1.– It is known that the polynomial $P(x) = x^5 + 5x - 1$ has a real root α, such that $0.1 < \alpha < 0.2$. Calculate for which values of the constant C is the iterative algorithm

$$x^{k+1} = x^k - C\,P(x^k)$$

convergent. How is the initial value of the sequence be chosen?

2.– Write an algorithm that allows to compute $x^{1/n}$, n nonzero integer, without evaluating any root.

3.– Given the equation $f(x) = 1 - x^2$:

a) What conditions must the constant m satisfy for the Whittaker's method

$$x^{k+1} = x^k - \frac{f(x^k)}{m}$$

to be convergent to the root $x = 1$?

b) Does the above condition guarantee that the method converges to the root $x = 1$ for any initial value x^0? Why? Give a numerical example of convergence and one of non-convergence, if any. (The convergence example should be non-trivial, in the sense that the initial value should not be chosen so that the method converges in a finite number of steps).

c) Apply the Aitken acceleration to the convergent example developed in the previous section.

d) If the conditions determined in a) are not satisfied, but Steffenson's method is applied, is the resulting algorithm convergent? Under what conditions? Give a numerical example of convergence.

4.– Analyze in detail whether any of the following iterative techniques may be suitable for calculating a triple root of the equation $f(x) = 0$:

a) $x^{k+1} = x^k - f'(x^k)/f''(x^k)$

b) $x^{k+1} = x^k - f''(x^k)/f'''(x^k)$

c) $x^{k+1} = x^k - f'''(x^k)/f^{(4}(x^k)$

5.– To obtain the simple roots of the equation $f(x)$ the following iterative method is proposed:

$$x_{i+1} = x_i - \frac{u\,(x_i)}{a\,u\,(x_i) + b} \quad ; \qquad u\,(x_i) = \frac{f\,(x_i)}{f'\,(x_i)} \quad ,$$

where a and b are two real constants:

a) Find the constants a and b for which the method is of third order.

b) Propose the formulation to calculate the square root of the positive real number s by solving the equation $x^2 - s = 0$ by Newton's method and by this third order method.

c) Apply both algorithms to the case $s = 2$, starting at $x_0 = 1$ and $x_0 = 5$.

d) What conclusions can be drawn? Were the results obtained predictable?

6.– Consider the equation $f(x) = tg(\pi x) - 6$. Let $x_0 = 0$ and $x_1 = 0.48$, we want to approximate $x = \frac{1}{\pi} \cdot \arctan(6) = 0.447431543$ using:

a) The Bisection method
b) The *Regula-Falsi* Method
c) The Secant Method

Analyze the results obtained for each case after ten iterations.

7.– We want to find the inverse of any number $b > 0$, without performing any division operation. Let β $(0.1 < \beta < 1)$ be the mantissa of the number b in decimal base, and let e be its exponent, this is,

$$b = \beta \cdot 10^e; \qquad 0.1 \leq \beta < 1.$$

Obviously, the part corresponding to the exponent can be easily inverted by a change of sign. To invert the mantissa, we propose to apply Newton's method to calculate the root of the function $f(x) = \beta - (1/x)$.

a) Apply Newton's method to the given function, simplify as much the expression so the final expression does no use any division.
b) Study graphically the behavior of the Newton's method in this case and find an initial value x_0 which guarantees the convergence of the algorithm.
c) Obtain through the proposed algorithm the inverse of $b = 0.39000 \cdot 10^{-1}$. Make comments on the results.

8.– In a structural problem it is required to solve the buckling problem of a pinned beam, with an elastic spring restraining the rotation in one of its supports. It is desired to solve this problem, for any values of the variables that define the system, namely, the length L of the beam, its bending stiffness EI, and the elastic constant K_ϕ of the spring.

It is deduced, by means of the equilibrium of the deformed beam, that the load that produces the instability of the straight geometry of the beam, also called buckling load P, is given by the lowest of the non-trivial solutions of the equation:

$$\tan(kL) = \frac{kL}{\dfrac{(kL)^2}{k_\phi} + 1}$$

where $k = \frac{P}{EI}$ and $k_\phi = \frac{K_\phi L}{EI}$. The aim is to solve this equation by means of a numerical method that works for arbitrary values of the data. For this purpose, the initial expression is written in the equivalent, but more compact form:

$$\tan(x) = \frac{x}{(cx)^2 + 1},$$

with the new unknown being $x = kL$, and $c = \sqrt{1/k_\phi}$ being a parameter that is calculated based on the data. Obviously, it is of interest to calculate the smallest non-trivial positive root of the above equation.

For this purpose, the following numerical algorithm is proposed: starting from a certain initial value, conveniently chosen , iterate until convergence by means of the formulas:

$$y^k = g(x^k); \qquad g(x) = \frac{x}{(cx)^2 + 1}$$

$$x^{k+1} = f^{-1}(y^k); \qquad f(x) = \tan(x)$$

where f^{-1} is in the inverse function of f:

a) To perform an analytical study of the asymptotic convergence of the algorithm as a function of the parameter c.
b) Draw approximately the functions $f(x)$ and $g(x)$, and bound an interval where the solution of the problem is found as a function of the parameter $c > 0$.
c) Propose a reasoned initial value that works for any value of c.
d) Iterate until convergence, for the following values of the parameter c; $c = 0.1$, $c = 0.5$, $c = 1$, $c = 100$, $c = 1000$. Comment on the results obtained.

