NUMERICAL METHODS AND PROGRAMMING 2024/2025

Non-Linear Equations (PROBLEMS 5)

1.— It is known that the polynomial $P(x) = x^5 + 5x - 1$ has a real root α, such that $0.1 < \alpha < 0.2$. Calculate for which values of the constant C is the iterative algorithm

$$x^{k+1} = x^k - C\ P(x^k)$$

convergent. How is the initial value of the sequence chosen?

Solution 1.

The method will converge to the root α if and only if: $|\Phi'(\alpha)| < 1$, with $\Phi = x_k - \mathcal{C}P(x_k)$.

$$\Phi'(x) = 1 - \mathcal{C}P'(x) = 1 - \mathcal{C}(5x^4 + 5) \Rightarrow \left|\Phi'(\alpha)\right| = \left|1 - \mathcal{C}(5\alpha^4 + 5)\right|$$

$$\begin{aligned} |\Phi'(\alpha)| < 1 \quad \Leftrightarrow \quad \left|1 - \mathcal{C}(5\alpha^4 + 5)\right| < 1 \quad \Leftrightarrow \quad & -1 < 1 - \mathcal{C}(5\alpha^4 + 5) < 1 \\ & -2 < -\mathcal{C}(5\alpha^4 + 5) < 0 \\ & 2 > \mathcal{C}5(\alpha^4 + 1) > 0 \\ & 0.4 > \mathcal{C}(\alpha^4 + 1) > 0 \end{aligned}$$

then, it has to be satisfied that $0 < \mathcal{C}(\alpha^4 + 1) < 0.4$. Since $\alpha \in (0.1, 0.2)$, convergence is guaranteed if:

$$\begin{cases} \mathcal{C} > 0 \\ \mathcal{C} < \dfrac{0.4}{0.2^4 + 1} \end{cases}$$

Therefore, convergence is guaranteed (for a sufficiently good initial approximation) when we use a value of $\mathcal{C}$ such that:

$$\boxed{0 < \mathcal{C} < 0.399361022364}$$

The best value of $\mathcal{C}$ would be the one in which the A.C.F. is $|\Phi'(\alpha)| = 0$, so the optimum value would be:

$$\mathcal{C}_{opt} = \frac{1}{5(\alpha^4 + 1)} = \frac{0.2}{\alpha^4 + 1}$$

Obviously, this value depends on the solution α, which we do not know. However, we do know that $\alpha \in (0.1, 0.2)$. For the possible values of α it is verified that $\alpha^4 << 1$, therefore, it can be estimated with good accuracy the value of $\mathcal{C}_{opt}$:

$$\alpha \approx 0.15 \Rightarrow \mathcal{C}_{opt} = \frac{0.2}{0.15^4 + 1} = 0.199898801232$$

We could use $\mathcal{C} = 0.2$ and the results should be good (almost quadratic convergence).

To choose the value of x_0:

$$\begin{cases} P(0.1) = -0.49999 \\ P(0.2) = 0.00032 \end{cases}$$

Thus, the value 0.2 itself could be a good initial approximation, or we could obtain it by the Regula-Falsi method as:

$$x_0 = \frac{0.2P(0.1) - 0.1P(0.2)}{P(0,1) - P(0.2)}$$

Iterating:

Table 1: Ex. 1: Iterations with $\mathcal{C} = 0.2$

k	x_k	$P(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	0.200000000000000	3.200000000001E-04	-3.195912498328E-04
1	0.199936000000000	-5.116724248744E-07	5.110193671554E-07
2	0.199936102334485	8.176292976003E-10	-8.165836895557E-10
3	0.199936102170959	-1.306510455379E-12	1.306843522286E-12
4	0.199936102171220	0.000000000000E+00	0.000000000000E+00
5	0.199936102171220	0.000000000000E+00	0.000000000000E+00

Table 2: Ex. 1: Iterations with $\mathcal{C} = 0.2/(1 + 0.15^4)) = 0.199898801231876$

k	x_k	$P(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	0.200000000000000	3.200000000001E-04	-3.195912498328E-04
1	0.199936032383606	-3.494956584138E-07	3.490495905645E-07
2	0.199936102247369	3.813531712638E-10	-3.808644510883E-10
3	0.199936102171137	-4.161115896295E-13	4.175548795615E-13
4	0.199936102171220	0.000000000000E+00	0.000000000000E+00
5	0.199936102171220	0.000000000000E+00	0.000000000000E+00

Table 3: Ex. 1: Iterations with $\mathcal{C}_{opt} = 0.2/(1 + \alpha^4)) = 0.199680918628237$

k	x_k	$P(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	0.200000000000000	3.200000000001E-04	-3.195912498328E-04
1	0.199936102106039	-3.264259973435E-10	3.260113290438E-10
2	0.199936102171220	0.000000000000E+00	0.000000000000E+00

It can be seen that for $\mathcal{C} = 0.2$ and for $\mathcal{C} = 0.2/(1 + 0.15^4))$ the convergence is fast (almost quadratic), but for $\mathcal{C}_{opt}$ it is really quadratic.

2.— Write an algorithm that allows to compute $x^{1/n}$, n nonzero integer, without evaluating any root.

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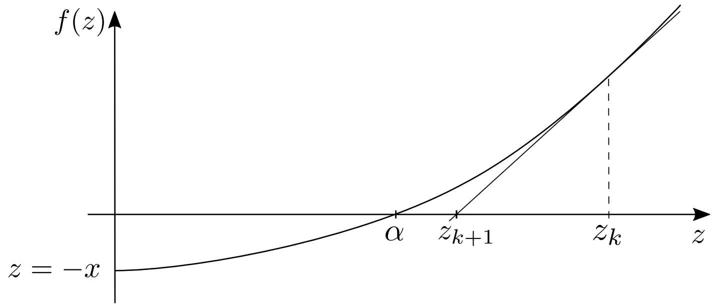
solution 2.

We want to obtain $z = x^{1/n}$, so $z^n = x \Leftrightarrow f(z) = z^n - x = 0$.

We solve $f(z) = 0$ by means of Newton's method.

$$z_{k+1} = \Phi(z_k) \qquad \Phi(z) = z - \frac{f(z)}{f'(z)} = z - \frac{z^n - x}{nz^{n-1}} = \left(1 - \frac{1}{n}\right) z + \frac{x}{nz^{n-1}}$$



We can see that

- The method will converge monotone decreasing when an initial approximation $z_0 > \alpha$ is chosen.
- if $z_0 < \alpha$, then $z_1 > \alpha$ and we are in the previous case.

Thus, we try to choose an initial approximation $z_0 > \alpha$ and as close as we can to α.

We can choose for example:

$$\begin{cases} z_0 = x & \text{if} \quad x > 1 \\ z_0 = 1 & \text{if} \quad x < 1 \end{cases}$$

Table 4: Ex. 2: Example with $n = 5$; $x = 0.5$

k	x_k	$f(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	1.000000000000000	5.000000000000E-01	-1.486983549970E-01
1	0.900000000000000	9.049000000000E-02	-3.382851949733E-02
2	0.872415790275873	5.379460453482E-03	-2.142583163333E-03
3	0.870558521981443	2.285573972949E-05	-9.142128733153E-06
4	0.870550563441640	4.178850598890E-10	-1.671540683645E-10
5	0.870550563296124	0.000000000000E+00	0.000000000000E+00
6	0.870550563296124	0.000000000000E+00	0.000000000000E+00

This method is actually the one used to obtain any integer root of a number in computers.

Table 5: Ex. 2: Example with $n = 5$; $x = 20$

k	x_k	$f(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	20.000000000000000	3.199980000000E+06	-9.985605433061E+00
1	16.000025000000000	1.048564192026E+06	-7.788498078456E+00
2	12.800081034774800	3.435882601211E+05	-6.030831987962E+00
3	10.240213835658400	1.125817469149E+05	-4.624747437436E+00
4	8.192534836021100	3.688553315662E+04	-3.499997760257E+00
5	6.554915815326280	1.208139932478E+04	-2.600485939705E+00
6	5.246099316012680	3.953585303008E+03	-1.881578857418E+00
7	4.202160425952500	1.290277074190E+03	-1.308163820297E+00
8	3.374556651685630	4.176063518333E+02	-8.535773943465E-01
9	2.730490912483530	1.317762007349E+02	-4.998047901552E-01
10	2.256353706721350	3.848384318400E+01	-2.393705769733E-01
11	1.959406182449020	8.881674568003E+00	-7.626316017428E-02
12	1.838895108771000	1.027361841733E+00	-1.006880488722E-02
13	1.820926038947400	1.988283514550E-02	-1.987493331568E-04
14	1.820564346798220	7.897121488298E-06	-7.897120246625E-08
15	1.820564203026100	1.254107928617E-12	-1.265654248073E-14
16	1.820564203026080	0.000000000000E+00	0.000000000000E+00
17	1.820564203026080	0.000000000000E+00	0.000000000000E+00

3.— Given the equation $f(x) = 1 - x^2$:

a) What conditions must the constant m satisfy for the Whittaker's method

$$x^{k+1} = x^k - \frac{f(x^k)}{m}$$

to be convergent to the root $x = 1$?

b) Does the above condition guarantee that the method converges to the root $x = 1$ for any initial value x^0? Why? Give a numerical example of convergence and one of non-convergence, if any. (The convergence example should be non-trivial, in the sense that the initial value should not be chosen so that the method converges in a finite number of steps).

c) Apply the Aitken acceleration to the convergent example developed in the previous section.

d) If the conditions determined in a) are not satisfied, but Steffenson's method is applied, is the resulting algorithm convergent? Under what conditions? Give a numerical example of convergence.

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solution 3.a)

Given the proposed functional iteration method $x_{k+1} = \Phi(x_k)$ with $\Phi(x_k) = x_k - \dfrac{f(x)}{m}$, for the method to be convergent to the solution $\alpha = 1$, we need:

$$|\Phi'(\alpha)| < 1 \quad \Leftrightarrow \quad \left|1 + \frac{2\alpha}{m}\right| < 1 \quad \Leftrightarrow \quad -1 < 1 + \frac{2\alpha}{m} < 1$$

$$\Leftrightarrow \quad -2 < \frac{2\alpha}{m} < 0$$

$$\Leftrightarrow \quad -1 < \frac{\alpha}{m} < 0$$

$\alpha = 1$, then the conditions will be:

$$\left.\begin{array}{l} \dfrac{\alpha}{m} < 0 \longrightarrow m < 0 \\ \\ -1 < \dfrac{\alpha}{m} \longrightarrow m < -1 \end{array}\right\} \Rightarrow \boxed{m < -1}$$

Additionally, x_0 must be sufficiently close to $\alpha = 1$ for the method to converge.

solution 3.b)

No, the above condition only guarantees convergence when we are close to the solution, meaning, for a sufficiently good initial approximation.

We can use an example with $m = -2$ starting with different initial approximations x_0:

Table 6: Ex. 3b: $m = -2 \quad ; \quad x_0 = 0.0$

k	x_k	$f(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	0.000000000000000	1.000000000000E+00	1.000000000000E+00
1	0.500000000000000	7.500000000000E-01	5.000000000000E-01
2	0.875000000000000	2.343750000000E-01	1.250000000000E-01
3	0.992187500000000	1.556396484375E-02	7.812500000000E-03
4	0.999969482421875	6.103422492743E-05	3.051757812500E-05
5	0.999999999534338	9.313225746155E-10	4.656612873077E-10
6	1.000000000000000	0.00000000000E+00	0.00000000000E+00
7	1.000000000000000	0.00000000000E+00	0.00000000000E+00

Table 7: Ex. 3b: $m = -2 \quad ; \quad x_0 = 3.5$

k	x_k	$f(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	3.500000000000E+00	-1.125000000000E+01	-2.500000000000E+00
1	-2.125000000000E+00	-3.515625000000E+00	3.125000000000E+00
2	-3.882812500000E+00	-1.407623291016E+01	4.882812500000E+00
3	-1.092092895508E+01	-1.182666892419E+02	1.192092895508E+01
4	-7.005427357601E+01	-4.906601246262E+03	7.105427357601E+01
5	-2.523354896707E+03	-6.367318934736E+06	2.524354896707E+06
6	-3.186182822265E+06	-1.015176097690E+13	3.186183822265E+06
7	-5.075883674630E+12	-2.576459507838E+25	5.075883674631E+12

It can be seen how the solutions are different depending on the initial approximation chosen, in the case $x_0 = 0.0$ the solution converges quadratically, however, with $x_0 = 3.5$ the solution diverges.

