Convergence conditions in an interval (Lipschitz Conditions)

1. **Lipschitzian or contractive function.**

   A function $\phi(x)$ is contractive in a closed interval I if and only if

   $$\left\{\exists\lambda \in [0,1) \;/\; |\phi(x) - \phi(\xi)| \leq \lambda|x - \xi| \quad \forall x, \xi \in I\right\}$$

   being λ the Lipschitz constant.

2. **Convergence in an interval.**

   Given ϕ, Lipschitzian in $I = [x_0 - \rho, x_0 + \rho]$, with Lipschitz constant λ and given inicial approximation x_0 such that $|x_0 - \phi(x_0)| \leq (1-\lambda)\rho$:

   $$\left\{\begin{array}{c} \phi \;/\; |\phi(x) - \phi(\xi)| \leq \lambda|x - \xi| \quad \forall x, \xi \in I \equiv [x_0 - \rho, x_0 + \rho] \\ \text{with } \rho > 0;\; \lambda \in [0,1) \text{ and} \\ x_0 \;/\; |x_0 - \phi(x_0)| \leq (1-\lambda)\rho \end{array}\right\}$$

   Then,

   (a) $\boxed{x_{k+1} = \phi(x_k) \;\in\; I \quad \Longrightarrow \quad |x_{k+1} - x_0| \;\leq\; \rho}$

   Demostration by induction.

   First it is demonstrated that $x_1 \in I$

   $$\begin{array}{rcl} x_1 = \phi(x_0) & \leadsto & |x_0 - \phi(x_0)| \;\leq\; (1-\lambda)\rho \;\Leftrightarrow \\ & & |x_1 - x_0| \;\leq\; (1-\lambda)\rho \;\leq\; \rho \;\Leftrightarrow\; x_1 \;\in\; I \end{array}$$

   Then it is assumed as satisfied for x_k and imposed for x_{k+1} as:

   $$\begin{array}{lcll} |x_{k+1} - x_k| & = & |\phi(x_k) - \phi(x_{k-1})| & \leq \;\lambda\; |x_k - x_{k-1}| \\ |x_k - x_{k-1}| & = & |\phi(x_{k-1}) - \phi(x_{k-2})| & \leq \;\lambda\; |x_{k-1} - x_{k-2}| \\ & \vdots & & \\ |x_2 - x_1| & = & |\phi(x_1) - \phi(x_0)| & \leq \;\lambda\; |x_1 - x_0| \end{array}$$

   Thus,

   $$|x_{k+1} - x_k| \;\leq\; \lambda^k\; |x_1 - x_0| \;\leq\; \lambda^k\; (1-\lambda)\rho \;\leq\; \rho$$

   $|x_{k+1} - x_0|$ bounds need to be obtained. So:

$$
\begin{aligned}
|x_{k+1}-x_0| &= |(x_{k+1}-x_k) + (x_k-x_{k-1}) + \ldots + (x_1-x_0)| \\
&\le \underbrace{|x_{k+1}-x_k|}_{\le \lambda^k(1-\lambda)\rho} + \underbrace{|x_k-x_{k-1}|}_{\le \lambda^{k-1}(1-\lambda)\rho} + \ldots + \underbrace{|x_1-x_0|}_{\le \lambda\ (1-\lambda)\rho} \\
&\le \underbrace{(\lambda^k+\lambda^{k-1}+\ldots+\lambda^0)}_{\left(\frac{1-\lambda^{k+1}}{1-\lambda}\right)}(1-\lambda)\rho
\end{aligned}
$$

This leads to:

$$
|x_{k+1}-x_0| \le \left(1-\lambda^{k+1}\right)\rho \le \rho
$$

(b) $\boxed{\lim_{k\to\infty} x_k = \alpha}$ (Demonstration that it is a Cauchy sequence)

$$
\begin{aligned}
|x_m-x_{m+p}| &= |(x_m-x_{m+1}) + (x_{m+1}-x_{m+2}) + \ldots + (x_{m+p-1}-x_{m+p})| \\
&\le \underbrace{|x_m-x_{m+1}|}_{\le \lambda^m(1-\lambda)\rho} + \underbrace{|x_{m+1}-x_{m+2}|}_{\le \lambda^{m+1}(1-\lambda)\rho} + \ldots + \underbrace{|x_{m+p-1}-x_{m+p}|}_{\le \lambda^{m+p-1}\ (1-\lambda)\rho} \\
&\le (1-\lambda)\ \rho\ \lambda^m(\underbrace{1+\lambda+\ldots+\lambda^{p-1}}_{\frac{1-\lambda^p}{1-\lambda}}) = (1-\lambda^p)\ \rho\ \lambda^m
\end{aligned}
$$

This implies that:

$$
\forall\, \varepsilon > 0,\ \exists\ N(\varepsilon)\ /\ |x_m-x_{m+p}| < \varepsilon, \quad \forall m > N(\varepsilon)
$$

If $N\ /\ \lambda^N < \varepsilon/\rho$ is stated then:

$$
|x_m-x_{m+p}| \le (1-\lambda^p)\ \rho\ \frac{\varepsilon}{\rho} \le \varepsilon
$$

And consequently it is a convergent Cauchy sequence.

(c) $\boxed{\alpha \text{ is the unique root of } f \text{ in } I}$

By *reductio ad absurdum* it is assumed that there is another root β. Then,

$$
\beta \in I \equiv [x_0-\rho,\ x_0+\rho];\ \beta = \phi(\beta)
$$

And,

$$
|\alpha-\beta| = |\phi(\alpha)-\phi(\beta)| \le \lambda|\alpha-\beta|.
$$

Consequently,

$$
|\alpha-\beta| \le \lambda|\alpha-\beta| \quad \text{with} \quad \lambda \in [0,1)
$$

Which is only possible if $\alpha = \beta$ and then the root would be unique.

