solution 7.a Methods of Euler and modified Euler:

Discretization:



$$x_{i+1} = x_i + h; \qquad h = \frac{-1}{n}$$

Euler:

$$\vec{u}_0 = \left\{ \begin{array}{c} 4 \\ 8 \\ 24 \\ 96 \end{array} \right\} \qquad \vec{u}_{i+1} = \vec{u}_i + h\,\vec{\varphi}(x_i, \vec{u}_i); \qquad \vec{u}_i = \left\{ \begin{array}{c} u_1^i \\ u_2^i \\ u_3^i \\ u_4^i \end{array} \right\}$$

$$\left\{ \begin{array}{l} u_1^{i+1} = u_1^i + h u_2^i \\ u_2^{i+1} = u_2^i + h u_3^i \\ u_3^{i+1} = u_3^i + h u_4^i \\ u_4^{i+1} = u_3^i + h(3(u_2^i)^2 + \frac{9}{2}(u_1^i)^3) \end{array} \right. \qquad \text{with} \qquad \left\{ \begin{array}{l} u_1^0 = 4 \\ u_2^0 = 8 \\ u_3^0 = 24 \\ u_4^0 = 96 \end{array} \right. \qquad h = \frac{-1}{n}$$

Modified Euler:

$$\vec{u}_0 = \left\{ \begin{array}{c} 4 \\ 8 \\ 24 \\ 96 \end{array} \right\} \qquad \vec{u}_{i+1} = \vec{u}_i + h\,\vec{\varphi}\Big(x_i + \frac{h}{2}, \vec{u}_i + \frac{h}{2}\vec{\varphi}(x_i, \vec{u}_i)\Big)$$

$$\vec{u}_i + \frac{h}{2}\vec{\varphi}(x_i, \vec{u}_i) = \left\{ \begin{array}{c} u_1^i \\ u_2^i \\ u_3^i \\ u_4^i \end{array} \right\} + \frac{h}{2} \left\{ \begin{array}{l} u_2^i \\ u_3^i \\ u_4^i \\ 3(u_2^i)^2 + \frac{9}{2}(u_1^i)^3 \end{array} \right\} = \left\{ \begin{array}{l} u_1^i + \frac{h}{2}u_2^i \\ u_2^i + \frac{h}{2}u_3^i \\ u_3^i + \frac{h}{2}u_4^i \\ u_4^i + \frac{h}{2}(3(u_2^i)^2 + \frac{9}{2}(u_1^i)^3) \end{array} \right\}$$

$$\left\{ \begin{array}{l} u_1^{i+1} = u_i^1 + h\Big(u_2^i + \frac{h}{2}u_3^i\Big) \\ u_2^{i+1} = u_2^i + h\Big(u_3^i + h u_4^i\Big) \\ u_3^{i+1} = u_3^i + h\Big(u_4^i + \frac{h}{2}(3(u_2^i)^2 + \frac{9}{2}(u_1^i)^3)\Big) \\ u_4^{i+1} = u_4^i + h\Big(3\big(u_2^i + \frac{h}{2}u_3^i\big)^2 + \frac{9}{2}\big(u_1^i + \frac{h}{2}u_2^i\big)^3\Big) \end{array} \right. \qquad \text{with} \qquad \left\{ \begin{array}{l} u_1^0 = 4 \\ u_2^0 = 8 \\ u_3^0 = 24 \\ u_4^0 = 96 \end{array} \right. \qquad h = \frac{-1}{n}$$

Solution 7.b To move from $x = 1$ to $x = 0.8$ in 2 steps we will use a step $h = \dfrac{0.8 - 1}{2} = -0.1$

Euler

i	0	1	2		
x_i	1.000000	0.900000	0.800000		
u_1^i	4.000000	3.200000	2.640000	$\longrightarrow$	4.96 %
u_2^i	8.000000	5.600000	4.160000	$\longrightarrow$	10.14 %
u_3^i	24.000000	14.400000	9.600000	$\longrightarrow$	17.05 %
u_4^i	96.000000	48.000000	23.846400	$\longrightarrow$	38.20 %

Modified Euler

i	0	1	2		
x_i	1.000000	0.900000	0.800000		
u_1^i	4.000000	3.320000	2.790600	$\longrightarrow$	−0.46 %
u_2^i	8.000000	6.080000	4.889480	$\longrightarrow$	−5.61 %
u_3^i	24.000000	16.800000	12.064589	$\longrightarrow$	−4.24 %
u_4^i	96.000000	61.132800	40.550081	$\longrightarrow$	−5.11 %