Performing the iterations again with $m = -4$ with two different initial approximations x_0:

It can be seen that for $m = -4$ the method converges linearly with a A.C.F.$= |1 + \frac{2}{-4}| = 0.5$ for both initial approximations.

solution 3.c) Aitken:

$$\begin{cases} \Delta x_k = x_{k+1} - x_k \\ \Delta^2 x_k = \Delta x_{k+1} - \Delta x_k = x_{k+2} - 2x_{k+1} + x_k \end{cases}$$

$$\hat{x}_k = x_k - \frac{(\Delta x_k)^2}{\Delta^2 x_k} = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}$$

Table 8: Ex. 3b: $m = -4 \quad ; \quad x_0 = 0.0$

k	x_k	$f(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	0.000000000000000	1.000000000000E+00	1.000000000000E+00
1	0.250000000000000	9.375000000000E-01	7.500000000000E-01
2	0.484375000000000	7.653808593750E-01	5.156250000000E-01
3	0.675720214843750	5.434021912515E-01	3.242797851563E-01
4	0.811570762656629	3.413528972009E-01	1.884292373434E-01
5	0.896908986956863	1.955542691160E-01	1.030910130431E-01
6	0.945797554235867	1.054669864015E-01	5.420244576413E-02
7	0.972164300836230	5.489657217960E-02	2.783569916377E-02
8	0.985888443881131	2.802397622164E-02	1.411155611887E-02
9	0.992894437936541	1.416063511468E-02	7.105562063459E-03
10	0.996434596715211	7.118094468994E-03	3.565403284789E-03
15	0.999888195448983	2.235966017771E-04	1.118045510173E-04
20	0.999996505729328	6.988529133056E-06	3.494270671478E-06
25	0.999999890803672	2.183926445243E-07	1.091963282018E-07
30	0.999999996587614	6.824771192626E-09	3.412385596313E-09
35	0.999999999893363	2.132740650751E-10	1.066370325375E-10
40	0.999999999996668	6.664890861430E-12	3.332445430715E-12
45	0.999999999999896	2.082778394197E-13	1.041389197098E-13
50	0.999999999999997	6.439293542826E-15	0.000000000000E+00
52	0.999999999999998	0.000000000000E+00	0.000000000000E+00
53	0.999999999999998	0.000000000000E+00	0.000000000000E+00

Table 9: Ex. 3b: $m = -4 \quad ; \quad x_0 = 3.5$

k	x_k	$f(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	3.500000000000E+00	-1.125000000000E+01	-2.500000000000E+00
1	6.875000000000E-01	5.273437500000E-01	3.125000000000E-01
2	8.193359375000E-01	3.286886215210E-01	1.806640625000E-01
3	9.015080928802E-01	1.872831584714E-01	9.849190711975E-02
4	9.483288824981E-01	1.006723306199E-01	5.167111750190E-02
5	9.734969651531E-01	5.230365883775E-02	2.650303484692E-02
6	9.865728798625E-01	2.667395271978E-02	1.342712013749E-02
7	9.932413680425E-01	1.347158480914E-02	6.758631957539E-03
8	9.966092642447E-01	6.769974421546E-03	3.390735755254E-03
9	9.983017578501E-01	3.393600273335E-03	1.698242149867E-03
10	9.991501579185E-01	1.698961931504E-03	8.498420815336E-04
15	9.999734205589E-01	5.315817573170E-05	2.657944109918E-05
20	9.999991693711E-01	1.661257153840E-06	8.306289218663E-07
25	9.999999740428E-01	5.191434859775E-08	2.595717463194E-08
30	9.999999991888E-01	1.622323386741E-09	8.111616933704E-10
35	9.999999999747E-01	5.069766828569E-11	2.534883414285E-11
40	9.999999999992E-01	1.584288256140E-12	7.921441280701E-13
45	1.000000000000E+00	4.951594689828E-14	2.475797344914E-14
50	1.000000000000E+00	0.000000000000E+00	0.000000000000E+00
52	1.000000000000E+00	0.000000000000E+00	0.000000000000E+00
53	1.000000000000E+00	0.000000000000E+00	0.000000000000E+00

Table 10: Ex. 3c: $m = -2$, using Aitken

k	x_k	Δx_k	$\Delta^2 x_k$	$\widehat{x}_k$	$r_k = (\alpha - \widehat{x}_k)/\alpha$
0	0.000000000000000	0.500000000000000	-0.125000000000000	2.000000000000000	-1.000000000000E+00
1	0.500000000000000	0.375000000000000	-0.257812500000000	1.045454545454550	-4.545454545455E-02
2	0.875000000000000	0.117187500000000	-0.109405517578125	1.000523012552300	-5.230125523013E-04
3	0.992187500000000	0.007781982421875	-0.007751465309411	1.000000119678610	-1.196786103552E-07
4	0.999969482421875	0.000030517112464	-0.000030516646802	1.000000000000000	-7.105427357601E-15
5	0.999999999534338	0.000000000465661	-0.000000000465661	1.000000000000000	0.00000000000E+00
6	1.000000000000000	0.000000000000000			
7	1.000000000000000				

Table 11: Ex. 3c: $m = -4$, using Aitken

k	x_k	Δx_k	$\Delta^2 x_k$	$\widehat{x}_k$	$r_k = (\alpha - \widehat{x}_k)/\alpha$
0	0.000000000000000	0.250000000000000	-0.015625000000000	4.000000000000000	-3.000000000000E+00
1	0.250000000000000	0.234375000000000	-0.043029785156250	1.526595744680850	-5.265957446809E-01
2	0.484375000000000	0.191345214843750	-0.055494667030871	1.144131931656760	-1.441319316568E-01
3	0.675720214843750	0.135850547812879	-0.050512323512645	1.041083952990530	-4.108395299053E-02
4	0.811570762656629	0.085338224300234	-0.036449657021231	1.011369968516790	-1.136996851679E-02
5	0.896908986956863	0.048888567279003	-0.022521820678640	1.003032379230400	-3.032379230403E-03
6	0.945797554235867	0.026366746600363	-0.012642603555462	1.000786649103700	-7.866491037041E-04
7	0.972164300836230	0.013724143044901	-0.006718148989491	1.000200610035620	-2.006100356167E-04
8	0.985888443881131	0.007005994055410	-0.003465835276741	1.000050672834670	-5.067283466831E-05
9	0.992894437936541	0.003540158778670	-0.001760635161421	1.000012735044770	-1.273504477139E-05
10	0.996434596715211	0.001779523617249	-0.000887381125025	1.000003192231880	-3.192231884608E-06
15	0.999888195448983	0.000055899150444	-0.000027947231511	1.000000003125500	-3.125501413237E-09
20	0.999996505729328	0.000001747132283	-0.000000873563852	1.000000000003050	-3.052225139299E-12
25	0.999999890803672	0.000000054598161	-0.000000027299078	1.000000000000000	0.000000000000E+00
30	0.999999996587614	0.000000001706193	-0.000000000853096	1.000000000000000	0.000000000000E+00
35	0.999999999893363				
40	0.999999999996668				
45	0.999999999999896				
50	0.999999999999997				
52	0.999999999999998				
53	0.999999999999998				

The differences with the previous example are significant since the convergence was already quadratic for $m = -2$. In the case $m = -4$ the convergence was linear and by using Aitken's acceleration the convergence improved significantly.

solution 3.d)

If the method is of first order but with A.C.F>1 (non-convergent Whittaker), we know that Steffensen's method:

$$x_{k+1} = \Phi(x_k) \qquad \text{with} \quad \Phi(x) = \frac{x\phi(\phi(x)) - \phi(x)^2}{\phi(\phi(x)) - 2\phi(x) + x}$$

is of second order, therefore the resulting algorithm will be convergent for an initial value close enough to the solution, and the convergence will be quadratic.

$$\phi(x) = x - \frac{1 - x^2}{m} \Rightarrow \phi(\phi(x)) = \phi(x) - \frac{1 - \phi(x)^2}{m} = \left(x - \frac{1 - x^2}{m}\right) - \frac{1 - \left(x - \frac{1 - x^2}{m}\right)^2}{m}$$

Table 12: Ex. 3d: Divergence with $m = 1$

k	x_k	$f(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	1.100000000000E+00	-2.100000000000E-01	-1.000000000000E-01
1	1.310000000000E+00	-7.161000000000E-01	3.100000000000E-01
2	2.026100000000E+00	-3.105081210000E+00	1.026100000000E+00
3	5.131181210000E+01	-2.532902060986E+01	4.131181210000E+00
4	3.046020181986E+01	-9.268238949064E+02	2.946020181986E+01
5	9.572840967263E+03	-9.163918418451E+05	9.562840967263E+02
6	9.173491259418E+06	-8.415294188652E+11	9.173481259418E+05
7	8.415303362143E+12	-7.081733067689E+13	8.415303362133E+11

Table 13: Ex. 3d: Application of Steffensen to the divergent case

k	x_k	$\phi(x_k)$	$\phi(\phi(x_k))$	$f(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	1.100000000000000	1.310000000000000	2.026100000000000	-2.10000000000E+00	-1.000000000000E-01
1	1.012863070539420	1.038754670201960	1.117765935068340	-2.589159966254E-02	-1.286307053945E-02
2	1.000242980964060	1.000729001931940	1.002187537239630	-4.860209678748E-04	-2.429809640629E-04
3	1.000000088523830	1.000000265571490	1.000000796714530	-1.770476594221E-07	-8.852382582525E-08
4	1.000000000000000	1.000000000000000	1.000000000000000	0.00000000000E+00	0.00000000000E+00

4.— Analyze in detail whether any of the following iterative techniques may be suitable for calculating a triple root of the equation $f(x) = 0$:

a) $x_{k+1} = x_k - f'(x_k)/f''(x_k)$

b) $x_{k+1} = x_k - f''(x_k)/f'''(x_k)$

c) $x_{k+1} = x_k - f'''(x_k)/f^{(4}(x_k)$

solution 4.

The root α of $f(x)$ has a multiplicity of 3. Therefore: $f(\alpha) = f'(\alpha) = f''(\alpha) = 0$. Additionally, for the method $x_{k+1} = \phi(x_k)$ to be convergent, it has to be satisfied that:

$$\left\{ \begin{array}{ll} \alpha = \phi(\alpha) & \text{fixed-point condition} \\ |\phi'(\alpha)| < 1 & \text{A.C.F.} < \ 1 \\ x_0 \text{ close enough to } \alpha & \end{array} \right. .$$

solution 4.a

The proposed method is an application of Newton's method to a function $g(x)$, being $g(x) = f'(x)$. Then it is not suitable, since α is a double root of $g(x)$, and therefore the method would decrease its order to first order, instead of being quadratic. Moreover, the denominator tends to 0 so it may produce numerical instabilities.

On the other hand, the method could converge to a value ρ such that $f'(\rho) = 0$ but $f(\rho) \neq 0$, and therefore different from the solution we are looking for.

solution 4.b

Following the same criteria of the previous example, the method is an application of Newton's method, but now to a function $h(x) = f''(x)$. In this case it is suitable and it will converge with quadratic order since α is a simple root of $h(x)$.

Even though, the method may converge to a γ such tat $f''(\gamma) = 0$ but $f(\gamma) \neq 0$ and $f'(\gamma) \neq 0$, and therefore, converge to a wrong solution.

solution 4.c

The proposed method is an application of Newton's method to a function $r(x) = f'''(x)$, So the method does not correspond to the problem at all.

It can not converge since:

$$\phi(\alpha) = \alpha - \frac{f'''(\alpha)}{f^{(4}(\alpha)} \neq \alpha$$

5.— To obtain the simple roots of equation $f(x)$ the following iterative method is proposed:

$$x_{i+1} = x_i - \frac{u\,(x_i)}{a\,u\,(x_i) + b} \quad ; \qquad u\,(x_i) = \frac{f\,(x_i)}{f'\,(x_i)} \quad ,$$

where a and b are two real constants:

a) Find the constants a and b for which the method is of third order.

b) Propose the formulation to calculate the square root of the positive real number s by solving the equation $x^2 - s = 0$ by Newton's method and by this third order method.

c) Apply both algorithms to the case $s = 2$, starting at $x_0 = 1$ and $x_0 = 5$

d) What conclusions can be drawn? Were the results obtained predictable?

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solution 5.a)

The proposed method can be expressed as:

$$x_{i+1} = \phi(x_i) \qquad \text{with} \qquad \phi(x) = x - \frac{f(x)}{a\, f(x) + b\, f'(x)}$$

Thus, the method will be of third order at least, if and only if: $\alpha = \phi(\alpha), \phi'(\alpha) = 0$ and $\phi''(\alpha) = 0$, for α such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.

$$\alpha = \phi(\alpha) \Leftrightarrow \alpha = \alpha - \frac{\cancel{f(\alpha)}}{a\, \cancel{f(\alpha)} + b\, f'(\alpha)} = \alpha, \qquad \forall b \neq 0$$

$$\phi'(x) = 1 - \left(\frac{f'(x)\,(af(x) + bf'(x)) - f(x)\,(af'(x) + bf''(x))}{(af(x) + bf'(x))^2}\right) = 1 - \frac{b\,(f'(x))^2 - bf(x)f''(x)}{(af(x) + bf'(x))^2}$$

$$\begin{aligned}\phi''(x) \quad = \quad -b \Bigg[& \frac{(2f'(x)f''(x) - f'(x)f''(x) - f(x)f'''(x))\,(af(x) + bf'(x))^2}{(af(x) + bf'(x))^4} - \\ & \frac{\left((f'(x))^2 - f(x)f''(x)\right) 2\,(af(x) + bf'(x))\,(af'(x) + bf''(x))}{(af(x) + bf'(x))^4}\Bigg]\end{aligned}$$

Therefore:

$$\begin{aligned}\phi'(\alpha) \quad &= \quad 1 - b\frac{(f'(\alpha))^2}{b^2 f'(\alpha)^2} = 1 - \frac{1}{b} = 0 \Leftrightarrow \boxed{b = 1} \\ \phi''(\alpha) \quad &= \quad -b\left[\frac{(f'(\alpha)f''(\alpha))\,b^2\,(f'(\alpha))^2}{b^4\,(f'(\alpha))^4} - \frac{(f'(\alpha))^2\ 2\,b\,f'(\alpha)\,(af(\alpha) + bf'(\alpha))^2}{b^4\,(f'(\alpha))^4}\right] = \\ & -\frac{f''(\alpha)}{bf'(\alpha)} + \frac{2}{b^2}\frac{af'(\alpha) + bf''(\alpha)}{f'(\alpha)} = \frac{2a}{b^2} + \frac{f''(\alpha)}{bf'(\alpha)} = 0 \Leftrightarrow \boxed{a = -\frac{b}{2}\frac{f''(\alpha)}{f'(\alpha)}}\end{aligned}$$

So the third order method will be:

$$\phi(x) = x - \frac{f(x)}{f'(x) - \dfrac{1}{2}\dfrac{f''(\alpha)}{f'(\alpha}f(x)}$$

For other values of a the method will be of second order. ($a = 0\ \Rightarrow$ Newton)

solution 5.b)

$$\begin{cases} f(x) = x^2 - 5 \quad \rightarrow \quad \alpha = \sqrt{s} \\ f'(x) = 2x \\ f''(x) = 2 \end{cases}$$

$$\text{Newton:} \quad x_{i+1} = \phi_N(x_i) \quad ; \quad \phi_N(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - s}{2x} = \frac{x}{2} + \frac{s}{2x}$$

$$\text{3rd order:} \quad x_{i+1} = \phi_3(x_i) \quad ; \quad \phi_3(x) = x - \frac{f(x)}{af(x)bf'(x)} = x - \frac{x^2 - s}{a(x^2 - s) + b(2x)}$$

$$\text{With} \begin{cases} b = 1 \\ a = \dfrac{-bf''(\alpha)}{2f'(\alpha)} = \dfrac{-2}{2(2\alpha)} = -\dfrac{1}{2\alpha} \end{cases}$$

We need the value of the solution α to adjust correctly the value of a, so the method is not very useful. In this particular case, we know that $\alpha = \sqrt{s}$ so it could be used.

