ANÁLISIS TENSORIAL EN COORDENADAS CURVILÍNEAS: NOTACIÓN Y FORMULARIO

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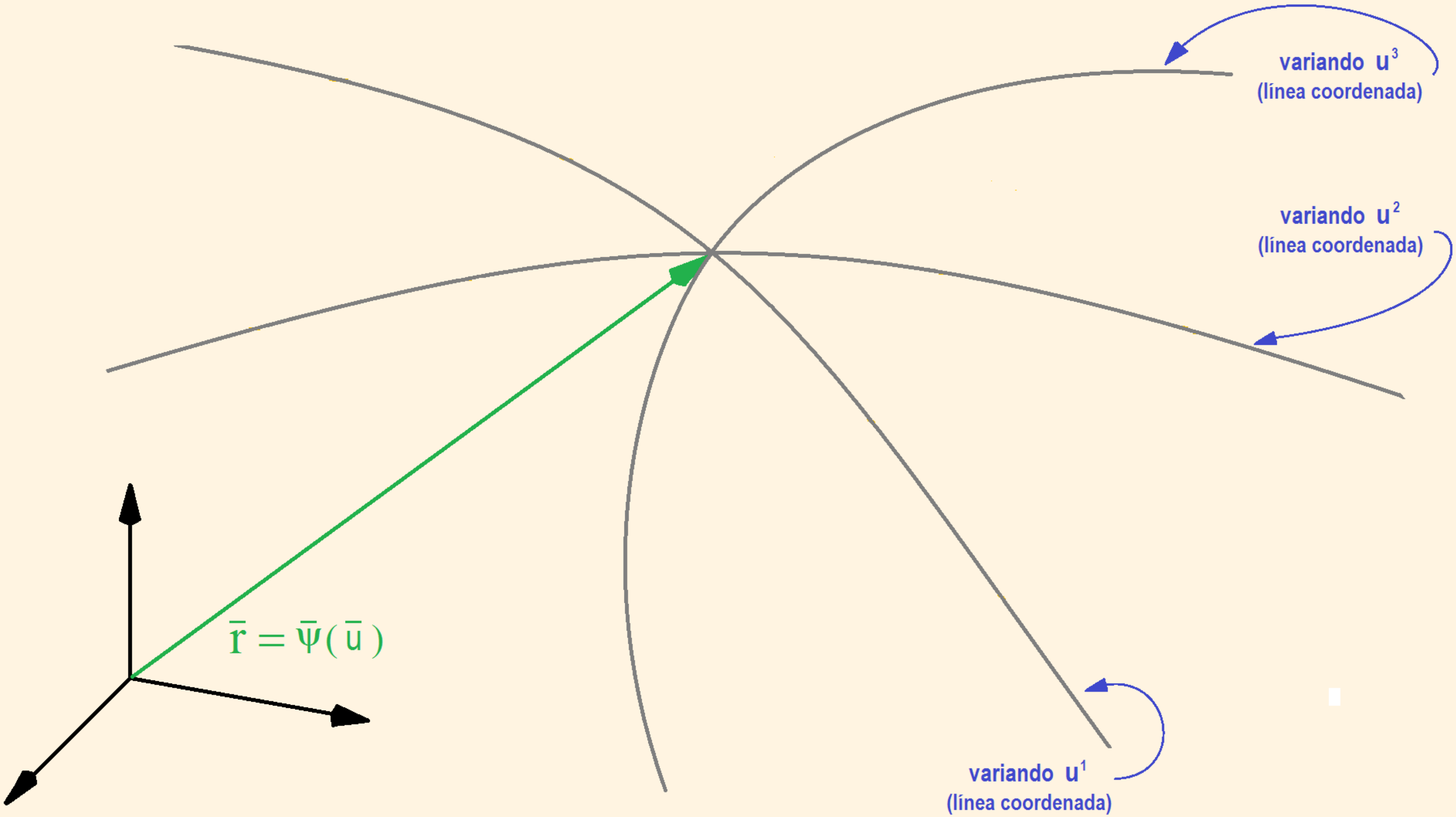
ÍNDICE