Solution 7.c Analytic solution

$$u(x) = (1 - x/2)^{-2} \Rightarrow \begin{cases} u' &= (1 - x/2)^{-3} \\ u'' &= \frac{3}{2}(1 - x/2)^{-4} \\ u''' &= 3(1 - x/2)^{-5} \end{cases}$$

i	0	1	2
x_i	1.000000	0.900000	0.800000
u_1^i	4.000000	3.305785	2.777784
u_2^i	8.000000	6.010518	4.629630
u_3^i	24.000000	16.392323	11.574074
u_4^i	96.000000	59.608447	38.580247

We observe that the modified Euler method works much better than the Euler method.

8.— The following third order ODE is to be solved:

$$u''' = -3\,u\,u'', \qquad x \in [-1, 1],$$

with boundary conditions

$$u(-1) = 1, \qquad u'(-1) = -1, \qquad u''(-1) = 2,$$

by means of a One-Step Method.

a) Propose and develop completely the application of Euler's method and the Improved Euler's method. Explicit how is the order of the ODE reduced and how are the calculations performed in each case.

b) Using both methods, obtain the values of u, u' and u'' at $x = 0$ moving from $x = -1$ in two steps.

c) Analyze the results obtained, comparing them and with those of the analytic solution

$$u(x) = (2 + x)^{-1}.$$

Solution 8.a We define

$$\vec{u} = \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right\} \qquad \text{with} \qquad \left\{ \begin{array}{l} u_1 = u \\ u_2 = u' \\ u_3 = u'' \end{array} \right\}$$

Then,

$$\vec{u}\,' = \left\{ \begin{array}{c} u' \\ u'' \\ u''' \end{array} \right\} = \left\{ \begin{array}{l} u_2 \\ u_3 \\ -3u_1u_3 \end{array} \right\}$$

which is equivalent to the 1st order system,

$$\left\{ \begin{array}{l} \dfrac{d\vec{u}}{dx} = \vec{\varphi}(x,\vec{u}) \quad \text{with} \quad \vec{\varphi}(x,\vec{u}) = \left\{ \begin{array}{l} u_2 \\ u_3 \\ -3u_1u_3 \end{array} \right\} \\ \vec{u}(-1) = \left\{ \begin{array}{r} 1 \\ -1 \\ 2 \end{array} \right\} \end{array} \right.$$

<u>Euler:</u>

$$\vec{u}_{i+1} = \vec{u}_i + h\,\vec{\varphi}(x_i,\vec{u}_i)$$

$$x_0 = -1, \quad x_i = -1 + i\ h, \quad \left\{ \begin{array}{c} u_1^0 \\ u_2^0 \\ u_3^0 \end{array} \right\} = \left\{ \begin{array}{r} 1 \\ -1 \\ 2 \end{array} \right\}$$

$$\left\{ \begin{array}{c} u_1^{i+1} \\ u_2^{i+1} \\ u_3^{i+1} \end{array} \right\} = \left\{ \begin{array}{c} u_1^i \\ u_2^i \\ u_3^i \end{array} \right\} + h \left\{ \begin{array}{l} u_2^i \\ u_3^i \\ -3u_1^iu_3^i \end{array} \right\}$$

<u>Modified Euler:</u>

$$\vec{u}_{i+1} = \vec{u}_i + h\,\vec{\varphi}\Big(x_i + \frac{h}{2}, \vec{u}_i + \frac{h}{2}\underbrace{\vec{\varphi}(x_i,\vec{u}_i)}_{\vec{K}_0^i}\Big)$$

$$x_0 = -1, \quad x_i = -1 + i\ h, \quad \left\{ \begin{array}{c} u_1^0 \\ u_2^0 \\ u_3^0 \end{array} \right\} = \left\{ \begin{array}{r} 1 \\ -1 \\ 2 \end{array} \right\}$$

$$\vec{K}_0^i = \left\{ \begin{array}{l} u_2^i \\ u_3^i \\ -3u_1^i u_3^i \end{array} \right\}, \qquad \vec{u}_i + \frac{h}{2}\vec{K}_0^i = \left\{ \begin{array}{l} u_1^i + \frac{1}{2}h\ u_2^i \\ u_2^i + \frac{1}{2}h\ u_3^i \\ u_3^i - \frac{3}{2}h\ u_1^i\ u_3^i \end{array} \right\}$$

$$\left\{ \begin{array}{c} u_1^{i+1} \\ u_2^{i+1} \\ u_3^{i+1} \end{array} \right\} = \left\{ \begin{array}{c} u_1^i \\ u_2^i \\ u_3^i \end{array} \right\} + h \left\{ \begin{array}{l} u_2^i + \frac{1}{2}h\ u_3^i \\ u_3^i - \frac{3}{2}h\ u_1^i\ u_3^i \\ -3\Big(u_1^i + \frac{1}{2}h\ u_2^i\Big)\Big(u_3^i - \frac{3}{2}h\ u_1^i\ u_3^i\Big) \end{array} \right\}$$