In practice, the method could be modified estimating and updating the value of a in each iteration as:

$$a = -\frac{b\,f''(\alpha)}{2\,f'(\alpha)} \approx -\frac{b\,f''(x_k)}{2\,f'(x_k)}$$

Such that:

$$x_{i+1} = \phi(x_i), \quad \phi(x) = x - \frac{f(x)}{\dfrac{-f''(x)}{2\,f'(x)}f(x) + f'(x)} = x - \frac{2\,f(x)f'(x)}{2\,(f'(x))^2 - f(x)\,f''(x)}$$

This method is useful and will probably be of third order (should be checked) but requires the determination of $f''(x)$

solution 5.c)

Table 14: Ex. 5c: Application of Newton-Raphson with $x_0 = 1$

$s = 2$

k	x_k	$f(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	1.000000000000000	$-1.000000000000E+00$	$2.928932188135E-01$
1	1.500000000000000	$2.500000000000E-01$	$-6.066017177982E-02$
2	1.416666666666670	$6.944444444445E-03$	$-1.734606680942E-03$
3	1.414215686274510	$6.007304882871E-06$	$-1.501825092953E-06$
4	1.414213562374690	$4.510614104447E-12$	$-1.127542503809E-12$
5	1.414213562373100	$0.000000000000E+00$	$0.000000000000E+00$
6	1.414213562373100	$0.000000000000E+00$	$0.000000000000E+00$

solution 5.d) We can see that:

Table 15: Ex. 5c: Application of Newton-Raphson with $x_0 = 5$

$s = 2$

k	x_k	$f(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	5.000000000000000	$2.300000000000E+01$	$-2.535533905933E+00$
1	2.700000000000000	$5.290000000000E+00$	$-9.091883092037E-01$
2	1.720370370370370	$9.596742112483E-01$	$-2.164855550413E-01$
3	1.441455368177650	$7.779357844816E-02$	$-1.926286561617E-02$
4	1.414470981367770	$7.281571315052E-04$	$-1.820227167415E-04$
5	1.414213585796880	$6.625247950254E-08$	$-1.656311976461E-08$
6	1.414213562373100	$0.000000000000E+00$	$0.000000000000E+00$
7	1.414213562373100	$0.000000000000E+00$	$0.000000000000E+00$

Table 16: Ex. 5c: Application of the third order method with $x_0 = 1$

$s =$ 2
$a =$ -0.353553390593274 $[-(1/2)(f''(\alpha)/(f'(\alpha))]$
$b =$ 1

k	x_k	$f(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	1.000000000000000	$-1.000000000000E+00$	$2.928932188135E-01$
1	1.424889447588830	$3.030993785001E-02$	$-7.548990831217E-03$
2	1.414213410845510	$-4.285847143848E-07$	$1.071461843694E-07$
3	1.414213562373100	$0.000000000000E+00$	$0.000000000000E+00$
4	1.414213562373100	$0.000000000000E+00$	$0.000000000000E+00$

Table 17: Ex. 5c: Application of the third order method with $x_0 = 5$

$s =$ 2
$a =$ -0.353553390593274 $[-(1/2)(f''(\alpha)/(f'(\alpha))]$
$b =$ 1

k	x_k	$f(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	5.000000000000000	$2.300000000000E+01$	$-2.535533905933E+00$
1	-7.310841140187210	$5.144839817705E+01$	$6.169545346404E+00$
2	-5.742839203198120	$3.098020211379E+01$	$5.060800543845E+00$
3	-4.362188040799800	$1.702868450330E+01$	$4.084532744460E+00$
4	-3.207303614914170	$8.286796478242E+00$	$3.267906135430E+00$
5	-2.320487136236160	$3.384660549437E+00$	$2.640832189689E+00$
6	-1.740686924813230	$1.029990970216E+00$	$2.230851528458E+00$
7	-1.472845873963900	$1.692749684525E-01$	$2.041459305123E+00$
8	-1.416524880948660	$6.542738346614E-03$	$2.001634349038E+00$
9	-1.414217332173550	$1.066262008198E-05$	$2.000002665651E+00$
10	-1.414213562383140	$2.842259760882E-11$	$2.000000000007E+00$
11	-1.414213562373100	$0.000000000000E+00$	$2.000000000000E+00$
12	-1.414213562373100	$0.000000000000E+00$	$2.000000000000E+00$

$$\left\{\begin{array}{lcl} x_0 = 1 & \Longrightarrow & \left\{\begin{array}{l}\text{Newton-Raphson converges with quadratic order}\\ \text{The new method converges with order 3 but the advantages}\\ \qquad \text{are not significant}\end{array}\right. \\ x_0 = 5 & \Longrightarrow & \left\{\begin{array}{l}\text{Newton-Raphson converges even though } x_0 \text{ is not close } \alpha\\ \text{The new algorithm does not converge to } +\sqrt{2} \text{ and converges to}\\ \qquad \text{other root } (-\sqrt{2}) \text{ by accident}\end{array}\right.\end{array}\right.$$

In general:

- In practice, the difference between an algorithm of order 2 and other of order 3 is not significant, in terms of number or iterations.
- Usually, higher order methods have smaller convergence radius.

In any case, the proposed algorithm is not useful in practice, since we need to know the value of the solution in order to adjust its coefficients and to be able to apply it. We could convert it into a practical 3rd order algorithm (see note at the end of section b) but in general it will not compensate the additional work (calculation of the second derivative) if we compare the results with those provided by the Newton-Raphson method.

6.— Consider the equation $f(x) = tg(\pi x) - 6$. Let $x_0 = 0$ and $x_1 = 0.48$, we want to approximate $x = \frac{1}{\pi}$ arctan(6) = 0.447431543 using:

a) The Bisection method

b) The *Regula-Falsi* Method

c) The Secant Method

Analyze the results obtained for each case after ten iteration

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solution 6.

$$f(x) = \tan(\pi\, x) - 6$$

$$x_0 = 0; \quad x_1 = 0.48 \longrightarrow \left\{\begin{array}{l} f(x_0) = -6 \\ f(x_1) \approx 9.895 \end{array}\right.$$

$$\alpha = \frac{1}{\pi}\,\text{arctan}(6) = 0.447431543$$

solution 6.a Bisection

Starting at $x_0 = 0; \quad z_0 = 0.48$:

$$\mu_k = \frac{x_k + z_k}{2} \quad \text{if} \quad f(x_k)f(\mu_k) \quad \text{is} \quad \left\{\begin{array}{lcl} < 0 & \longrightarrow & x_{k+1} = x_k;\ z_{k+1} = \mu_k \\ = 0 & \longrightarrow & \alpha = \mu_k \\ > 0 & \longrightarrow & x_{k+1} = \mu_k;\ z_{k+1} = z_k \end{array}\right.$$

We expect a similar to linear convergence with ACF≈ 1/2.

Table 18: Ex. 6a:Application of Bisection Method

k	x_k	z_k	μ_k	$f(\mu_k)$	$f(x_k)\,f(\mu_k)$	$r_k = \dfrac{\alpha - \mu_k}{\alpha}$
0	0.000000000000000	0.480000000000000	0.240000000000000	$-5.061E+00$	$3.037E+01$	$4.636E-01$
1	0.240000000000000	0.480000000000000	0.360000000000000	$-3.875E+00$	$1.961E+01$	$1.954E-01$
2	0.360000000000000	0.480000000000000	0.420000000000000	$-2.105E+00$	$8.158E+00$	$6.131E-02$
3	0.420000000000000	0.480000000000000	0.450000000000000	$3.138E-01$	$-6.605E-01$	$-5.740E-03$
4	0.420000000000000	0.450000000000000	0.435000000000000	$-1.171E+00$	$2.466E+00$	$2.778E-02$
5	0.435000000000000	0.450000000000000	0.442500000000000	$-5.245E-01$	$6.143E-01$	$1.102E-02$
6	0.442500000000000	0.450000000000000	0.446250000000000	$-1.343E-01$	$7.047E-02$	$2.641E-03$
7	0.446250000000000	0.450000000000000	0.448125000000000	$8.167E-02$	$-1.097E-02$	$-1.550E-03$
8	0.446250000000000	0.448125000000000	0.447187500000000	$-2.824E-02$	$3.794E-03$	$5.454E-04$
9	0.447187500000000	0.448125000000000	0.447656250000000	$2.623E-02$	$-7.407E-04$	$-5.022E-04$
10	0.447187500000000	0.447656250000000	0.447421875000000	$-1.124E-03$	$3.173E-05$	$2.161E-05$
15	0.447421875000000	0.447436523437500	0.447429199218750	$-2.725E-04$	$3.061E-07$	$5.239E-06$
20	0.447431488037109	0.447431945800781	0.447431716918945	$2.018E-05$	$-1.296E-10$	$-3.881E-07$
25	0.447431530952454	0.447431545257568	0.447431538105011	$-6.026E-07$	$8.640E-13$	$1.159E-08$
30	0.447431543022394	0.447431543469429	0.447431543245911	$-4.979E-09$	$1.542E-16$	$9.574E-11$
35	0.447431543287821	0.447431543301791	0.447431543294806	$7.043E-10$	$-7.578E-20$	$-1.354E-11$
40	0.447431543288694	0.447431543289131	0.447431543288912	$1.928E-11$	$-1.176E-22$	$-3.706E-13$
45	0.447431543288735	0.447431543288749	0.447431543288742	$-5.471E-13$	$7.352E-25$	$1.066E-14$
47	0.447431543288745	0.447431543288749	0.447431543288747	$4.441E-14$	$-7.139E-27$	$0.000E+00$
48	0.447431543288745	0.447431543288747	0.447431543288746	$-5.418E-14$	$8.710E-27$	$0.000E+00$

solution 6.b Regula-Falsi

Starting at $x_0 = 0;\quad z_0 = 0.48$:

$$\mu_k = x_k - \frac{f(x_k)}{f(x_k) - f(z_k)}(x_k - z_k) = \frac{z_k\, f(x_k) - x_k\, f(z_k)}{f(x_k) - f(z_k)}$$

$$\text{If} \quad f(\mu_k)f(x_k) \begin{cases} < 0 \;\longrightarrow\; x_{k+1} = x_k;\; z_{k+1} = \mu_k \\ = 0 \;\longrightarrow\; \alpha = \mu_k \\ > 0 \;\longrightarrow\; x_{k+1} = \mu_k;\; z_{k+1} = z_k \end{cases}$$

We expect convergence similar to linear.

solution 6.c Secant

Starting at $x_0 = 0;\quad x_1 = 0.48$:

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} = \frac{x_{k-1}\, f(x_k) - x_k\, f(x_{k-1})}{f(x_k) - f(x_{k-1})}$$

We expect superlinear convergence of order $\dfrac{1+\sqrt{5}}{2} \approx 1.618$.

It can be seen how both, Bisection and *Regula-Falsi* converge slowly.

The Secant method does not converge for $x_0 = 0,\ x_1 = 0.48$. However, it is fast for $x_0 = 0.4,\ x_1 = 0.48$.

Table 19: Ex. 6b: Application of *Regula-Falsi* method

k	x_k	z_k	μ_k	$f(\mu_k)$	$f(x_k)\,f(\mu_k)$	$r_k = \dfrac{\alpha - \mu_k}{\alpha}$
0	0.000000000000000	0.480000000000000	0.181194241690512	$-5.360E+00$	$3.216E+01$	$5.950E-01$
1	0.181194241690512	0.480000000000000	0.286187165822290	$-4.742E+00$	$2.542E+01$	$3.604E-01$
2	0.286187165822290	0.480000000000000	0.348981227423906	$-4.053E+00$	$1.922E+01$	$2.200E-01$
3	0.348981227423906	0.480000000000000	0.387052621184471	$-3.301E+00$	$1.338E+01$	$1.349E-01$
4	0.387052621184471	0.480000000000000	0.410304719883756	$-2.546E+00$	$8.403E+00$	$8.298E-02$
5	0.410304719883756	0.480000000000000	0.424566482926005	$-1.860E+00$	$4.734E+00$	$5.110E-02$
6	0.424566482926005	0.480000000000000	0.433336313016866	$-1.295E+00$	$2.408E+00$	$3.150E-02$
7	0.433336313016866	0.480000000000000	0.438737408614391	$-8.685E-01$	$1.125E+00$	$1.943E-02$
8	0.438737408614391	0.480000000000000	0.442066949081701	$-5.664E-01$	$4.919E-01$	$1.199E-02$
9	0.442066949081701	0.480000000000000	0.444120661758188	$-3.623E-01$	$2.052E-01$	$7.400E-03$
10	0.444120661758188	0.480000000000000	0.445387878162835	$-2.287E-01$	$8.286E-02$	$4.568E-03$
15	0.447134701747684	0.480000000000000	0.447248279025501	$-2.123E-02$	$7.284E-04$	$4.096E-04$
20	0.447404916875548	0.480000000000000	0.447415104357734	$-1.910E-03$	$5.909E-06$	$3.674E-05$
25	0.447429154821945	0.480000000000000	0.447430068666389	$-1.714E-04$	$4.759E-08$	$3.296E-06$
30	0.447431329035498	0.480000000000000	0.447431411010301	$-1.538E-05$	$3.829E-10$	$2.956E-07$
35	0.447431524069526	0.480000000000000	0.447431531422935	$-1.379E-06$	$3.081E-12$	$2.652E-08$
40	0.447431541564719	0.480000000000000	0.447431542224344	$-1.237E-07$	$2.479E-14$	$2.379E-09$
45	0.447431543134096	0.480000000000000	0.447431543193266	$-1.110E-08$	$1.995E-16$	$2.134E-10$
50	0.447431543274874	0.480000000000000	0.447431543280182	$-9.956E-10$	$1.605E-18$	$1.914E-11$
55	0.447431543287502	0.480000000000000	0.447431543287978	$-8.930E-11$	$1.292E-20$	$1.717E-12$
60	0.447431543288635	0.480000000000000	0.447431543288678	$-8.007E-12$	$1.040E-22$	$1.541E-13$
65	0.447431543288737	0.480000000000000	0.447431543288740	$-7.283E-13$	$8.532E-25$	$1.410E-14$
70	0.447431543288746	0.480000000000000	0.447431543288746	$-7.105E-14$	$7.384E-27$	$0.000E+00$
71	0.447431543288746	0.480000000000000	0.447431543288746	$-4.619E-14$	$3.282E-27$	$0.000E+00$

Table 20: Ex. 6c: Application of Secant method

k	x_k	$r_k = \dfrac{\alpha - \mu_k}{\alpha}$
0	0.400000000000000	$1.060084922491E-01$
1	0.480000000000000	$-7.278980930103E-02$
2	0.418240449937923	$6.524147389400E-02$
3	0.429444232066420	$4.020125870008E-02$
4	0.457230361057407	$-2.190014967795E-02$
5	0.444112050716810	$7.418995423384E-03$
6	0.446817662750469	$1.372009969984E-03$
7	0.447469927757018	$-8.578847165919E-05$
8	0.447431099172228	$9.925909904007E-07$
9	0.447431542967413	$7.181742978091E-10$
10	0.447431543288749	$-5.995204332976E-15$
11	0.447431543288747	$0.000000000000E+00$
12	0.447431543288747	$0.000000000000E+00$

7.— We want to find the inverse of any number $b > 0$, without performing any division operation. Let β $(0.1 < \beta < 1)$ be the mantissa of the number b in decimal base, and let e be its exponent, this is,

$$b = \beta \cdot 10^e; \qquad 0.1 \leq \beta < 1.$$

Obviously, the part corresponding to the exponent can be easily inverted by a change of sign. To invert the mantissa, we propose to apply Newton's method to calculate the root of the function $f(x) = \beta - (1/x)$.

a) Apply Newton's method to the given function, simplify as much the expression so the final expression does no use any division.

b) Study graphically the behavior of the Newton's method in this case and find an initial value x_0 which guarantees the convergence of the algorithm.

c) Obtain through the proposed algorithm the inverse of $b = 0.39000 \cdot 10^{-1}$. Make comments on the results.

