LIPSCHITZ CONDITIONS FOR FUNCTIONAL ITERATION METHODS $x_{k+1} = \Phi(x_k)$.

Let x_k be the sequence of values computed by applying the functional iteration method:

$$x_{k+1} = \Phi(x_k), \qquad k = 0, 1, 2, \ldots$$

The convergence of this iterative method is ruled by the so called "Lipschitz conditions" that can be stated with different approaches. Some of them are presented now.

If the existence of the root α is not known *a priori*, then the convergence of the iterative method can be studied by means of the following two approaches of the Lipschitz conditions:

A. General form: $\Phi(x)$ is a contractive function

Let's consider the closed interval I defined by the known values $x_0 \in \mathbb{R}$ and $\rho \in \mathbb{R}^+$ ($I \equiv [x_0 - \rho, x_0 + \rho]$) and let $\lambda \in \mathbb{R}$ ($0 \leq \lambda < 1$) be defined and known too.

$$\text{If } \left\{ \begin{array}{c} |\Phi(x) - \Phi(z)| \leq \lambda |x - z|, \quad \forall x, z \in I; \\ \\ |x_0 - \Phi(x_0)| \leq (1 - \lambda)\rho; \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} x_k \in I, \quad \forall k; \\ \\ \exists \lim_{k \to \infty} x_k = \alpha; \\ \\ \exists^* \alpha \in I / \Phi(\alpha) = \alpha; \\ \\ |x_{k+1} - \alpha| \leq \lambda |x_k - \alpha|, \quad \forall k; \end{array} \right\}$$

B. More restrictive approach: $\Phi(x)$ is a differentiable function

Let's consider the closed interval I defined by the known values $x_0 \in \mathbb{R}$ and $\rho \in \mathbb{R}^+$ ($I \equiv [x_0 - \rho, x_0 + \rho]$) and let $\lambda \in \mathbb{R}$ ($0 \leq \lambda < 1$) be defined and known too.

$$\text{If } \left\{ \begin{array}{c} |\Phi'(x)| \leq \lambda, \quad \forall x \in I; \\ \\ |x_0 - \Phi(x_0)| \leq (1 - \lambda)\rho; \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} x_k \in I, \quad \forall k; \\ \\ \exists \lim_{k \to \infty} x_k = \alpha; \\ \\ \exists^* \alpha \in I / \Phi(\alpha) = \alpha; \\ \\ |x_{k+1} - \alpha| \leq \lambda |x_k - \alpha|, \quad \forall k; \end{array} \right\}$$

This approach is more restrictive than the one stated in **A**, but it is generally more difficult to verify.

If the existance of the root α is known *a priori*, then the convergence of the iterative method can be studied by using any of the following approaches of the Lipschitz conditions:

C. General form: $\Phi(x)$ is a contractive function

Let's consider the closed interval I defined by $\alpha \in \mathbb{R}$ and $\rho \in \mathbb{R}^+$ ($I \equiv [\alpha - \rho, \alpha + \rho]$) and let $\lambda \in \mathbb{R}$ ($0 \leq \lambda < 1$) be defined and known too.

$$\text{If} \left\{ \begin{array}{c} \exists\ \alpha \ / \ \alpha = \Phi(\alpha) \\ \\ |\Phi(x) - \Phi(z)| \leq \lambda |x - z|, \quad \forall x, z \in I; \\ \\ x_0 \in I; \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} x_k \in I, \quad \forall k; \\ \\ \exists \lim_{k \to \infty} x_k = \alpha; \\ \\ \exists^* \alpha; \\ \\ |x_{k+1} - \alpha| \leq \lambda |x_k - \alpha|, \quad \forall k; \end{array} \right\}$$

D. General form: $\Phi(x)$ is a contractive function

Let's consider the closed interval I defined by $\alpha \in \mathbb{R}$ and $\rho \in \mathbb{R}^+$ ($I \equiv [\alpha - \rho, \alpha + \rho]$) and let $\lambda \in \mathbb{R}$ ($0 \leq \lambda < 1$) be defined and known too.

$$\text{If} \left\{ \begin{array}{c} \exists\ \alpha \ / \ \alpha = \Phi(\alpha) \\ \\ |\Phi(x) - \Phi(\alpha)| \leq \lambda |x - \alpha|, \quad \forall x \in I; \\ \\ x_0 \in I; \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} x_k \in I, \quad \forall k; \\ \\ \exists \lim_{k \to \infty} x_k = \alpha; \\ \\ \exists^* \alpha; \\ \\ |x_{k+1} - \alpha| \leq \lambda |x_k - \alpha|, \quad \forall k; \end{array} \right\}$$

E. More restrictive approach: $\Phi(x)$ is a differentiable function

Let's consider the closed interval I defined by $\alpha \in \mathbb{R}$ and $\rho \in \mathbb{R}^+$ ($I \equiv [\alpha - \rho, \alpha + \rho]$) and let $\lambda \in \mathbb{R}$ ($0 \leq \lambda < 1$) be defined and known too.

$$\text{If} \left\{ \begin{array}{c} \exists\ \alpha \ / \ \alpha = \Phi(\alpha) \\ \\ |\Phi'(x)| \leq \lambda, \quad \forall x \in I; \\ \\ x_0 \in I; \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} x_k \in I, \quad \forall k; \\ \\ \exists \lim_{k \to \infty} x_k = \alpha; \\ \\ \exists^* \alpha; \\ \\ |x_{k+1} - \alpha| \leq \lambda |x_k - \alpha|, \quad \forall k; \end{array} \right\}$$

This approach is more restrictive than the one stated in **C** and **D**, but it is generally more difficult to verify.