(d) $\boxed{\text{Convergence is, at least, linear}}$

$$
|x_{k+1}-\alpha| = |\phi(x_k)-\phi(\alpha)| \le \lambda\ |x_k-\alpha|
$$

And,

$$|x_{k+1} - \alpha| \ \leq \ \lambda \ |x_k - \alpha| \quad \text{with} \quad \lambda \in [0, 1)$$

So it is proved that the convergence is linear.

3. **Convergence in an interval considering rounding errors propagation**

Given the functional iteration algorithm:

$$x_{k+1} \ = \ \phi(x_k)$$

It can be computed in practice:

$$\widehat{x}_{k+1} \ = \ \phi(\widehat{x}_k) \ + \ \varepsilon_k$$

Let be ε such that $\text{máx}|\varepsilon_k| \ \leq \ \varepsilon$
For the algorithm to converge,

$$\lim_{k\to\infty} \widehat{x}_k \ = \ \alpha.$$

Given α such that $\alpha = \phi(\alpha)$, with $\rho > 0$ in an interval $I = [\alpha - \rho, \alpha + \rho]$. If $\phi(x)$ is Lipschitzian in I with constant $\lambda \in [0, 1)$ and even more $\widehat{x}_0 \in [\alpha - \rho_0, \alpha + \rho_0]$ with $\rho_0 \in (0, \rho - \frac{\varepsilon}{1-\lambda})$ this validity range of ρ_0 is accepted so that $\widehat{x}_1 \in I$. Thus,

$$|\alpha - \widehat{x}_1| = |\alpha - (\phi(x_0) + \varepsilon_0)| \leq |\alpha - \phi(x_0)| + |\varepsilon_0| \leq (1 - \lambda)\rho_0 + |\varepsilon_0| \leq (1 - \lambda)\rho_0 + \varepsilon$$

$$|\alpha - \widehat{x}_1| \leq (1 - \lambda)\rho_0 + \varepsilon < (1 - \lambda)\rho$$

Then,

$$\rho_0 < \rho - \frac{\varepsilon}{1 - \lambda}$$

And it can be demonstrated that:

1. $\widehat{x}_k \in I \ \forall k$
2. $|\widehat{x}_k - \alpha| \ \leq \ \dfrac{\varepsilon}{1 - \lambda} + \lambda^k \left(\rho_0 - \dfrac{\varepsilon}{1 - \lambda}\right)$

And in the limit:
$k \to \infty \implies \widehat{x}_k \in \left[\alpha - \dfrac{\varepsilon}{1 - \lambda}, \ \alpha + \dfrac{\varepsilon}{1 - \lambda}\right]$
It is therefore possible to converge to the solution but within the limits of machine accuracy.
Moreover, the slower an algorithm is $(\lambda \to 1)$, the higher this error rate will be. Slower algorithms are also more inaccurate.

Demostration:
Let's assume that $\widehat{x}_0 \in [\alpha - \rho_0, \alpha + \rho_0]$

$$|\widehat{x}_k - \alpha| = |\phi(\widehat{x}_{k-1}) + \varepsilon_{k-1} - \phi(\alpha)| \leq |\phi(\widehat{x}_{k-1}) - \phi(\alpha)| + |\varepsilon_{k-1}| \leq \lambda|\widehat{x}_{k-1} - \alpha| + \varepsilon$$

$$|\widehat{x}_1 - \alpha| \leq \lambda|\widehat{x}_0 - \alpha| + \varepsilon$$

$$|\widehat{x}_2 - \alpha| \leq \lambda^2|\widehat{x}_0 - \alpha| + \lambda\varepsilon + \varepsilon$$

$$\vdots$$

$$|\widehat{x}_k - \alpha| \leq \lambda^k|\widehat{x}_0 - \alpha| + \lambda^{k-1}\varepsilon + \ldots + \lambda\varepsilon + \varepsilon$$

$$|\widehat{x}_k - \alpha| \leq \lambda^k|\widehat{x}_0 - \alpha| + \varepsilon(\lambda^{k-1} + \ldots + \lambda + 1)$$

$$|\widehat{x}_k - \alpha| \leq \lambda^k|\widehat{x}_0 - \alpha| + \varepsilon\frac{1-\lambda^k}{1-\lambda}$$

$$|\widehat{x}_k - \alpha| \leq \lambda^k\rho_0 + \varepsilon\ \frac{1-\lambda^k}{1-\lambda}$$

$$|\widehat{x}_k - \alpha| \leq \lambda^k\left(\rho_0 - \frac{\varepsilon}{1-\lambda}\right) + \frac{\varepsilon}{1-\lambda} \leq \rho_0 + \frac{\varepsilon}{1-\lambda} = \rho \quad \forall k$$

Thus, $|\widehat{x}_k - \alpha| \leq \rho$ and $\widehat{x}_k \in I\ \forall k$

In the limit:

$$\lim_{k\to\infty} |\widehat{x}_k - \alpha| \leq \frac{\varepsilon}{1-\lambda} \implies \alpha - \frac{\varepsilon}{1-\lambda} \leq \lim_{k\to\infty} \widehat{x}_k \leq \alpha + \frac{\varepsilon}{1-\lambda}$$

Then the iterative algorithm converges to the solution α but with a non-zero upper bound on the error.