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solution 7.

$$b = \beta\ 10^e \qquad 0.1 < \beta < 1 \quad \Longrightarrow \quad b^{-1} = \beta^{-1}\ 10^{-e}$$

To obtaine β^{-1} we propose $f(x) = \beta - \dfrac{1}{x} = 0$; $\alpha = \dfrac{1}{\beta} \in (1, 10)$

solution 7.a Newton-Raphson:

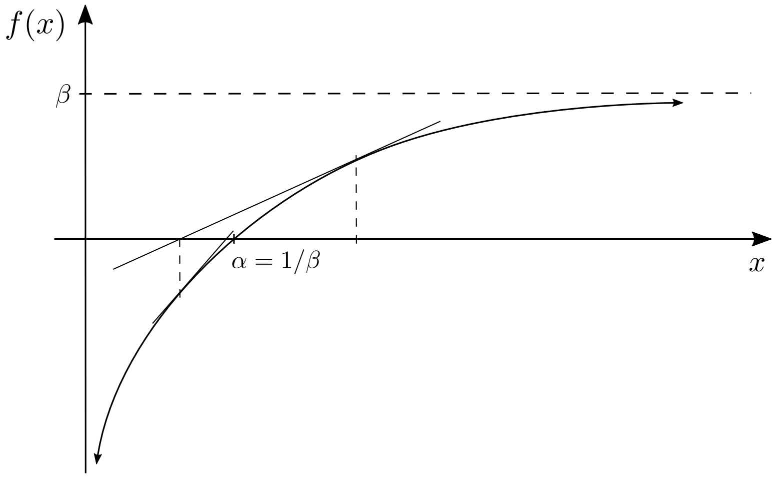
$$\left.\begin{array}{l} f(x) = \beta - \dfrac{1}{x} \\[2ex] f'(x) = \dfrac{1}{x^2} \end{array}\right\} \quad \Longrightarrow \quad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = \phi(x_k)$$

$$\begin{aligned} \phi(x) &= x - \frac{f(x)}{f'(x)} = x - \frac{\beta - 1/x}{1/x^2} \\ &= x - (\beta\, x^2 - x) = 2\, x - \beta\, x^2 \\ &= x(2 - \beta\, x) \end{aligned}$$

Then $x_{k+1} = \phi(x_k)$ with $\phi(x) = x(2 - \beta\, x)$.

The method will be quadratic since $f'(\alpha) \neq 0$ for $\alpha = 1/\beta$

solution 7.b)



$$\text{We see that } \begin{cases} x_k < \alpha \rightsquigarrow x_{k+1} > x_k \quad \text{and} \quad x_k < \alpha \quad \Rightarrow \quad \text{OK} \\ x_k > \alpha \rightsquigarrow x_{k+1} > x_k \quad \Rightarrow \quad \text{Previous case} \end{cases}$$

So it seems reasonable to start with $x_0 < \alpha$. Since $1 < \alpha < 10$ we can choose $x_0 = 1$.

solution 7.c)

For $b = 0.39999\,10^{-1} \quad \Rightarrow \quad \beta = 0.39$.

We start at $x_0 = 1$ and iterate as $x_{k+1} = x_k(2 - \beta\, x_k)$

Table 21: Ex. 7c: Application of Newton-Raphson

$\beta = 0.39$

k	x_k	$f(x_k)$	$r_k = \dfrac{\alpha - \mu_k}{\alpha}$
0	1.000000000000000	$-6.100000000000E-01$	$6.100000000000E-01$
1	1.610000000000000	$-2.311180124224E-01$	$3.721000000000E-01$
2	2.209081000000000	$-6.267692764548E-02$	$1.384584100000E-01$
3	2.514946842821210	$-7.622718291025E-03$	$1.917073129973E-02$
4	2.563160212978030	$-1.433843022007E-04$	$3.675169385664E-04$
5	2.564102217772560	$-5.267680019960E-08$	$1.350687001134E-07$
6	2.564102564102520	$-7.105427357601E-15$	$1.820765760385E-14$
7	2.564102564102560	$0.000000000000E+00$	$0.000000000000E+00$
8	2.564102564102560	$0.000000000000E+00$	$0.000000000000E+00$

The convergence is quadratic and we obtain 16 significant digits in just 8 iterations.

8.— In a structural problem it is required to solve the buckling problem of a pinned beam, with an elastic spring restraining the rotation in one of its supports. It is desired to solve this problem, for any values of the variables that define the system, namely, the length L of the beam, its bending stiffness EI, and the elastic constant K_ϕ of the spring.

It is deduced, by means of the equilibrium of the deformed beam, that the load that produces the instability of the straight geometry of the beam, also called buckling load P, is given by the lowest of the non-trivial solutions of the equation:

$$\tan(kL) = \frac{kL}{\dfrac{(kL)^2}{k_\phi} + 1}$$

where $k = \frac{P}{EI}$ and $k_\phi = \frac{K_\phi L}{EI}$. The aim is to solve this equation by means of a numerical method that works for arbitrary values of the data. For this purpose, the initial expression is written in the equivalent, but more compact form:

$$\tan(x) = \frac{x}{(cx)^2 + 1},$$

with the new unknown being $x = kL$, and $c = \sqrt{1/k_\phi}$ being a parameter that is calculated based on the data. Obviously, it is of interest to calculate the smallest non-trivial positive root of the above equation.

For this purpose, the following numerical algorithm is proposed: starting from a certain initial value, conveniently chosen , iterate until convergence by means of the formulas:

$$y^k = g(x^k); \qquad g(x) = \frac{x}{(cx)^2 + 1}$$

$$x^{k+1} = f^{-1}(y^k); \qquad f(x) = \tan(x)$$

where f^{-1} is in the inverse function of f:

a) To perform an analytical study of the asymptotic convergence of the algorithm as a function of the parameter c.

b) Draw approximately the functions $f(x)$ and $g(x)$, and bound an interval where the solution of the problem is found as a function of the parameter $c > 0$.

c) Propose a reasoned initial value that works for any value of c.

d) Iterate until convergence, for the following values of the parameter c; $c = 0.1$, $c = 0.5$, $c = 1$, $c = 100$, $c = 1000$. Comment on the results obtained.

---

solution 8.

solution 8.a

We seek the solution of $f(x) = g(x)$ with

$$\left\{ \begin{array}{l} f(x) = \tan(x) \\ \\ g(x) = \dfrac{x}{(c\,x)^2 + 1} \end{array} \right\} \quad \text{by means of the algorithm:} \quad \left\{ \begin{array}{l} y_k = g(x_k) \\ \\ x_{k+1} = f^{-1}(y_k) \end{array} \right.$$

What we actually do is solve the problem as: $x = f^{-1}(g(x))$ by means of fixed-point iterations:

$$x_{k+1} = \phi(x_k) \quad \text{with} \quad \phi(x) = f^{-1}(g(x)) \qquad \left\{ \begin{array}{l} f(x) = \tan(x) \\ g(x) = \dfrac{x}{(c\,x)^2 + 1} \end{array} \right.$$

The method will be convergent if $|\phi'(\alpha)| < 1$ at $\alpha = \phi(\alpha)$

$$\begin{aligned} \phi'(x) &= \frac{d}{dx}\left[f^{-1}(g(x))\right] = \frac{d}{dy} f^{-1}(y)\Big|_{y=g(x)} \frac{dg(x)}{dx} \\ &= \frac{1}{\left[\left. \dfrac{df(z)}{dz} \right|_{z=f^{-1}(y)} \right] \Bigg|_{y=g(x)}} \frac{dg(x)}{dx} = \frac{1}{f'(f^{-1}(g(x)))}\, g'(x) \end{aligned}$$

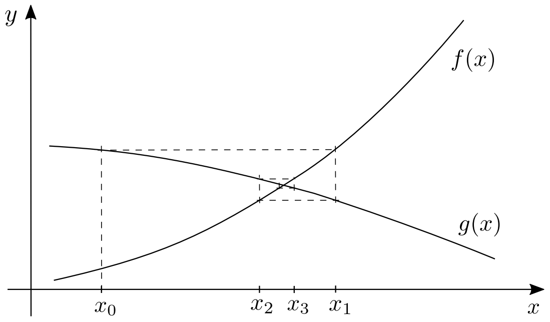
Then:

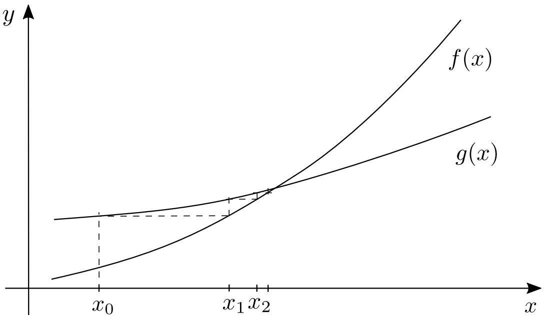
$$\phi'(\alpha) = \frac{1}{f'(f^{-1}(g(\alpha)))}\, g'(\alpha) = \frac{1}{f'(\alpha)}\, g'(\alpha)$$

Therefore,

$$|\phi'(\alpha)| = \left| \frac{g'(\alpha)}{f'(\alpha)} \right| \qquad \text{and} \qquad |\phi'(\alpha)| < 1 \quad \Longleftrightarrow \quad \boxed{|g'(\alpha)| \;<\; |f'(\alpha)|}$$

To verify the condition, we must draw $g(x)$ and $f(x)$. The next section analyzes those curves. The iterations are performed as follows:



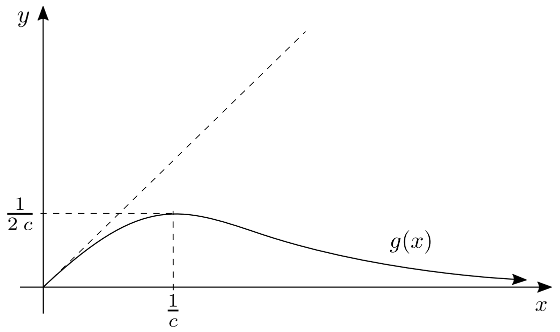


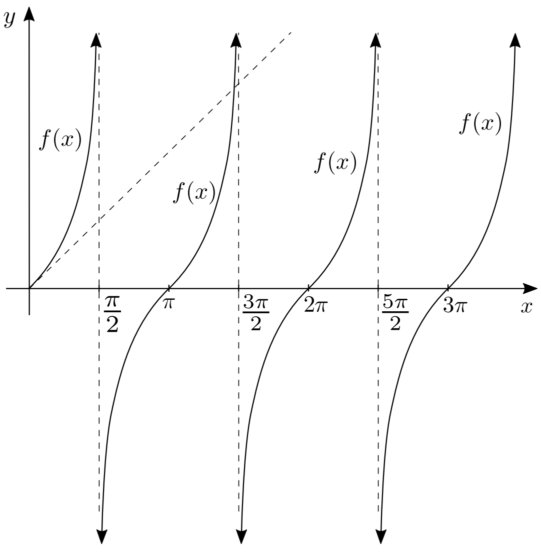
solution 8.b

$$g(x) = \frac{x}{(c\,x)^2 + 1} \qquad \longrightarrow \qquad g'(x) = \frac{((c\,x)^2 + 1) - x(2\,c^2 x)}{((c\,x)^2 + 1)^2}$$

Then:

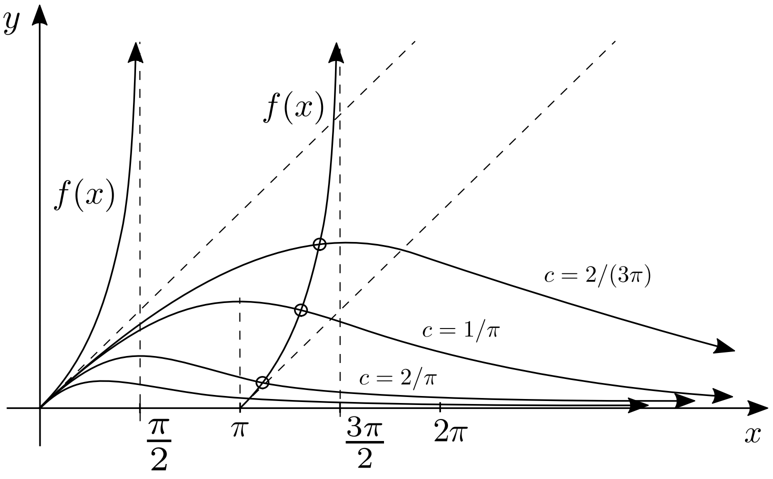
$$\left\{ \begin{array}{l} g'(0) = 1 \\ g'(x) = 0 \;\Rightarrow\; x = \dfrac{1}{c} \;\Rightarrow\; g\,(1/c) = \dfrac{1}{2\,c} \end{array} \right.$$





The root we are searching (lowest positive non-zero) is always in the interval $(\pi,\, 3\pi/2) \quad \forall c > 0$

For different values of c different situations arise:



We see that $f'(\alpha) > 1$ always.