Coordenadas curvilíneas (I)



COORDENADAS CURVILÍNEAS (en $\mathbb{R}^3$)

Coordenadas curvilíneas (II)

EXPRESIÓN GENERAL:

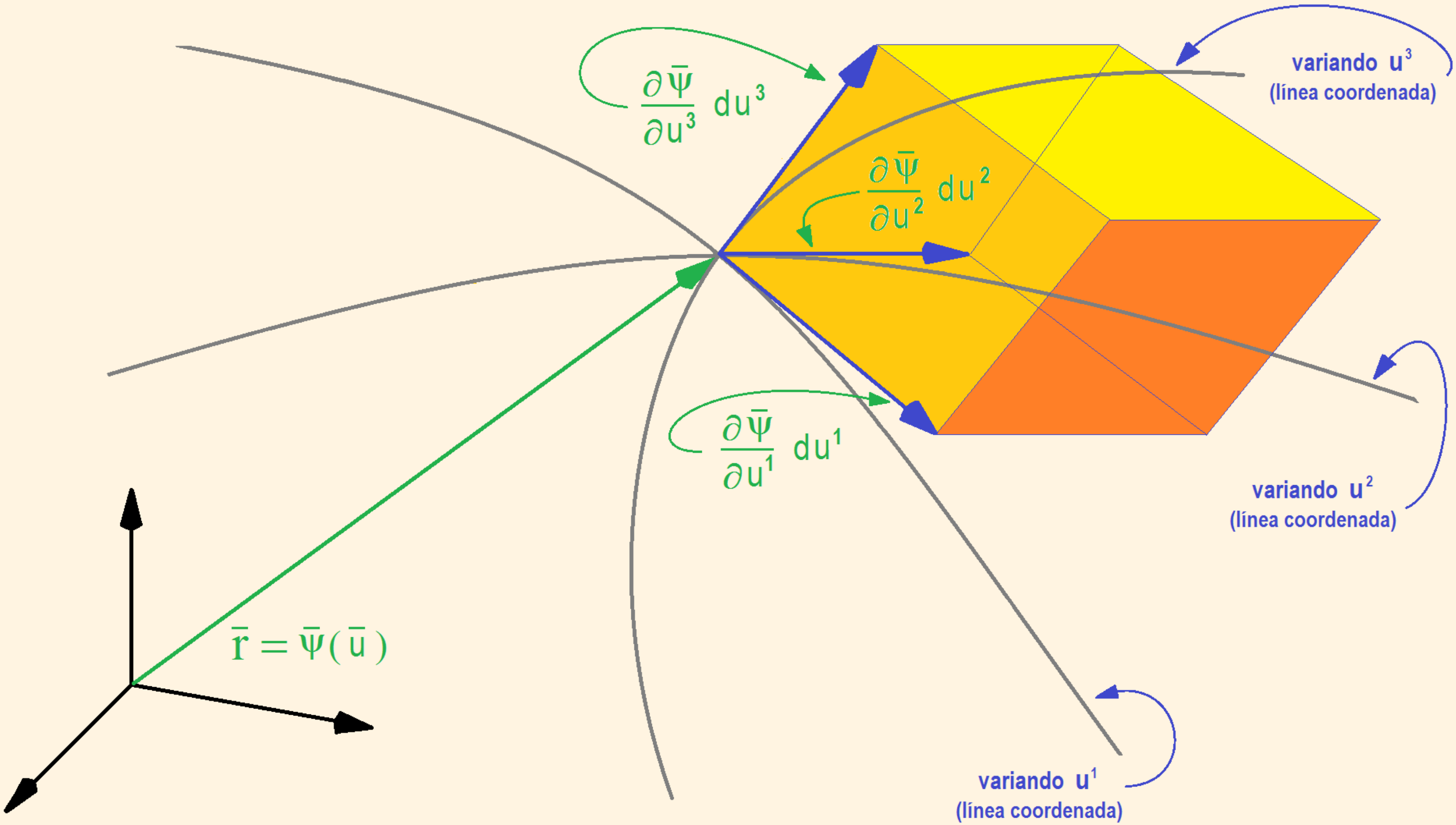
$$\vec{r} = \vec{\psi}(\bar{u}) \quad \text{donde} \quad \left\{ \begin{array}{c} \vec{\psi}(\bar{u}) = \vec{e}_p \;\; \psi^p(\bar{u}) \\ \Updownarrow \\ \vec{\psi}(\bar{u}) \; = \; \underset{\sim}{E} \;\; \bar{\psi}(\bar{u}) \end{array} \right\} \quad \text{con} \quad \bar{u} = \left\{ \begin{array}{c} u^{\,1} \\ \vdots \\ u^{\,\nu} \end{array} \right\}$$