Solution 8.b $h = 0.5 \longrightarrow x_1 = -0.5, \quad x_2 = 0$

Euler:

$$\left\{ \begin{array}{c} u_1^0 \\ u_2^0 \\ u_3^0 \end{array} \right\} = \left\{ \begin{array}{r} 1 \\ -1 \\ 2 \end{array} \right\}$$

$$\left\{ \begin{array}{c} u_1^1 \\ u_2^1 \\ u_3^1 \end{array} \right\} = \left\{ \begin{array}{r} 1 \\ -1 \\ 2 \end{array} \right\} + 0.5 \left\{ \begin{array}{l} -1 \\ \ \ 2 \\ (-3)\ (1)\ \ (2) \end{array} \right\} = \left\{ \begin{array}{r} 0.5 \\ 0 \\ -1 \end{array} \right\}$$

$$\left\{ \begin{array}{c} u_1^2 \\ u_2^2 \\ u_3^2 \end{array} \right\} = \left\{ \begin{array}{r} 0.5 \\ 0 \\ -1 \end{array} \right\} + 0.5 \left\{ \begin{array}{l} \ \ 0 \\ -1 \\ (-3)\ (0.5)\ (-1) \end{array} \right\} = \left\{ \begin{array}{r} 0.5 \\ -0.5 \\ -0.25 \end{array} \right\}$$

Then

$$\left. \begin{array}{rcl} u_E(0) & \approx & 0.5 \\ u(0) & = & 0.5 \end{array} \right\} \quad \Longrightarrow \quad r = 0\,\% \quad \text{(the exact value is obtained)}$$

Modified Euler:

$$\left\{ \begin{array}{c} u_1^0 \\ u_2^0 \\ u_3^0 \end{array} \right\} = \left\{ \begin{array}{r} 1 \\ -1 \\ 2 \end{array} \right\}$$

$$\left\{ \begin{array}{c} u_1^1 \\ u_2^1 \\ u_3^1 \end{array} \right\} = \left\{ \begin{array}{r} 1 \\ -1 \\ 2 \end{array} \right\} + 0.5 \left\{ \begin{array}{l} -1 + \frac{1}{2}(0.5)(2) \\ \ \ 2 - \frac{3}{2}(0.5)(2) \\ -3\Big(-1 + \frac{1}{2}(0.5)(-1)\Big)\Big(2 - \frac{3}{2}(0.5)(1)(2)\Big) \end{array} \right\} = \left\{ \begin{array}{r} 0.75 \\ -0.75 \\ 1.4375 \end{array} \right\}$$

$$\left\{\begin{array}{c} u_1^2 \\ u_2^2 \\ u_3^2 \end{array}\right\} = \left\{\begin{array}{c} 0.75 \\ -0.75 \\ 1.4375 \end{array}\right\} + 0.5 \left\{\begin{array}{l} -0.75 + \frac{1}{2}(0.5)(1.4375) \\ \ \ 1.4375 - \frac{3}{2}(0.5)(0.75)(1.4375) \\ -3\Big(0.75 + \frac{1}{2}(0.5)(-0.75)\Big)\Big(1.4375 - \frac{3}{2}(0.5)(0.75)(1.4375)\Big) \end{array}\right\}$$

$$\left\{\begin{array}{c} u_1^2 \\ u_2^2 \\ u_3^2 \end{array}\right\} = \left\{\begin{array}{r} 0.55468750 \\ -0.43554688 \\ 0.90686035 \end{array}\right\}$$

Then,

$$\left.\begin{array}{lcl} u_{EM}(0) & \approx & 0.55468750 \\ u(0) & = & 0.5 \end{array}\right\} \implies r = -10.94\,\%$$

Solution 8.c The modified Euler method is 2nd order while Euler's method is 1st order. The exact Euler result in this case is accidental, and is due to the value of $h = 0.5$ being too large. In general, the 2nd order method will be more accurate.