For $g(x)$ the maximum possible slope will be located at the point that $g''(x) = 0$ (inflection), or at $x = 0$

$$\begin{aligned} g''(x) &= \frac{-2\,c^2x(1+c^2x^2)^2 - 2(1+c^2x^2)2\,c^2x(1-c^2x^2)}{(1+c^2x^2)^4} \\ &= \frac{-2\,c^2x(1+c^2x^2) - 4\,c^2x(1-c^2x^2)}{(1+c^2x^2)^3} = \frac{2\,c^2x(3-c^2x^2)}{(1+c^2x^2)^3} \end{aligned}$$

Then $g''(x) = 0 \quad \Rightarrow \quad \left\{ \begin{array}{ll} x = 0 & \rightarrow \quad g'(0) = 1 \\ x = \dfrac{\sqrt{3}}{c} & \rightarrow \quad g'\!\left(\dfrac{\sqrt{3}}{c}\right) = -\dfrac{1}{8} \end{array} \right.$

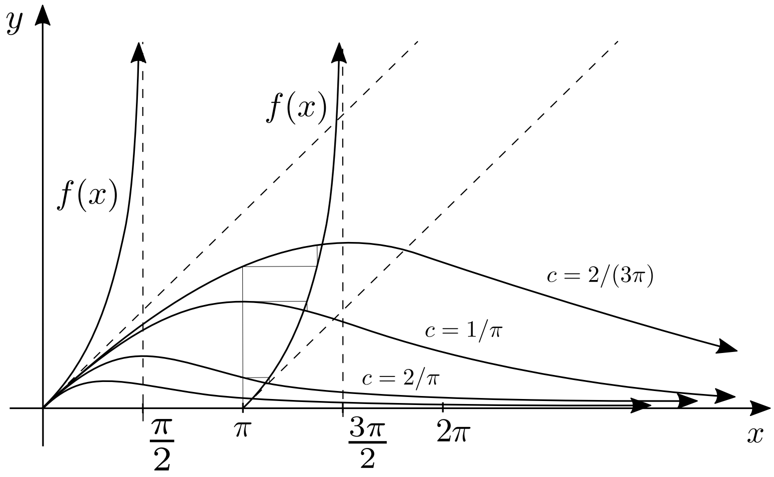
Given the shape of $g(x)$, it is satisfied that $|g'(\alpha)| < 1$

Thus:

$$\boxed{|g'(\alpha)| < |f'(\alpha)| \text{ in all cases } \Rightarrow \text{ The algorithm converges } \forall c}$$

solution 8.c

From the last plot, it seems that the method will work as far as we start with $x_0 = \pi$, since:



solution 8.d

Is is necessary in this point to have the caution that $f^{-1}(y_k)$ has to give values of the arctangent between (π y $3\pi/2$) and not in between (0 y $\pi/2$). We will program $\boxed{f^{-1}(y_k) = \arctan(y_k) + \pi}$. This is relevant since the function $f(x) = \tan(x)$ has more than one inverse. Since we want to work within $(\pi,\ 3\pi/2)$ we need to use the expression of the inverse in that interval.

Table 22: Ex. 8d: Application of the Fixed-Point iteration methods ($c = 0.1$)

k	x_k	$y_k = g(x_k)$	$f(x_k)$	$f(x_k) - g(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	3.141592653589790	2.859382875468550	0.000000000000000	$-2.859E+00$	$2.939E-01$
1	4.375958446715440	3.672677063148310	2.859382875468550	$-8.133E-01$	$1.650E-02$
2	4.446552401691690	3.712519434872770	3.672677063148300	$-3.984E-02$	$6.359E-04$
3	4.449274819624000	3.714041450086730	3.712519434872770	$-1.522E-03$	$2.404E-05$
4	4.449377738800720	3.714098967760040	3.714041450086720	$-5.752E-05$	$9.081E-07$
5	4.449381626622730	3.714101140488050	3.714098967760040	$-2.173E-06$	$3.430E-08$
6	4.449381773482860	3.714101222561500	3.714101140488050	$-8.207E-08$	$1.296E-09$
7	4.449381779030410	3.714101225661770	3.714101222561490	$-3.100E-09$	$4.895E-11$
8	4.449381779239960	3.714101225778880	3.714101225661760	$-1.171E-10$	$1.849E-12$
9	4.449381779247880	3.714101225783300	3.714101225778880	$-4.425E-12$	$6.983E-14$
10	4.449381779248180	3.714101225783470	3.714101225783310	$-1.639E-13$	$0.000E+00$
11	4.449381779248190	3.714101225783470	3.714101225783460	$0.000E+00$	$0.000E+00$
12	4.449381779248190	3.714101225783480	3.714101225783480	$0.000E+00$	$0.000E+00$

We see that the method converges linearly with a reasonable convergence rate since $\phi'(\alpha) = \dfrac{|g'(\alpha)|}{|f'(\alpha)|}$ and $|g'(\alpha)|$ is small in comparison with $|f'(\alpha)|$ in general.

Table 23: Ex. 8d: Application of the Fixed-Point iteration methods ($c = 0.5$)

k	x_k	$y_k = g(x_k)$	$f(x_k)$	$f(x_k) - g(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	3.141592653589790	0.906036700900580	0.000000000000000	$-9.060E-01$	$1.795E-01$
1	3.877732955782450	0.814786168455241	0.906036700900580	$9.125E-02$	$-1.276E-02$
2	3.825284748827560	0.821193604576772	0.814786168455240	$-6.407E-03$	$9.343E-04$
3	3.829123593373330	0.820723421569361	0.821193604576772	$4.702E-04$	$-6.836E-05$
4	3.828842714862010	0.820757817413532	0.820723421569361	$-3.440E-05$	$5.002E-06$
5	3.828863266685940	0.820755300643011	0.820757817413532	$2.517E-06$	$-3.660E-07$
6	3.828861762916590	0.820755484793998	0.820755300643010	$-1.842E-07$	$2.678E-08$
7	3.828861872946860	0.820755471319735	0.820755484793998	$1.347E-08$	$-1.959E-09$
8	3.828861864895980	0.820755472305642	0.820755471319735	$-9.859E-10$	$1.434E-10$
9	3.828861865485060	0.820755472233504	0.820755472305642	$7.214E-11$	$-1.049E-11$
10	3.828861865441960	0.820755472238782	0.820755472233503	$-5.279E-12$	$7.676E-13$
11	3.828861865445110	0.820755472238396	0.820755472238782	$3.862E-13$	$-5.618E-14$
12	3.828861865444880	0.820755472238424	0.820755472238396	$-2.853E-14$	$4.219E-15$
13	3.828861865444900	0.820755472238422	0.820755472238424	$0.000E+00$	$0.000E+00$
14	3.828861865444900	0.820755472238422	0.820755472238422	$0.000E+00$	$0.000E+00$

Table 24: Ex. 8d: Application of the Fixed-Point iteration methods ($c = 1$)

k	x_k	$y_k = g(x_k)$	$f(x_k)$	$f(x_k) - g(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	3.141592653589790	0.289025482222236	0.000000000000000	$-2.890E-01$	$7.752E-02$
1	3.422950922702660	0.269172011522757	0.289025482222236	$1.985E-02$	$-5.092E-03$
2	3.404532597965240	0.270397527108780	0.269172011522757	$-1.226E-03$	$3.158E-04$
3	3.405674967431480	0.270321228132215	0.270397527108780	$7.630E-05$	$-1.965E-05$
4	3.405603865586900	0.270325975916208	0.270321228132215	$-4.748E-06$	$1.223E-06$
5	3.405608290053650	0.270325680470714	0.270325975916208	$2.954E-07$	$-7.611E-08$
6	3.405608014727870	0.270325698855683	0.270325680470714	$-1.838E-08$	$4.736E-09$
7	3.405608031860830	0.270325697711624	0.270325698855683	$1.144E-09$	$-2.947E-10$
8	3.405608030794680	0.270325697782817	0.270325697711624	$-7.119E-11$	$1.834E-11$
9	3.405608030861030	0.270325697778386	0.270325697782817	$4.430E-12$	$-1.141E-12$
10	3.405608030856900	0.270325697778662	0.270325697778386	$-2.758E-13$	$7.105E-14$
11	3.405608030857160	0.270325697778645	0.270325697778662	$1.721E-14$	$-4.441E-15$
12	3.405608030857140	0.270325697778646	0.270325697778645	$-9.992E-16$	$0.000E+00$
13	3.405608030857140	0.270325697778646	0.270325697778646	$0.000E+00$	$0.000E+00$

Table 25: Ex. 8d: Application of the Fixed-Point iteration methods ($c = 100$)

k	x_k	$y_k = g(x_k)$	$f(x_k)$	$f(x_k) - g(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	3.141592653589790	0.000031830666106	0.000000000000000	$-3.183E-05$	$1.013E-05$
1	3.141624484255890	0.000031830343607	0.000031830666106	$3.225E-10$	$-1.027E-10$
2	3.141624483933390	0.000031830343611	0.000031830343607	$-3.497E-15$	$0.000E+00$
3	3.141624483933390	0.000031830343611	0.000031830343611	$5.579E-17$	$0.000E+00$

Table 26: Ex. 8d: Application of the Fixed-Point iteration methods ($c = 1000$)

k	x_k	$y_k = g(x_k)$	$f(x_k)$	$f(x_k) - g(x_k)$	$r_k = (\alpha - x_k)/\alpha$
0	3.141592653589790	0.000000318309854	0.000000000000000	$-3.183E-07$	$1.013E-07$
1	3.141592971899650	0.000000318309822	0.000000318309854	$3.223E-14$	$-1.021E-14$
2	3.141592971899610	0.000000318309822	0.000000318309821	$-1.912E-16$	$0.000E+00$
3	3.141592971899610	0.000000318309822	0.000000318309821	$-1.912E-16$	$0.000E+00$

In fact, when $c \to \infty$, $|g'(\alpha)| \to 0$ and the method is almost quadratic, as it can be seen in the case $c = 1000$.

Thus, it is understandable that the method works better for higher values of c. Moreover when $c \to \infty$ then $\alpha \to \pi$, so that x_0 is a very good approximation of the root.

9.— The classified flow curve, used in hydrological studies, is the curve obtained by classifying the average daily flows of any hydrological year according to the number of days of the year in which this flow has been exceeded or equaled. An analytical expression of this curve has been proposed by Coutagne. According to this author, this curve takes the form:

$$q(t) = Q_{mc} + \frac{(Q - Q_{mc})(1+\nu)}{T^{\nu}}(T-t)^{\nu},$$

where $q(t)$ is the flow rate equaled or exceeded during t days over the course of a T day observation period days, Q is the mean annual flow rate, Q_{mc} is the minimum characteristic flow (the flow equaled or exceeded on all days of the year except for the 10 driest days), Q_{mc} is the minimum characteristic flow rate driest days), Q_s is the semi-permanent flow (that equaled or exceeded on half the days of the year), and ν is the irregularity coefficient whose value is defined as the largest of the roots of the function

$$f(\nu) = 2^{\nu}\kappa - (\nu + 1), \qquad \text{donde} \qquad \kappa = \frac{Q_s - Q_{mc}}{Q - Q_{mc}}.$$

The Newton method is proposed to calculate the irregularity coefficient. It is known that for the cases to be studied the value of the κ coefficient takes values that do not differ excessively from 1.

a) Draw the function $f(f(\nu)$ and set out the proposed iterative scheme fully develop and simplify as much as possible its expression.

b) Analyze the convergence of the method and explain for which values of κ more iterations will be needed, taking into account that we want to obtain the largest possible value of the irregularity coefficient. Study for which initial values the algorithm converges.

c) Apply the developed iterative scheme to the following cases:

  1) $Q_s = 100\ m^3/s$, $Q_{mc} = 1\ m^3/s$, $Q = 98.0000000000\ m^3/s$.

  2) $Q_s = 100\ m^3/s$, $Q_{mc} = 1\ m^3/s$, $Q = 94.2663845753\ m^3/s$.

In both cases, the initial values $\nu_0 = 0$ and $\nu_0 = 1$ will be used. Compare and comment the results.

d) Propose another algorithm to solve the last case of the previous section and apply it with the two proposed initial values. Comment the results.

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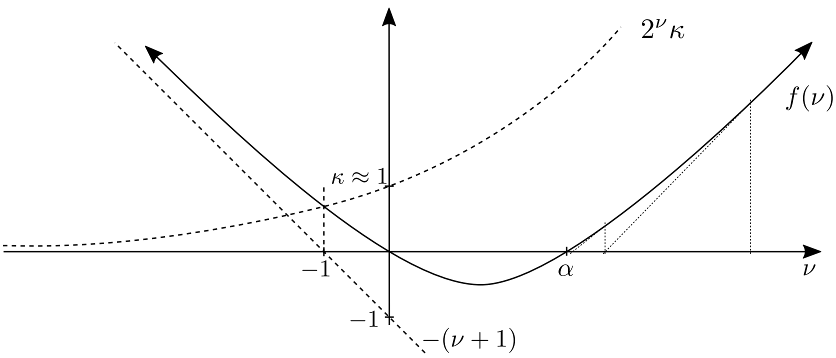
solution 9.

We seek the roots of equation:

$$f(\nu) = 2^\nu \kappa - (\nu + 1) \quad \text{with} \quad \kappa \approx 1$$

solution 9.a



To use the Newton-Raphson method we need:

$$\begin{array}{l} f(\nu) = 2^\nu\kappa - (\nu+1) \\ f'(\nu) = 2^\nu\kappa\ln(2) - 1 \end{array}$$

so:

$$\begin{aligned} \nu_{k+1} = \phi(\nu_k); \quad \phi(\nu) &= \nu - \frac{f(\nu)}{f'(\nu)} = \nu - \frac{2^\nu\kappa - (\nu+1)}{2^\nu\kappa\ln(2) - 1} \\ &= \frac{\nu 2^\nu\kappa\ln(2) \cancel{-\nu} - 2^\nu\kappa \cancel{+\nu} + 1}{2^\nu\kappa\ln(2) - 1} \\ &= \frac{2^\nu\kappa(\nu\ln(2) - 1) + 1}{2^\nu\kappa\ln(2) - 1} \end{aligned}$$

solution 9.b

By the look of the function, the method will converge always for initial values higher than the main root.

The method will need more iterations when $f'(\alpha) \rightarrow 0$. This is, for higher values of κ.

The method would even degenerate in a linear method when the root is double, this is, when:

$$\left.\begin{array}{rll} f(\nu) = & 2^\nu\kappa - (\nu+1) & = 0 \\ f'(\nu) = & 2^\nu\kappa\ln(2) - 1 & = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \nu = \dfrac{1}{\ln(2)} - 1 = 0,44269504089 \\ \kappa = \dfrac{1+\nu}{2\nu} = 1,06147569085 \end{array}\right.$$

solution 9.c

Analyzing the cases:

$$\begin{cases} \kappa_1 &= \dfrac{100-1}{98-1} = \dfrac{99}{97} = 1,0206185567 \\[2ex] \kappa_2 &= \dfrac{100-1}{94,2663845753-1} = \dfrac{99}{93,2663845753} = 1,06147569085 \end{cases}$$

Starting at $\nu_0 = 0$ and at $\nu_0 = 1$.