EXPRESIÓN GENERAL EN LA BASE CANÓNICA (ORTONORMAL): (*)

$$\bar{r} = \bar{\psi}(\bar{u}) \quad \text{con} \quad \bar{\psi}(\bar{u}) = \left\{ \begin{array}{c} \psi^{1}(\bar{u}) \\ \vdots \\ \psi^{n}(\bar{u}) \end{array} \right\}, \quad \bar{u} = \left\{ \begin{array}{c} u^{1} \\ \vdots \\ u^{\nu} \end{array} \right\}$$

(*) Por tanto, $\vec{r} = \bar{r}$ y $\vec{\psi}(\bar{u}) = \bar{\psi}(\bar{u})$.

Coordenadas curvilíneas (IV)



VECTORES NATURALES Y DIFERENCIAL DE VOLUMEN

Coordenadas curvilíneas (V)

VECTORES NATURALES:

$$\vec{r} = \vec{\psi}(\bar{u})$$

$$\Downarrow$$

$$\vec{h}_i = \frac{\partial \vec{\psi}}{\partial u^i} \quad \Longleftrightarrow \quad \boxed{\vec{h}_i = \frac{\partial \vec{r}}{\partial u^i}} \quad \equiv$$ **vectores naturales** (*)

(*) Se requiere que sean linealmente independientes $\Longrightarrow$ forman una base.
(cuando $\nu < n$ forman una base de la variedad de dimension ν.)

Coordenadas curvilíneas (VI)

EXPRESIÓN DE UN VECTOR DIFERENCIAL:

$$\boxed{d\vec{r} = \vec{h}_i \, du^i} \quad \text{con} \quad \vec{h}_i = \frac{\partial \vec{r}}{\partial u^i}$$

$$\Updownarrow$$

$$\boxed{d\vec{r} = \underset{\sim}{H} \; d\bar{u}} \quad \text{con} \quad \underset{\sim}{H} = \left[\vec{h}_1 \quad \cdots \quad \vec{h}_\nu \right], \quad d\bar{u} = \left\{ \begin{matrix} du^1 \\ \vdots \\ du^\nu \end{matrix} \right\}$$

Cambio de variables curvilíneas (I)

CAMBIO DE VARIABLE:

$$\boxed{\bar{u} = \bar{\phi}(\bar{v})}$$

$$\Downarrow$$

$$\boxed{\vec{r} = \vec{\psi}(\bar{u})\Big|_{\bar{u}=\bar{\phi}(\bar{v})}}$$

$$\Downarrow$$

$$\vec{h}\,'_\alpha = \frac{\partial \vec{r}}{\partial v^\alpha} = \frac{\partial \vec{r}}{\partial u^i}\;\frac{\partial u^i}{\partial v^\alpha} \quad \Longrightarrow \quad \boxed{\vec{h}\,'_\alpha = \vec{h}_i\;\frac{\partial u^i}{\partial v^\alpha}}$$

Cambio de variables curvilíneas (IIa)

