**1.–** Write an inversion algorithm for lower triangular matrices of size $n$ and of size $n$ and half-bandwidth $n$. The initial matrix must be stored in a vector, and its inverse must be stored over the initial matrix. initial matrix.

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**2.–** Write an algorithm for solving linear systems of the form $\underset{\sim}{K}\bar{u} = \bar{f}$, where $\underset{\sim}{K}$ is a symmetric positive definite matrix of bandwidth $s$, stored in a vector. positive definite, of bandwidth $s$, stored in a vector. Use the most appropriate storage scheme and solution method.

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**3.–** Generalize the above algorithm for solving $m$ systems of $m$ equations with the same matrix and different independent terms $\{\bar{f}^1, \bar{f}^2, \ldots, \bar{f}^m\}$, these being stored in a vector.

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**4.–** To analyze a certain engineering problem it is necessary to to solve systems of linear equations $\underset{\sim}{A}\ x = \bar{b}$, where the matrix $\underset{\sim}{A}$ is symmetric, positive definite, and its shape is:

$$\underset{\sim}{A} = \begin{bmatrix} a_{11} & & & & & & \\ a_{21} & a_{22} & & & & SIM. & \\ a_{31} & 0 & a_{33} & & & & \\ a_{41} & 0 & 0 & a_{44} & & & \\ a_{51} & 0 & 0 & 0 & a_{55} & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ a_{n1} & 0 & 0 & 0 & \ldots & 0 & a_{nn} \end{bmatrix} \qquad \begin{array}{l} 1000 \leq n \leq 5000 \\ |a_{ii}| \gg |a_{ij}| \quad \forall j \neq i \end{array}$$

In order to solve the system of equations a direct method is considered, storing the intermediate operations over the initial matrix.

- **a)** What is the least expensive storage scheme to be used? Why?
- **b)** What direct method is convenient in this case? Why?
- **c)** Could it be advantageous, in terms of computational time and storage, to use an iterative algorithm? Which one? Why?
- **d)** Could it be advantageous, in terms of computational time and storage, to use a semi-iterative algorithm? Which one? Why?
- **e)** If the equations and/or the unknowns are rearranged conveniently, is it possible to reduce the computational cost and storage associated with the use of a direct method? How?

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**5.–** Derive the sufficient condition for the algorithm:

$$\underset{\sim}{B}^{k+1} = \underset{\sim}{B}^k[2\underset{\sim}{I} - \underset{\sim}{A}\underset{\sim}{B}^k]; \qquad k = 0, \ldots,$$

to converge to the inverse of the matrix $\underset{\sim}{A}$, being $\underset{\sim}{B}^0$ an initial approximation to $\underset{\sim}{A}^{-1}$ such that:

$$\underset{\sim}{A}\underset{\sim}{B}^0 = \underset{\sim}{I} + \underset{\sim}{E}; \qquad (\underset{\sim}{E} = \text{error matrix})$$

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**6.**– To solve the following system of equations:

$$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 9 \end{Bmatrix}$$

The over-relaxed Gauss-Seidel method is considered.

a) Study the convergence of the algorithm in terms of the over-relaxation coefficient $\alpha$ used ($\alpha =$ is constant in every iteration).
b) Is there an optimal value of the over-relaxation coefficient (constant in all iterations) for faster convergence? (Hint: plot the spectral radius of the corresponding matrix as a function of the over-relaxation coefficient).
c) Perform the first five iterations for $\alpha = 1$ and for $\alpha = \frac{32}{31}$, starting at initial approximation $x_1^0 = 0$, $x_2^0 = 0$.
d) In practice, could an analysis such as the one described be performed for a system with several thousands of equations? Why?

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**7.**– Let be a continuous beam consisting of $n$ spans. The spans and supports are numbered sequentially, so that span $e \in \{1, \ldots, n\}$ goes from support $e-1$ to support $e$. Let $E$ be the modulus of elasticity of the material and let $I_e$ and $L_e$, the moment of inertia of the cross-section and length corresponding to the $e$-th span respectively. Given the moments $\{M_i\}_{i=0,\ldots,n}$ acting on the supports we wish to calculate the corresponding $\{w_i\}_{i=0,\ldots,n}$ produced.

**a)** Set out the system of linear equations that must be solved to calculate the spins.
**b)** Check that the matrix of the above system is symmetric on positive definite.
**c)** Propose and develop completely the most suited direct method for solving the system.
**d)** Write a FORTRAN program with the selected method that allows to solve this type of problems
**e)** Propose the solution of the system by the iterative method of Gauss-Seidel. Can the convergence of the method be guaranteed? Interpret how does the method works from the point of view of the structural problem.

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**8.**– Let $\underset{\sim}{A}$ be a symmetric and regular matrix of size $n$. We want to solve the solution to the following linear system of equations:

$$\underset{\sim}{A}\bar{x} = \bar{b}$$

given $\bar{b}$.

**a)** Write the Gaussian algorithm without pivoting **avoiding useless operations**. Calculate the number of operations needed to solve the problem.
**b)** Can the Gauss algorithm with pivoting be used keeping the symmetry? Why?

**Hint:** Recall that at each step of the elimination process, when the terms of the $k$-th column below the pivot are canceled, only the following submatrix is recalculated.

$$\begin{pmatrix} a_{k+1,k+1} & \cdots & a_{k+1,n} \\ \vdots & \ddots & \vdots \\ a_{n,k+1} & \cdots & a_{n,n} \end{pmatrix}.$$

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