Table 27: Ex. 9c: Application of Newton-Raphson (simple root).

κ = 1.02061855670103 κ = 1.02061855670103

k	ν_k	$f(\nu_k)$	$r_k = \frac{\alpha-\nu_k}{\alpha}$	k	ν_k	$f(\nu_k)$	$r_k = \frac{\alpha-\nu_k}{\alpha}$
0	1.000000000000000	$4.124E-02$	$-1.293E-01$	0	0.000000000000000	$2.062E-02$	$1.000E+00$
1	0.900604181831465	$4.735E-03$	$-1.704E-02$	1	0.070476064450964	$1.238E-03$	$6.423E-02$
2	0.885838234464342	$9.946E-05$	$-3.657E-04$	2	0.075289897819311	$5.973E-06$	$3.114E-04$
3	0.885514515243613	$4.747E-08$	$-1.747E-07$	3	0.075313350983297	$1.421E-10$	$7.409E-09$
4	0.885514360575589	$1.066E-14$	$-3.930E-14$	4	0.075313351541267	$0.000E+00$	$0.000E+00$
5	0.885514360575554	$0.000E+00$	$0.000E+00$	5	0.075313351541267	$0.000E+00$	$0.000E+00$
6	0.885514360575554	$0.000E+00$	$0.000E+00$				

Table 28: Ex. 9c: Application of Newton-Raphson (double root).

κ = 1.06147569080289 κ = 1.06147569080289

k	ν_k	$f(\nu_k)$	$r_k = \frac{\alpha-\nu_k}{\alpha}$	k	ν_k	$f(\nu_k)$	$r_k = \frac{\alpha-\nu_k}{\alpha}$
0	1.000000000000000	$1.230E-01$	$-1.259E+00$	0	0.000000000000000	$6.148E-02$	$1.000E+00$
1	0.739243373569765	$3.268E-02$	$-6.698E-01$	1	0.232649980218788	$1.457E-02$	$4.745E-01$
2	0.596045300412636	$8.447E-03$	$-3.464E-01$	2	0.340220019476420	$3.555E-03$	$2.315E-01$
3	0.520728270357000	$2.149E-03$	$-1.762E-01$	3	0.392064047412284	$8.781E-04$	$1.143E-01$
4	0.482063364671728	$5.421E-04$	$-8.890E-02$	4	0.417527612860319	$2.182E-04$	$5.682E-02$
5	0.462468727529052	$1.361E-04$	$-4.464E-02$	5	0.430147909909289	$5.440E-05$	$2.831E-02$
6	0.452604473391758	$3.411E-05$	$-2.235E-02$	6	0.436430562178420	$1.358E-05$	$1.412E-02$
7	0.447655437756846	$8.537E-06$	$-1.118E-02$	7	0.439565054815421	$3.393E-06$	$7.041E-03$
8	0.445176677671291	$2.136E-06$	$-5.576E-03$	8	0.441130586674439	$8.479E-07$	$3.505E-03$
9	0.443936249144733	$5.340E-07$	$-2.774E-03$	9	0.441912901009423	$2.119E-07$	$1.737E-03$
10	0.443315802252678	$1.335E-07$	$-1.373E-03$	10	0.442303897979993	$5.296E-08$	$8.542E-04$
11	0.443005580289624	$3.337E-08$	$-6.721E-04$	11	0.442499261702816	$1.322E-08$	$4.129E-04$
12	0.442850588941604	$8.327E-09$	$-3.220E-04$	12	0.442596720831132	$3.291E-09$	$1.927E-04$
13	0.442773360900202	$2.067E-09$	$-1.475E-04$	13	0.442645019850909	$8.084E-10$	$8.359E-05$
14	0.442735282830582	$5.025E-10$	$-6.150E-05$	14	0.442668337038374	$1.884E-10$	$3.092E-05$
15	0.442717266959403	$1.125E-10$	$-2.080E-05$	15	0.442678516827421	$3.591E-11$	$7.924E-06$
16	0.442709965207591	$1.848E-11$	$-4.309E-06$	16	0.442681652454445	$3.407E-12$	$8.412E-07$
17	0.442708179022127	$1.106E-12$	$-2.743E-07$	17	0.442682019635504	$4.707E-14$	$1.178E-08$
18	0.442708057572102	$0.000E+00$	$0.000E+00$	18	0.442682024851027	$0.000E+00$	$0.000E+00$
19	0.442708057572102	$0.000E+00$	$0.000E+00$	19	0.442682024851027	$0.000E+00$	$0.000E+00$
20	0.442708057572102	$0.000E+00$	$0.000E+00$				

solution 9.d

In the case where the root is practically double, it can be seen that the Newton-Raphson method presents an atypical behavior. In this case it happens that:

$$\left.\begin{array}{r} f(\nu)=0 \\ f'(\nu)=0 \end{array}\right\} \Rightarrow \text{double root} \Rightarrow \left\{\begin{array}{l} \text{Newton-Rapshon becomes first order} \\ \text{numerical instabilities close to the solution} \end{array}\right.$$

We propose to solve $g(\nu) = f'(\nu) = 0$ by Newton-Raphson.

$$\begin{cases} g(\nu) = 2^{\nu} k \ln(2) - 1 \\ g'(\nu) = 2^{\nu} k (\ln(2))^2 \end{cases}$$

$$\nu_{k+1} = \phi(\nu_k) \qquad \text{with} \qquad \phi(\nu) = \nu - \frac{g(\nu)}{g'(\nu)} \;=\; \nu - \frac{2^{\nu}\kappa \ln(2) - 1}{2^{\nu}\kappa(\ln(2))^2}$$
$$= \; \nu - \frac{1}{\ln(2)} + \frac{1}{2^{\nu}\kappa(\ln(2))^2}$$

Table 29: Ex. 9d: Application of Modified Newton-Raphson (double root).

κ	$= 1.06147569080289$			κ	$= 1.06147569080289$		
k	ν_k	$g(\nu_k)$	$r_k = \frac{\alpha-\nu_k}{\alpha}$	k	ν_k	$g(\nu_k)$	$r_k = \frac{\alpha-\nu_k}{\alpha}$
0	1.000000000000000	$4.715E-01$	$-1.293E-01$	0	0.000000000000000	$-2.642E-01$	$1.000E+00$
1	0.537717887565050	$6.808E-02$	$3.928E-01$	1	0.518130816019064	$5.368E-02$	$-1.704E-01$
2	0.445756784487373	$2.124E-03$	$4.966E-01$	2	0.444633307975891	$1.344E-03$	$-4.378E-03$
3	0.442698287526226	$2.250E-06$	$5.001E-01$	3	0.442696342399847	$9.021E-07$	$-2.940E-06$
4	0.442695040951332	$2.532E-12$	$5.001E-01$	4	0.442695040948267	$4.068E-13$	$-1.326E-12$
5	0.442695040947680	$0.000E+00$	$5.001E-01$	5	0.442695040947680	$0.000E+00$	$0.000E+00$
6	0.442695040947680	$0.000E+00$	$5.001E-01$	6	0.442695040947680	$0.000E+00$	$0.000E+00$

Finally we check that $f(\nu) = 0$ since that is not imposed initially.

10.— Given the non-linear system of equations with n equations and n unknowns

$$\boldsymbol{K}(\boldsymbol{x})\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}),$$

where the matrix $\boldsymbol{K}(\boldsymbol{x})$ and the vector $\boldsymbol{f}(\boldsymbol{x})$ depend on the unknowns $\boldsymbol{x}$:

a) Diseñar y escribir, de forma suficientemente explícita, algoritmos que permitan resolver el sistema anterior por:

- Fixed Point interations,
- Newton-Raphson,
- Newton,
- Whittaker,

indicating, if applicable, which methods are suitable to solve the derived linear systems.

b) Discuss in a reasoned manner, and taking into account all relevant aspects (storage needs, speed of convergence, etc.) which of the designed algorithms would be preferable in the following cases:

- The matrix $\boldsymbol{K}(\boldsymbol{x})$ is SYMMETRIC, POSITIVE DEFINITE and BANDED for every value of $\boldsymbol{x}$,
- The matrix $\boldsymbol{K}(\boldsymbol{x})$ and the vector $\boldsymbol{f}(\boldsymbol{x})$ are lowly sensible with respecto to $\boldsymbol{x}$ (thus, in order to obtain significant variations in its components it is necessary to modify greatly the value $\boldsymbol{x}$).

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solution 10.

We want to solve the non-linear system of equations:

$$\underset{\sim}{K}(\bar{x})\ \bar{x} = \bar{f}(\bar{x}); \qquad \begin{array}{lcll} \underset{\sim}{K}(\bar{x}) & = & [K_{ij}(\bar{x})] & ;\quad i=1,\ldots,n\ ;\quad j=1,\ldots,n \\ \bar{f}(\bar{x}) & = & \{f_i(\bar{x})\} & ;\quad i=1,\ldots,n \\ \bar{x} & = & \{x_i\} & ;\quad i=1,\ldots,n \end{array}$$

solution 10.a

Fixed-Point iterations

1) If $\underset{\sim}{K}(\bar{x})$ is non-singular ($\det(\underset{\sim}{K}(\bar{x}) \neq 0$) we can propose

$$\boxed{\bar{x}_{k+1} = \bar{\phi}(\bar{x}_k) \quad \text{with} \quad \bar{\phi}(\bar{x}) = \underset{\sim}{K}^{-1}(\bar{x})\ \bar{f}(\bar{x})}$$

In practice, to obtain $\bar{x}_{k+1}$ from $\bar{x}_k$ we will solve the linear system:

$$\underset{\sim}{K}(\bar{x}_k)\ \ \bar{x}_{k+1}\ =\ \bar{f}(\bar{x}_k)$$

2) In general we can propose

$$\bar{g}(\bar{x}) = \bar{0} \quad \text{with} \quad \bar{g}(\bar{x}) = \underset{\sim}{K}(\bar{x})\ \bar{x} - \bar{f}(\bar{x})$$

$$\underbrace{\underset{\sim}{B}\,\bar{g}(\bar{x})\ +\ \bar{x}}_{\bar{\phi}(\bar{x})}\ =\ \bar{x} \qquad \Longleftrightarrow \qquad \bar{g}(\bar{x}) = \bar{0} \qquad \Big(\text{if } \det(\underset{\sim}{B}) \neq 0\Big)$$

$$\boxed{\bar{x}_{k+1} = \bar{\phi}(\bar{x}_k) \quad \text{with} \quad \bar{\phi}(\bar{x}) = \underset{\sim}{B}\Big[\underset{\sim}{K}(\bar{x})\ \bar{x}\ -\ \bar{f}(\bar{x})\Big] + \bar{x}}$$

The matrix $\underset{\sim}{B}$ is chosen such that $\det(\underset{\sim}{B}) \neq 0$.

If possible, $\underset{\sim}{B} \approx -\left(\left.\dfrac{d\bar{g}}{d\bar{x}}\right|_{\bar{x}=\bar{\alpha}}\right)^{-1}$ being $\bar{\alpha}$ the solution.

Newton-Raphson

$$\boxed{\bar{x}_{k+1} = \bar{\phi}(\bar{x}_k) \quad \text{with} \quad \bar{\phi}(\bar{x}) = \bar{x}\ -\ \left[\frac{d\bar{g}}{d\bar{x}}\right]^{-1} \bar{g}(\bar{x})}$$

In practice, to obtain $\bar{x}_{k+1}$ as a function of $\bar{x}_k$:

1. We solve the system $\left[\left.\frac{d\bar{g}}{d\bar{x}}\right|_{\bar{x}=\bar{x}_k} \right] \Delta\bar{x}_k = -\bar{g}(\bar{x}_k)$

2. And update the approximate solution as: $\bar{x}_{k+1} = \bar{x}_k \ + \ \Delta\bar{x}_k$

where:

$$\begin{cases} \bar{g}(\bar{x}) = \{g_i(\bar{x})\};\ i = 1, \ldots, n \\ g_i(\bar{x}) = \sum\limits_{j=1,n} K_{i,j}(\bar{x})\, x_j \ - \ f_i(\bar{x}) \\ \left[\frac{d\bar{g}}{d\bar{x}}\right] = \underset{\sim}{A}(\bar{x}) = [a_{i,j}(\bar{x})]; \quad i = 1, \ldots, n; \quad j = 1, \ldots, n \\ a_{i,j}(\bar{x}) = \frac{\partial g_i(\bar{x})}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\sum\limits_{\ell=1,n} K_{i,\ell}(\bar{x})\, x_\ell - f_i(\bar{x}) \right) = \sum\limits_{\ell=1,n} \frac{\partial K_{i,\ell}(\bar{x})}{\partial x_j}\, x_\ell + K_{i,j}(\bar{x}) - \frac{\partial f_i(\bar{x})}{\partial x_j} \end{cases}$$

Newton (Simple)

$$\boxed{\bar{x}_{k+1} = \bar{\phi}(\bar{x}_k) \quad \text{simple} \quad \bar{\phi}(\bar{x}) = \bar{x} \ - \ \left(diag\left[\frac{d\bar{g}}{d\bar{x}}\right]\right)^{-1} \bar{g}(\bar{x})}$$

In practice,

$$(x_i)_{k+1} = (x_i)_k - \frac{g_i(\bar{x}_k)}{a_{i,i}(\bar{x}_k)}; \qquad i = 1, \ldots, n$$

with

$$a_{i,i}(\bar{x}) = \sum_{\ell=1,n} \frac{\partial K_{i,\ell}}{\partial x_i}\, x_\ell + K_{i,i}(\bar{x}) - \frac{\partial f_i(\bar{x})}{\partial x_i}$$

Whittaker

$$\boxed{\bar{x}_{k+1} = \bar{\phi}(\bar{x}_k) \qquad \text{with} \qquad \bar{\phi}(\bar{x}) = \bar{x} \ - \ \underset{\sim}{C}^{-1}\, \bar{g}(\bar{x})}$$

$\underset{\sim}{C}$ is chosen as $det(\underset{\sim}{C}) \neq 0$. If possible, $\underset{\sim}{C} \approx \left(\left.\frac{d\bar{g}}{d\bar{x}}\right|_{\bar{x}=\bar{\alpha}} \right)$
In practice, to obtain $\bar{x}_{k+1}$ as a function of $\bar{x}_k$:

1. We solve the system: $\underset{\sim}{C}\Delta\bar{x}_k = -\bar{g}(\bar{x}_k)$

2. And update the approximation as: $\bar{x}_{k+1} = \bar{x}_k + \Delta\bar{x}_k$

We see that the method is mainly the same as Fixed-Point iterations with $\underset{\sim}{B} = -\underset{\sim}{C}^{-1}$

General comments

- We have to solve a system of equations in almost every case.
- In the first Fixed-point iterations method the best method will depend on which type of matrix is $\underset{\sim}{K}(\bar{x})$. If we know nothing from it we will have to use Gauss-elimination with pivoting.