CAMBIO DE BASE DIRECTO:

$$\boxed{\vec{h}\,'_\alpha = \vec{h}_i \, c^i_{\ \alpha}} \quad \text{con} \quad \boxed{c^i_{\ \alpha} = \frac{\partial u^i}{\partial v^\alpha}}$$

$$\Updownarrow$$

$$\boxed{\underset{\sim}{H}\,' = \underset{\sim}{H}\,\underset{\sim}{C}} \quad \text{con} \quad \boxed{\underset{\sim}{C} = \frac{d\bar{u}}{d\bar{v}} = \begin{bmatrix} \frac{\partial u^1}{\partial v^1} & \cdots & \frac{\partial u^1}{\partial v^\nu} \\ \vdots & \ddots & \vdots \\ \frac{\partial u^\nu}{\partial v^1} & \cdots & \frac{\partial u^\nu}{\partial v^\nu} \end{bmatrix}}$$

Cambio de variables curvilíneas (IIb)

CAMBIO DE BASE INVERSO:

$$\boxed{\vec{h}_i = \vec{h}'_\alpha \, \gamma^{\alpha}_{\ i}} \quad \text{con} \quad \boxed{\gamma^{\alpha}_{\ i} = \frac{\partial v^\alpha}{\partial u^i}} \quad \Longleftrightarrow \quad \begin{cases} \gamma^{\alpha}_{\ i} \, c^{i}_{\ \beta} = \delta^{\alpha}_{\ \beta} \\ c^{i}_{\ \alpha} \, \gamma^{\alpha}_{\ j} = \delta^{i}_{\ j} \end{cases} \quad (*)$$

$$\Updownarrow$$

$$\boxed{\underset{\sim}{H} = \underset{\sim}{H}' \, \underset{\sim}{C}^{-1}} \quad \text{con} \quad \boxed{\underset{\sim}{C}^{-1} = \frac{d\bar{v}}{d\bar{u}}} \quad \Longleftrightarrow \quad \begin{cases} \underset{\sim}{C}^{-1} \, \underset{\sim}{C} = \underset{\sim}{I} \\ \underset{\sim}{C} \, \underset{\sim}{C}^{-1} = \underset{\sim}{I} \end{cases}$$

(*) $\gamma^{i}_{\ j} = \frac{1}{c} \, \mathrm{cofactor}(c^{j}_{\ i}), \quad c = \det(\underset{\sim}{C}), \quad \underset{\sim}{C}^{-1} = \left[\gamma^{i}_{\ j}\right].$

CAMBIO DE BASE DE LOS COMP. DE UN VECTOR DIFERENCIAL:

$$\left.\begin{aligned} d\vec{r} &= \vec{h}_i \; du^i \\ \vec{h}'_\alpha &= \vec{h}_i \; \frac{\partial u^i}{\partial v^\alpha} \end{aligned}\right\} \Longrightarrow d\vec{r} = \vec{h}'_\alpha \; dv^\alpha \quad \text{con} \quad \left\{\begin{aligned} du^i &= \frac{\partial u^i}{\partial v^\alpha} \; dv^\alpha \\ dv^\alpha &= \frac{\partial v^\alpha}{\partial u^i} \; du^i \quad (*) \end{aligned}\right.$$

$$\Updownarrow$$

$$\left.\begin{aligned} d\vec{r} &= \underset{\sim}{H} \; d\bar{u} \\ \underset{\sim}{H}' &= \underset{\sim}{H} \; \frac{d\bar{u}}{d\bar{v}} \end{aligned}\right\} \Longrightarrow d\vec{r} = \underset{\sim}{H}' \; d\bar{v} \quad \text{con} \quad \left\{\begin{aligned} d\bar{u} &= \frac{d\bar{u}}{d\bar{v}} \; d\bar{v} \\ d\bar{v} &= \frac{d\bar{v}}{d\bar{u}} \; d\bar{u} \quad (*) \end{aligned}\right.$$

(*) Cambio **CONTRAVARIANTE**

Tensor Métrico (Ia)