9.– The classified flow curve, used in hydrological studies, is the curve obtained by classifying the average daily flows of any hydrological year according to the number of days of the year in which this flow has been exceeded or equaled. An analytical expression of this curve has been proposed by Coutagne. According to this author, this curve takes the form:

$$q(t) = Q_{mc} + \frac{(Q - Q_{mc})(1+\nu)}{T^{\nu}}(T-t)^{\nu},$$

where $q(t)$ is the flow rate equaled or exceeded during t days over the course of a T day observation period days, Q is the mean annual flow rate, Q_{mc} is the minimum characteristic flow (the flow equaled or exceeded on all days of the year except for the 10 driest days), Q_{mc} is the minimum characteristic flow rate driest days), Q_s is the semi-permanent flow (that equaled or exceeded on half the days of the year), and ν is the irregularity coefficient whose value is defined as the largest of the roots of the function

$$f(\nu) = 2^{\nu}\kappa - (\nu + 1), \qquad \text{donde} \qquad \kappa = \frac{Q_s - Q_{mc}}{Q - Q_{mc}}.$$

The Newton method is proposed to calculate the irregularity coefficient. It is known that for the cases to be studied the value of the κ coefficient takes values that do not differ excessively from 1.

a) Draw the function $f(f(\nu)$ and set out the proposed iterative scheme fully develop and simplify as much as possible its expression.
b) Analyze the convergence of the method and explain for which values of κ more iterations will be needed, taking into account that we want to obtain the largest possible value of the irregularity coefficient. Study for which initial values the algorithm converges.
c) Apply the developed iterative scheme to the following cases:
  1) $Q_s = 100\ m^3/s$, $Q_{mc} = 1\ m^3/s$, $Q = 98.0000000000\ m^3/s$.
  2) $Q_s = 100\ m^3/s$, $Q_{mc} = 1\ m^3/s$, $Q = 94.2663845753\ m^3/s$.

  In both cases, the initial values $\nu_0 = 0$ and $\nu_0 = 1$ will be used. Compare and comment the results.
d) Propose another algorithm to solve the last case of the previous section and apply it with the two proposed initial values. Comment the results.

10.– Given the non-linear system of equations with n equations and n unknowns

$$\underset{\sim}{K}(\bar{x})\bar{x} = \bar{f}(\bar{x}),$$

where the matrix $\underset{\sim}{K}(\bar{x})$ and the vector $\bar{f}(\bar{x})$ depend on the unknowns $\bar{x}$:

a) Design and write, explicitly, algorithms that allow to solve the above system by::
- Fixed Point interations,
- Newton-Raphson,

- Newton,
- Whittaker,

indicating, if applicable, which methods are suitable to solve the derived linear systems.

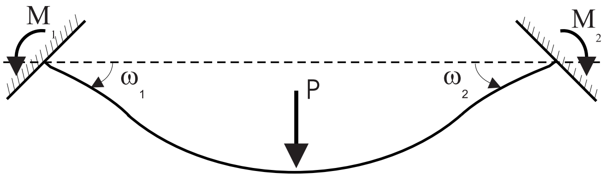
b) Discuss in a reasoned manner, and taking into account all relevant aspects (storage needs, speed of convergence, etc.) which of the designed algorithms would be preferable in the following cases:

- The matrix $\underset{\sim}{K}(\bar{x})$ is SYMMETRIC, POSITIVE DEFINITE and BANDED for every value of $\bar{x}$,
- The matrix $\underset{\sim}{K}(\bar{x})$ and the vector $\bar{f}(\bar{x})$ are lowly sensible with respecto to $\bar{x}$ (thus, in order to obtain significant variations in its components it is necessary to modify greatly the value $\bar{x}$).

11.– A constant section beam supports a point load applied at the center of the span. The ends of the beam are imperfectly clamped. Therefore, a twist occurs at each end, although its value is logically lower than that which would occur in pinned joint. It can be assumed that the moment opposing the twist at each end is proportional to the square root of the rotated angle. The positive constants θ, μ_1 and μ_2 are defined so that

$$\theta = \frac{PL^2}{16EI}, \qquad M_1 = 6\frac{EI}{L}\mu_1\sqrt{\omega_1}, \qquad M_2 = 6\frac{EI}{L}\mu_2\sqrt{\omega_2},$$

where E is the elasticity modulus of the beam, I is the inertia moment of the cross-section, L is the length of the span, P is the point load applied on the center of the span, M_1 and M_2 are the moments generated by the imperfect clamps, and ω_1 and ω_2, are the rotations produced at the ends of the beam. The sign convention is the one shown in the following figure.



a) Prove that the rotations ω_1 and ω_2 can be obtained by solving the non-linear system

$$\underset{\sim}{K}\bar{\omega} = \bar{\varphi}(\bar{\omega}), \qquad \text{with} \quad \bar{\omega} = \begin{Bmatrix} \omega_1 \\ \omega_2 \end{Bmatrix},$$

being

$$\underset{\sim}{K} = \begin{bmatrix} +4 & -2 \\ -2 & +4 \end{bmatrix}, \qquad \bar{\varphi}(\omega) = \begin{Bmatrix} 2\theta - 6\mu_1\sqrt{\omega_1} \\ 2\theta - 6\mu_2\sqrt{\omega_2} \end{Bmatrix}.$$

b) Solve the problem by Fixed Point Iterations

$$\bar{\omega}^{k+1} = \underset{\sim}{K}^{-1}\bar{\varphi}(\bar{\omega}^k)$$

in the case of $\mu_1 = 1/300$, $\mu_2 = 23/450$, and $\theta = 10^{-2}$. As initial approximations, $\omega_1^0 = \omega_2^0 = \theta/2$ will be used. (if the clamped ends would not produce any resistance to the rotation, the rotations would be half the values.).

c) Propose and solve the same problem by means of the Newton-Raphson method using the same initial approximation.

d) Propose and solve the same problem by means of the Simplified Newton method using the same initial approximation.

e) Compare the results and the efficiency of the three proposed methods in this case.