- In Newton-Raphson the method will depend on $\underset{\sim}{A}(\bar{x}) = \left[\dfrac{d\bar{g}}{d\bar{x}}\right]$. If we know nothing from it we will have to use Gauss-elimination with pivoting.
- In Whittaker, in general, a matrix $\underset{\sim}{C}$ which treatment is easy will be used. The best method will depend on the matrix we choose. If we choose $\underset{\sim}{C}$ without special properties (symmetric, positive definite) we will have to use Gauss-elimination with pivoting.

solution 10.b

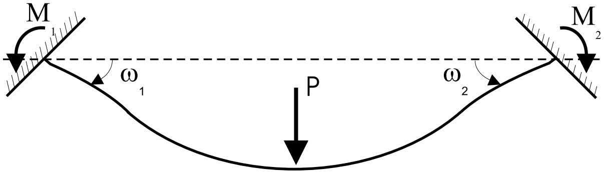
Analyzing the algorithms for the established conditions:

1. If $\underset{\sim}{K}$ is symmetric, positive definite and banded, the first method of Fixed-Point iterations will be advantageous (in principle) since Cholesky's method can be used to solve the systems. Convergence will be linear. The Newton-Raphson method will converge quadratically, but its cost per iteration will be much higher, since $\underset{\sim}{A}(\bar{x})$ will be a non-symmetric and undefined (in general) full matrix. It is possible for Newton's method to be competitive against Fixed-point iterations, if indeed $diag\left(\underset{\sim}{A}(\bar{x})\right)^{-1} \approx \underset{\sim}{A}^{-1}(\bar{x})$.
2. If $\underset{\sim}{K}(\bar{x})$ and $\bar{f}(\bar{x})$ are insensitive with respect to the value of $\bar{x}$ then $a_{i,j}(\bar{x})$ then $a_{i,j}(\bar{x})$. The best is Whitakker with $\underset{\sim}{C} = \underset{\sim}{K}(\bar{x}_0)$. This case is very common in engineering and for this reason this method will work very well.

11.— A constant section beam supports a point load applied at the center of the span. The ends of the beam are imperfectly clamped. Therefore, a twist occurs at each end, although its value is logically lower than that which would occur in pinned joint. It can be assumed that the moment opposing the twist at each end is proportional to the square root of the rotated angle. The positive constants θ, μ_1 and μ_2 are defined so that

$$\theta = \frac{PL^2}{16EI}, \qquad M_1 = 6\frac{EI}{L}\mu_1\sqrt{\omega_1}, \qquad M_2 = 6\frac{EI}{L}\mu_2\sqrt{\omega_2},$$

where E is the elasticity modulus of the beam, I is the inertia moment of the cross-section, L is the length of the span, P is the point load applied on the center of the span, M_1 and M_2 are the moments generated by the imperfect clamps, and ω_1 and ω_2, are the rotations produced at the ends of the beam. The sign convention is the one shown in the following figure.



Se pide:

a) Prove that the rotations ω_1 and ω_2 can be obtained by solving the non-linear system

$$\underset{\sim}{K}\bar{\omega} = \bar{\varphi}(\bar{\omega}), \qquad \text{with} \quad \bar{\omega} = \left\{ \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right\},$$

being

$$\underset{\sim}{K} = \left[\begin{array}{cc} +4 & -2 \\ -2 & +4 \end{array} \right], \qquad \bar{\varphi}(\omega) = \left\{ \begin{array}{c} 2\theta - 6\mu_1\sqrt{\omega_1} \\ 2\theta - 6\mu_2\sqrt{\omega_2} \end{array} \right\}.$$

b) Solve the problem by Fixed Point Iterations

$$\bar{\omega}^{k+1} = \underset{\sim}{K}^{-1}\bar{\varphi}(\bar{\omega}^k)$$

in the case of $\mu_1 = 1/300$, $\mu_2 = 23/450$, and $\theta = 10^{-2}$. As initial approximations, $\omega_1^0 = \omega_2^0 = \theta/2$ will be used. (if the clamped ends would not produce any resistance to the rotation, the rotations would be half the values.).

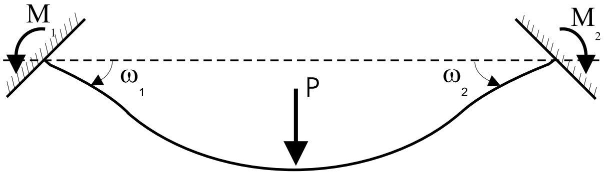
c) Propose and solve the same problem by means of the Newton-Raphson method using the same initial approximation.

d) Propose and solve the same problem by means of the Simplified Newton method using the same initial approximation.

e) Compare the results and the efficiency of the three proposed methods in this case.

solution 11.

solution 11.a



$$\text{with} \left\{ \begin{array}{l} \theta = \dfrac{PL^2}{16EI} \\ M_1 = 6\dfrac{EI}{L}\mu_1\sqrt{\omega_1} \\ M_2 = 6\dfrac{EI}{L}\mu_2\sqrt{\omega_2} \end{array} \right.$$

$$\left\{ \begin{array}{l} \omega_1 = \dfrac{PL^2}{16EI} - M_1 \dfrac{L}{3EI} - M_2 \dfrac{L}{6EI} \\ \omega_2 = \dfrac{PL^2}{16EI} - M_1 \dfrac{L}{6EI} - M_2 \dfrac{L}{3EI} \end{array} \right\} \Longleftrightarrow \frac{L}{6EI} \left[\begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right] \left\{ \begin{array}{c} M_1 \\ M_2 \end{array} \right\} = \left\{ \begin{array}{c} \frac{PL^2}{16EI} - \omega_1 \\ \frac{PL^2}{16EI} - \omega_2 \end{array} \right\}$$

$$\begin{aligned} \left\{ \begin{array}{c} M_1 \\ M_2 \end{array} \right\} &= \frac{6EI}{L} \left[\begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right]^{-1} \left\{ \begin{array}{c} \theta - \omega_1 \\ \theta - \omega_2 \end{array} \right\} = \frac{6EI}{L} \frac{1}{3} \left[\begin{array}{rr} 2 & -1 \\ -1 & 2 \end{array} \right]^{-1} \left\{ \begin{array}{c} \theta - \omega_1 \\ \theta - \omega_2 \end{array} \right\} \\ &= \frac{EI}{L} \left[\begin{array}{rr} 4 & -2 \\ -2 & 4 \end{array} \right] \left\{ \begin{array}{c} \theta - \omega_1 \\ \theta - \omega_2 \end{array} \right\} \end{aligned}$$

Then

$$\frac{EI}{L} \left[\begin{array}{rr} 4 & -2 \\ -2 & 4 \end{array} \right] \left\{ \begin{array}{c} \theta - \omega_1 \\ \theta - \omega_2 \end{array} \right\} = \frac{6EI}{L} \left\{ \begin{array}{c} \mu_1 \sqrt{\omega_1} \\ \mu_2 \sqrt{\omega_2} \end{array} \right\}$$

And therefore:

$$\left[\begin{array}{rr} 4 & -2 \\ -2 & 4 \end{array} \right] \left\{ \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right\} = \left\{ \begin{array}{c} 2\theta - 6\mu_1 \sqrt{\omega_1} \\ 2\theta - 6\mu_2 \sqrt{\omega_2} \end{array} \right\}$$

solution 11.b Fixed-Point iterations

$$\left[\begin{array}{rr} 4 & -2 \\ -2 & 4 \end{array} \right] \left\{ \begin{array}{c} (\omega_1)_{k+1} \\ (\omega_2)_{k+1} \end{array} \right\} = \left\{ \begin{array}{c} 2\theta - 6\mu_1 \sqrt{(\omega_1)_k} \\ 2\theta - 6\mu_2 \sqrt{(\omega_2)_k} \end{array} \right\} \Longleftrightarrow$$

$$\left\{ \begin{array}{c} (\omega_1)_{k+1} \\ (\omega_2)_{k+1} \end{array} \right\} = \left[\begin{array}{rr} 4 & -2 \\ -2 & 4 \end{array} \right]^{-1} \left\{ \begin{array}{c} 2\theta - 6\mu_1 \sqrt{(\omega_1)_k} \\ 2\theta - 6\mu_2 \sqrt{(\omega_2)_k} \end{array} \right\} = \frac{1}{12} \left[\begin{array}{cc} 4 & 2 \\ 2 & 4 \end{array} \right] \left\{ \begin{array}{c} 2\theta - 6\mu_1 \sqrt{(\omega_1)_k} \\ 2\theta - 6\mu_2 \sqrt{(\omega_2)_k} \end{array} \right\}$$

$$\boxed{\left\{ \begin{array}{c} (\omega_1)_{k+1} \\ (\omega_2)_{k+1} \end{array} \right\} = \left\{ \begin{array}{c} \theta - 2\mu_1 \sqrt{(\omega_1)_k} - \mu_2 \sqrt{(\omega_2)_k} \\ \theta - \mu_1 \sqrt{(\omega_1)_k} - 2\mu_2 \sqrt{(\omega_2)_k} \end{array} \right\}}$$

Solving the case: $\left\{ \begin{array}{l} \mu_1 = 1/300 \\ \mu_2 = 23/450 \\ \theta = 10^{-}2 \end{array} \right\}$ starting from $\left\{ \begin{array}{l} (\omega_1)_0 = \theta/2 \\ (\omega_2)_0 = \theta/2 \end{array} \right.$

solution 11.c Newton-Raphson

$$\bar{f}(\bar{\omega}) = \underset{\sim}{K} \bar{\omega} - \bar{\varphi}(\bar{\omega}) = \bar{0} \quad \longrightarrow \quad \bar{\omega}_{k+1} = \bar{\omega}_k - \left[\left. \frac{d\bar{f}}{d\bar{\omega}} \right|_{\bar{\omega} = \bar{\omega}_k} \right]^{-1} \bar{f}(\bar{\omega}_k)$$

Table 30: Ex. 11b: Application of Fixed-Point iterations.

$\theta =$ 0.010000000000000

$\omega_1 =$ 0.003333333333333

$\omega_2 =$ 0.051111111111111

$$K = \begin{matrix} 4.000000000000000 & -2.000000000000000 \\ -2.000000000000000 & 4.000000000000000 \end{matrix}$$

$$K^{-1} = \begin{matrix} 0.333333333333333 & 0.166666666666667 \\ 0.166666666666667 & 0.333333333333333 \end{matrix}$$

k	$\bar{\omega}_k$	$\bar{f}(\bar{\omega}_k)$	$\underset{\sim}{K}\bar{\omega}_k - \bar{f}(\bar{\omega}_k)$	$\bar{e}_k = \bar{\omega} - \bar{\omega}_k$	$\|\bar{e}_k\|$
0	0.005000000000000 0.005000000000000	0.018585786437627 −0.001684607956387	−0.008585786437627 0.011684607956388	$-1.400E-03$ $1.400E-03$	$1.980E-03$
1	0.005914494153144 0.002536095087475	0.018461885029896 0.004556371748844	0.000123901407731 −0.006240979705232	$-4.855E-04$ $-1.064E-03$	$1.169E-03$
2	0.006913356968106 0.004595771421264	0.018337068014848 −0.000789583211465	0.000124817015048 0.005345954960309	$5.134E-04$ $9.958E-04$	$1.120E-03$
3	0.005980758803038 0.002792983598653	0.018453292684049 0.003793069665178	−0.000116224669202 −0.004582652876644	$-4.192E-04$ $-8.070E-04$	$9.094E-04$
4	0.006783275838880 0.004339905335734	0.018352787100721 −0.000202573752882	0.000100505583329 0.003995643418060	$3.833E-04$ $7.399E-04$	$8.333E-04$
5	0.006083833408093 0.002991273265826	0.018440021358083 0.003227622933342	−0.000087234257362 −0.003430196686224	$-3.162E-04$ $-6.087E-04$	$6.859E-04$
6	0.006684610941585 0.004149211204128	0.018364810599156 0.000246259529525	0.000075210758927 0.002981363403817	$2.846E-04$ $5.492E-04$	$6.186E-04$
7	0.006162646787973 0.003142888276368	0.018429949454575 0.002807816255578	−0.000065138855419 −0.002561556726053	$-2.374E-04$ $-4.571E-04$	$5.151E-04$
8	0.006611285860788 0.004007596994289	0.018373803719007 0.000586287503800	0.000056145735568 0.002221528751778	$2.113E-04$ $4.076E-04$	$4.591E-04$
9	0.006222315823636 0.003257729787768	0.018422366858407 0.002496532113871	−0.000048563139399 −0.001910244610070	$-1.777E-04$ $-3.423E-04$	$3.856E-04$
10	0.006556877638447 0.003902571847691	0.018380509013493 0.000842359192201	0.000041857844914 0.001654172921670	$1.569E-04$ $3.026E-04$	$3.408E-04$
20	0.006435572888894 0.003668577755659	0.018395559550635 0.001425572500555	0.000009576493625 0.000377592744291	$3.557E-05$ $6.858E-05$	$7.726E-05$
30	0.006408104915720 0.003615623106032	0.018398987206082 0.001560117507769	0.000002186244734 0.000086165084919	$8.105E-06$ $1.562E-05$	$1.760E-05$
40	0.006401849115242 0.003603564286366	0.018399768877288 0.001590893521652	0.000000499010948 0.000019665393328	$1.849E-06$ $3.564E-06$	$4.015E-06$
50	0.006400422006467 0.003600813439725	0.018399947250061 0.001597921327007	0.000000113896358 0.000004488418959	$4.220E-07$ $8.134E-07$	$9.164E-07$
60	0.006400096317769 0.003600185657352	0.018399987960324 0.001599525548439	0.000000025996048 0.000001024445432	$9.632E-08$ $1.857E-07$	$2.092E-07$
70	0.006400021983715 0.003600042374706	0.018399997252038 0.001599891709403	0.000000005933410 0.000000233821992	$2.198E-08$ $4.237E-08$	$4.774E-08$
80	0.006400005017616 0.003600009671705	0.018399999372798 0.001599975283438	0.000000001354257 0.000000053368148	$5.018E-09$ $9.672E-09$	$1.090E-08$
90	0.006400001145234 0.003600002207495	0.018399999856846 0.001599994358624	0.000000000309099 0.000000012180888	$1.145E-09$ $2.207E-09$	$2.487E-09$
100	0.006400000261391 0.003600000503845	0.018399999967326 0.001599998712397	0.000000000070550 0.000000002780199	$2.614E-10$ $5.038E-10$	$5.676E-10$
150	0.006400000000162 0.003600000000312	0.018399999999980 0.001599999999202	0.000000000000044 0.000000000001722	$1.619E-13$ $3.121E-13$	$3.516E-13$
194	0.006400000000000 0.003600000000000	0.018400000000000 0.001599999999999	0.000000000000000 0.000000000000003	$2.429E-16$ $4.684E-16$	$5.276E-16$
195	0.006400000000000 0.003600000000000	0.018400000000000 0.001600000000001	0.000000000000000 −0.000000000000002	$-2.099E-16$ $-4.046E-16$	$4.558E-16$