TENSOR MÉTRICO:

$$\left.\begin{aligned} d\vec{r}_1 &= \vec{h}_i \;\; du_1^i \\ d\vec{r}_2 &= \vec{h}_j \;\; du_2^j \end{aligned}\right\} \Longrightarrow \boxed{d\vec{r}_1 \cdot d\vec{r}_2 = du_1^i \;\; g_{ij} \;\; du_2^j} \quad \text{con} \quad \boxed{g_{ij} = \vec{h}_i \cdot \vec{h}_j}$$

$$\Updownarrow$$

$$\left.\begin{aligned} d\vec{r}_1 &= \underset{\sim}{H} \;\; d\bar{u}_1 \\ d\vec{r}_2 &= \underset{\sim}{H} \;\; d\bar{u}_2 \end{aligned}\right\} \Longrightarrow \boxed{d\vec{r}_1 \cdot d\vec{r}_2 = d\bar{u}_1^T \;\; \underset{\sim}{G} \;\; d\bar{u}_2} \quad \text{con} \quad \boxed{\underset{\sim}{G} = \underset{\sim}{H}^T \; \underset{\sim}{H}} \quad (*)$$

(*) $\underset{\sim}{G} = \left[g_{ij}\right]$, $\underset{\sim}{G} = \underset{\sim}{G}^T$, $\underset{\sim}{G}$ es Def +. En general, la base $\underset{\sim}{H}$ no será ortonormal $\Longrightarrow$ $\underset{\sim}{G} \neq \underset{\sim}{I}$.

Tensor Métrico (Ib)

INVERSA DEL TENSOR MÉTRICO:

$$\begin{cases} g^{\alpha i}\, g_{i\beta} = \delta^{\alpha}{}_{\beta} \\[2ex] g_{i\alpha}\, g^{\alpha j} = \delta_{i}{}^{j} \end{cases} \qquad (*)$$

$$\Updownarrow$$

$$\begin{cases} \underset{\sim}{G}^{-1}\, \underset{\sim}{G} = \underset{\sim}{I} \\[2ex] \underset{\sim}{G}\, \underset{\sim}{G}^{-1} = \underset{\sim}{I} \end{cases}$$

(*) $g^{ij} = \frac{1}{g}\ \mathrm{cofactor}(g_{ji}), \quad g = \det(\underset{\sim}{G}), \quad \underset{\sim}{G}^{-1} = \left[g^{ij}\right].$

Tensor Métrico (II)

CAMBIO DE BASE DE LOS COMPONENTES DEL TENSOR MÉTRICO:

$$\left.\begin{array}{l} g_{ij} = \vec{h}_i \cdot \vec{h}_j \\ \vec{h}'_{\alpha} = \vec{h}_i \dfrac{\partial u^i}{\partial v^\alpha} \end{array}\right\} \Longrightarrow g'_{\alpha\beta} = \vec{h}'_{\alpha} \cdot \vec{h}'_{\beta} \quad \text{con} \quad \left\{\begin{array}{ll} \boxed{g'_{\alpha\beta} = \dfrac{\partial u^i}{\partial v^\alpha}\, g_{ij}\, \dfrac{\partial u^j}{\partial v^\beta}} & (*) \\ g_{ij} = \dfrac{\partial v^\alpha}{\partial u^i}\, g'_{\alpha\beta}\, \dfrac{\partial v^\beta}{\partial u^j} & \end{array}\right.$$

$$\Updownarrow$$

$$\left.\begin{array}{l} \underset{\sim}{G} = \underset{\sim}{H}^T \underset{\sim}{H} \\ \underset{\sim}{H}' = \underset{\sim}{H} \dfrac{d\bar{u}}{d\bar{v}} \end{array}\right\} \Longrightarrow \underset{\sim}{G}' = \left(\underset{\sim}{H}'\right)^T \underset{\sim}{H}' \quad \text{con} \quad \left\{\begin{array}{ll} \boxed{\underset{\sim}{G}' = \left(\dfrac{d\bar{u}}{d\bar{v}}\right)^T \underset{\sim}{G}\, \dfrac{d\bar{u}}{d\bar{v}}} & (*) \\ \underset{\sim}{G} = \left(\dfrac{d\bar{v}}{d\bar{u}}\right)^T \underset{\sim}{G}'\, \dfrac{d\bar{v}}{d\bar{u}} & \end{array}\right.$$