$$
\begin{aligned}
\bar{f}(\bar{\omega}_k) &= \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \begin{Bmatrix} (\omega_1)_k \\ (\omega_2)_k \end{Bmatrix} - \begin{Bmatrix} 2\theta - 6\mu_1\sqrt{(\omega_1)_k} \\ 2\theta - 6\mu_2\sqrt{(\omega_2)_k} \end{Bmatrix} \\
&= \begin{Bmatrix} -2\theta + 4(\omega_1)_k + 6\mu_1\sqrt{(\omega_1)_k} - 2(\omega_2)_k \\ -2\theta + 4(\omega_2)_k + 6\mu_2\sqrt{(\omega_2)_k} - 2(\omega_1)_k \end{Bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\left[\frac{d\bar{f}}{d\bar{\omega}}\Big|_{\bar{\omega}=\bar{\omega}_k}\right] &= \underset{\sim}{K} - \frac{d\bar{\varphi}(\bar{\omega})}{d\bar{\omega}}\Big|_{\bar{\omega}=\bar{\omega}_k} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} 3\mu_1\sqrt{(\omega_1)_k} & 0 \\ 0 & 3\mu_2\sqrt{(\omega_2)_k} \end{bmatrix} \\
&= \begin{bmatrix} 4 + 3\mu_1\sqrt{(\omega_1)_k} & -2 \\ -2 & 4 + 3\mu_2\sqrt{(\omega_2)_k} \end{bmatrix}
\end{aligned}
$$

$$
\begin{Bmatrix} (\omega_1)_{k+1} \\ (\omega_2)_{k+1} \end{Bmatrix} = \begin{Bmatrix} (\omega_1)_k \\ (\omega_2)_k \end{Bmatrix} - \begin{bmatrix} 4 + 3\mu_1\sqrt{(\omega_1)_k} & -2 \\ -2 & 4 + 3\mu_2\sqrt{(\omega_2)_k} \end{bmatrix}^{-1} \begin{Bmatrix} -2\theta + 4(\omega_1)_k + 6\mu_1\sqrt{(\omega_1)_k} - 2(\omega_2)_k \\ -2\theta + 4(\omega_2)_k + 6\mu_2\sqrt{(\omega_2)_k} - 2(\omega_1)_k \end{Bmatrix}
$$

The method works as follows:

$$
\boxed{
\begin{array}{l}
\begin{Bmatrix} (\omega_1)_{k+1} \\ (\omega_2)_{k+1} \end{Bmatrix} = \begin{Bmatrix} (\omega_1)_k \\ (\omega_2)_k \end{Bmatrix} + \begin{Bmatrix} (\Delta\omega_1)_k \\ (\Delta\omega_2)_k \end{Bmatrix}, \qquad \text{with} \\[2ex]
\begin{bmatrix} 4 + 3\mu_1\sqrt{(\omega_1)_k} & -2 \\ -2 & 4 + 3\mu_2\sqrt{(\omega_2)_k} \end{bmatrix} \begin{Bmatrix} (\Delta\omega_1)_k \\ (\Delta\omega_2)_k \end{Bmatrix} = - \begin{Bmatrix} -2\theta + 4(\omega_1)_k + 6\mu_1\sqrt{(\omega_1)_k} - 2(\omega_2)_k \\ -2\theta + 4(\omega_2)_k + 6\mu_2\sqrt{(\omega_2)_k} - 2(\omega_1)_k \end{Bmatrix}
\end{array}
}
$$

Solving the same case as the previous section but with Newton-Raphson.

Table 31: Ex. 11c: Application of Newton-Raphson.

$\theta = 0.010000000000000$

$\omega_1 = 0.003333333333333$

$\omega_2 = 0.051111111111111$

$$K = \begin{matrix} 4.00000000 & -2.00000000 \\ -2.00000000 & 4.00000000 \end{matrix}$$

k	$\bar{\omega}_k$	$\bar{f}(\bar{\omega}_k)$	$\bar{f}(\bar{\omega}_k) = \underset{\sim}{K}\bar{\omega}_k - \bar{f}(\bar{\omega}_k)$	$\left[\frac{d\bar{f}}{d\bar{\omega}}\right]^{-1}$	$\left[\frac{d\bar{f}}{d\bar{\omega}}\right]^{-1} \bar{f}(\bar{\omega}_k)$	$\bar{e}_k = \bar{\omega} - \bar{\omega}_k$	$\lvert\bar{e}_k\rvert$
0	0.005000000000000 0.005000000000000	$1.859E-02$ $-1.685E-03$	$-8.586E-03$ $1.168E-02$	$2.863E-01$ $9.282E-02$	$-1.373E-03$ $1.449E-03$	$-1.400E-03$ $1.400E-03$	$1.98E-03$
1	0.006373415155247 0.003551052208646	$1.840E-02$ $1.726E-03$	$-1.177E-05$ $-2.681E-04$	$2.844E-01$ $8.652E-02$	$-2.655E-05$ $-4.887E-05$	$-2.658E-05$ $-4.895E-05$	$5.57E-05$
2	0.006399961719597 0.003599922774353	$1.840E-02$ $1.600E-03$	$-3.455E-09$ $-4.297E-07$	$2.845E-01$ $8.680E-02$	$-3.828E-08$ $-7.723E-08$	$-3.828E-08$ $-7.723E-08$	$8.62E-08$
3	0.006399999999906 0.003599999999810	$1.840E-02$ $1.600E-03$	$-7.154E-15$ $-1.058E-12$	$2.845E-01$ $8.680E-02$	$-9.390E-14$ $-1.901E-13$	$-9.390E-14$ $-1.901E-13$	$2.12E-13$
4	0.006400000000000 0.003600000000000	$1.840E-02$ $1.600E-03$	$0.000E+00$ $0.000E+00$	$2.845E-01$ $8.680E-02$	$0.000E+00$ $0.000E+00$	$0.000E+00$ $0.000E+00$	$0.00E+00$

solution 11.d Newton simple

$$\bar{f}(\bar{\omega}) = \underset{\sim}{K}\bar{\omega} - \bar{\varphi}(\bar{\omega}) = \bar{0} \quad \longrightarrow \quad \bar{\omega}_{k+1} = \bar{\omega}_k - \left(diag \left[\frac{d\bar{f}}{d\bar{\omega}}\Big|_{\bar{\omega}=\bar{\omega}_k} \right] \right)^{-1} \bar{f}(\bar{\omega}_k)$$

Then:

$$\begin{Bmatrix} (\omega_1)_{k+1} \\ (\omega_2)_{k+1} \end{Bmatrix} = \begin{Bmatrix} (\omega_1)_k \\ (\omega_2)_k \end{Bmatrix} - \begin{bmatrix} 4+3\mu_1\sqrt{(\omega_1)}_k & 0 \\ 0 & 4+3\mu_2\sqrt{(\omega_2)}_k \end{bmatrix}^{-1} \begin{Bmatrix} -2\theta + 4(\omega_1)_k + 6\mu_1\sqrt{(\omega_1)}_k - 2(\omega_2)_k \\ -2\theta + 4(\omega_2)_k + 6\mu_2\sqrt{(\omega_2)}_k - 2(\omega_1)_k \end{Bmatrix}$$

$$\boxed{\begin{aligned} (\omega_1)_{k+1} &= (\omega_1)_k - \frac{-2\theta + 4(\omega_1)_k + 6\mu_1\sqrt{(\omega_1)}_k - 2(\omega_2)_k}{4+3\mu_1\sqrt{(\omega_1)}_k} \\ (\omega_2)_{k+1} &= (\omega_2)_k - \frac{-2\theta + 4(\omega_2)_k + 6\mu_2\sqrt{(\omega_2)}_k - 2(\omega_1)_k}{4+3\mu_2\sqrt{(\omega_2)}_k} \end{aligned}}$$

Solving the same case as the previous section but with Newton Simple.

solution 11.e Newton simple

Conclusions:

- The Newton-Raphson method converges quadratically, and finds the solution in 4 iterations.
- The Newton Simple method works reasonably well, although it does not exhibit quadratic convergence. The reason why it works satisfactorily is because in this case

  $$\left(diag \left[\frac{d\bar{f}}{d\bar{\omega}} \right] \right)^{-1} \approx \left[\frac{d\bar{f}}{d\bar{\omega}} \right]^{-1}$$

  So the approximations made are adequate. In nonlinear engineering computation problems this happens quite often, so this method is a (useful) alternative to the Newton-Raphson method. It requires more iterations to converge, but its computational cost is much lower since it is not necessary to solve systems of equations. This shortcoming is accentuated in large problems.
- The method of Fixed-Point iterations shows very slow linear convergence. The latter is common. Normally any other method is preferable.

Table 32: Ex. 11d: Application of Newton simple.

$\theta = 0.010000000000000$

$\omega_1 = 0.003333333333333$

$\omega_2 = 0.051111111111111$

$$\underset{\sim}{K} = \begin{matrix} 4.00000000 & -2.00000000 \\ -2.00000000 & 4.00000000 \end{matrix}$$

k	$\bar{\omega}_k$	$\bar{f}(\bar{\omega}_k)$	$\bar{f}(\bar{\omega}_k) = \underset{\sim}{K}\bar{\omega}_k - \bar{f}(\bar{\omega}_k)$	$diag\left[\frac{d\bar{f}}{d\bar{\omega}}\right]$	$\left[diag\left[\frac{d\bar{f}}{d\bar{\omega}}\right]\right]^{-1}\bar{f}(\bar{\omega}_k)$	$\bar{e}_k = \bar{\omega} - \bar{\omega}_k$	$\lvert\bar{e}_k\rvert$
0	0.005000000000000	$1.859E-02$	$-8.586E-03$	$4.141E+00$	$-2.073E-03$	$-1.400E-03$	$1.98E-03$
	0.005000000000000	$-1.685E-03$	$1.168E-02$	$0.000E+00$	$1.894E-03$	$1.400E-03$	
1	0.007073149698882	$1.832E-02$	$3.763E-03$	$4.119E+00$	$9.136E-04$	$6.731E-04$	$8.35E-04$
	0.003105749822606	$2.910E-03$	$-4.633E-03$	$0.000E+00$	$-6.862E-04$	$-4.943E-04$	
2	0.006159523097023	$1.843E-02$	$-1.376E-03$	$4.127E+00$	$-3.334E-04$	$-2.405E-04$	$3.08E-04$
	0.003791977427135	$1.116E-03$	$1.733E-03$	$0.000E+00$	$2.670E-04$	$1.920E-04$	
3	0.006492954386071	$1.839E-02$	$5.335E-04$	$4.124E+00$	$1.294E-04$	$9.295E-05$	$1.20E-04$
	0.003524936759631	$1.793E-03$	$-6.790E-04$	$0.000E+00$	$-1.032E-04$	$-7.506E-05$	
4	0.006363587693570	$1.840E-02$	$-2.064E-04$	$4.125E+00$	$-5.003E-05$	$-3.641E-05$	$4.60E-05$
	0.003628087219286	$1.528E-03$	$2.568E-04$	$0.000E+00$	$3.923E-05$	$2.809E-05$	
5	0.006413615288590	$1.840E-02$	$7.846E-05$	$4.125E+00$	$1.902E-05$	$1.362E-05$	$1.76E-05$
	0.003588853116780	$1.629E-03$	$-1.003E-04$	$0.000E+00$	$-1.529E-05$	$-1.115E-05$	
10	0.006399881668051	$1.840E-02$	$-6.696E-07$	$4.125E+00$	$-1.623E-07$	$-1.183E-07$	$1.49E-07$
	0.003600090733036	$1.600E-03$	$8.315E-07$	$0.000E+00$	$1.268E-07$	$9.073E-08$	
15	0.006400000962547	$1.840E-02$	$5.550E-09$	$4.125E+00$	$1.346E-09$	$9.625E-10$	$1.25E-09$
	0.003599999210083	$1.600E-03$	$-7.103E-09$	$0.000E+00$	$-1.084E-09$	$-7.899E-10$	
20	0.006399999991620	$1.840E-02$	$-4.742E-11$	$4.125E+00$	$-1.150E-11$	$-8.380E-12$	$1.06E-11$
	0.003600000006425	$1.600E-03$	$5.888E-11$	$0.000E+00$	$8.982E-12$	$6.425E-12$	
25	0.006400000000068	$1.840E-02$	$3.931E-13$	$4.125E+00$	$9.529E-14$	$6.816E-14$	$8.82E-14$
	0.003599999999944	$1.600E-03$	$-5.030E-13$	$0.000E+00$	$-7.673E-14$	$-5.594E-14$	
30	0.006399999999999	$1.840E-02$	$-3.362E-15$	$4.125E+00$	$-8.150E-16$	$-5.941E-16$	$7.49E-16$
	0.003600000000000	$1.600E-03$	$4.174E-15$	$0.000E+00$	$6.367E-16$	$4.554E-16$	
31	0.006400000000000	$1.840E-02$	$1.277E-15$	$4.125E+00$	$3.095E-16$	$2.212E-16$	$2.86E-16$
	0.003600000000000	$1.600E-03$	$-1.631E-15$	$0.000E+00$	$-2.487E-16$	$-1.813E-16$	