(*) Cambio (doblemente) **COVARIANTE**

DIFERENCIAL DE (HIPER-)VOLUMEN:

dV $\equiv$ volumen del (hiper-)paralelepípedo formado por los vectores $\vec{h}_i \, du^i$, para $i = 1, \ldots, \nu$

$$\Downarrow$$

$$\boxed{dV = \sqrt{\det(\underset{\sim}{G})} \;\; du^1 \, du^2 \; \cdots \; du^\nu}$$

ARCO DE UNA CURVA:

$$\bar{r} = \bar{\psi}(\bar{u})\Big|_{\bar{u}=\bar{\gamma}(t)} \longrightarrow \boxed{ds^2 = d\bar{r} \cdot d\bar{r}} \Longleftrightarrow ds = \sqrt{d\bar{r} \cdot d\bar{r}}$$

$$\Downarrow$$

$$\boxed{d\bar{r} = \vec{h}_i \; du^i = \underset{\sim}{H} \; d\bar{u}} \quad \text{con} \quad \boxed{d\bar{u} = \bar{\gamma}'(t) \; dt}$$

$$\Downarrow$$

$$\boxed{ds^2 = d\bar{u}^{\,T} \; \underset{\sim}{G} \; d\bar{u}} \quad (*)$$

(*) La matriz $\underset{\sim}{G} = \underset{\sim}{H}^{\,T} \underset{\sim}{H}$ es SIM. y DEF. POS.

Componentes contra y covariantes. Base Dual (I)

OBTENCIÓN DE LAS COMPONENTES DE UN VECTOR DIFERENCIAL:

$$du_p = d\vec{r} \cdot \vec{h}_p \quad \Longrightarrow \quad d\vec{r} = \vec{h}_i \, du^i \quad \text{con} \quad \begin{cases} du_p = \; g_{pi} \, du^i \\ \\ \boxed{du^i = \; g^{\,ip} \, du_p} \end{cases} \qquad (*)$$

$$\Updownarrow$$

$$d\underline{u} = \begin{Bmatrix} du_1 \\ \vdots \\ du_\nu \end{Bmatrix} = \begin{Bmatrix} d\vec{r} \cdot \vec{h}_1 \\ \vdots \\ d\vec{r} \cdot \vec{h}_\nu \end{Bmatrix} \quad \Longrightarrow \quad d\vec{r} = \underset{\sim}{H} \; d\bar{u} \quad \text{con} \quad \begin{cases} d\underline{u} = \underset{\sim}{G} \quad d\bar{u} \\ \\ \boxed{d\bar{u} = \underset{\sim}{G}^{-1} \, d\underline{u}} \end{cases} \qquad (*)$$

(*) Si las curvilíneas son ortonormales, $g_{pi} = \delta_{pi} \;\Rightarrow\; du_p = du^p \quad \Longleftrightarrow \quad \underset{\sim}{G} = \underset{\sim}{I} \;\Rightarrow\; d\underline{u} = d\bar{u}$

BASE DUAL. COMPONENTES CONTRAVARIANTES Y COVARIANTES

$$\left.\begin{array}{l} d\vec{r} = \vec{h}_i \; du^i \\ \vec{h}^{\,p} = \vec{h}_i \; g^{\,ip} \end{array}\right\} \Longrightarrow \quad d\vec{r} = \vec{h}^{\,p} \; du_p \quad \text{con} \quad \left\{\begin{array}{l} du^i = g^{\,ip} \; du_p \\ \boxed{du_p = g_{pi} \; du^i} \end{array}\right. \qquad (*)$$

$$\Updownarrow$$

$$\left.\begin{array}{l} d\vec{r} = \underset{\sim}{H} \; d\bar{u} \\ \widetilde{H} = \underset{\sim}{H} \; \underset{\sim}{G}^{-1} \end{array}\right\} \Longrightarrow \quad d\vec{r} = \widetilde{H} \; d\underline{u} \quad \text{con} \quad \left\{\begin{array}{l} d\bar{u} = \underset{\sim}{G}^{-1} \; d\underline{u} \\ \boxed{d\underline{u} = \underset{\sim}{G} \; d\bar{u}} \end{array}\right. \qquad (*)$$

(*) El vector $d\vec{r}$ se puede expresar en dos bases diferentes, de forma que $d\vec{r} = \vec{h}_i \, du^i = \vec{h}^{\,p} \, du_p$, donde

$$\left\{\begin{array}{llll} \{\vec{h}_i\} \equiv \textbf{BASE PRIMAL} & \rightarrow \{du^i\} \equiv \textbf{COMPONENTES CONTRAVARIANTES} & \text{(cambian de forma inversa a la base)} \\ \{\vec{h}^{\,p}\} \equiv \textbf{BASE DUAL} & \rightarrow \{du_p\} \equiv \textbf{COMPONENTES COVARIANTES} & \text{(cambian de forma directa con la base)} \end{array}\right.$$

INTERPRETACIÓN DE LA BASE DUAL

$$\boxed{\begin{array}{ll} \vec{h}_i \cdot \vec{h}_j = g_{ij}, & \vec{h}_i \cdot \vec{h}^{\,q} = \delta_i^{\;q} \\ \vec{h}^{\,p} \cdot \vec{h}_j = \delta^{\,p}_{\;j}, & \vec{h}^{\,p} \cdot \vec{h}^{\,q} = g^{\,pq} \end{array}}$$

$$\Updownarrow \qquad\qquad \text{(*)}$$

$$\boxed{\begin{array}{ll} \underset{\sim}{H}^{\,T}\, \underset{\sim}{H} = \underset{\sim}{G}, & \underset{\sim}{H}^{\,T}\, \widetilde{H} = \underset{\sim}{I} \\ \widetilde{H}^{\,T}\, \underset{\sim}{H} = \underset{\sim}{I}, & \widetilde{H}^{\,T}\, \widetilde{H} = \underset{\sim}{G}^{-1} \end{array}}$$

(*) Si las curvilíneas son ortonormales, $g_{ij} = \delta_{ij} \;\Rightarrow\; \vec{h}^{\,p} = \vec{h}_p \quad \Longleftrightarrow \quad \underset{\sim}{G} = \underset{\sim}{I} \;\Rightarrow\; \widetilde{H} = \underset{\sim}{H}$

COMPONENTES TENSORIALES Y FÍSICAS DE UN VECTOR

$$\vec{f} = \vec{h}_i \, f^{\,i} = \vec{h}^{\,p} \, f_p \quad \text{donde} \quad \begin{cases} \{\vec{h}_i\}, \ \{\vec{h}^{\,p}\} & \leftarrow \text{bases naturales: primal y dual} \\ \{f^{\,i}\}, \ \{f_p\} & \leftarrow \text{comp. tensoriales: contrav. y cov.} \end{cases}$$

$$\Updownarrow$$

$$\boxed{\vec{f} = \widehat{h}_k \, F^k} \quad \text{donde} \quad \begin{cases} \boxed{\widehat{h}_k = \dfrac{1}{|\vec{h}_k|} \, \vec{h}_k} & (*) \quad \leftarrow \textbf{base normalizada} \\ \boxed{F^{\,k} = |\vec{h}_k| \, f^{\,k}} & (*) \quad \leftarrow \textbf{componente física} \end{cases}$$

(*) Sin sumar en k.

Tensores homogéneos de orden superior (I)

PRODUCTO TENSORIAL:

$$\left.\begin{aligned} \vec{u} &= \vec{h}_i \; f^{\,i} \\ &= \vec{h}^{\,p} \; f_p \end{aligned}\right\} \quad \text{donde} \quad \begin{cases} \left\{\vec{h}_i\right\},\ \left\{\vec{h}^{\,p}\right\} & \leftarrow \text{bases naturales: primal y dual} \\ \left\{f^{\,i}\right\},\ \left\{f_p\right\} & \leftarrow \text{comp. tensoriales: contrav. y cov.} \\ \vec{f} & \leftarrow \textbf{vector} \equiv \textbf{TENSOR DE ORDEN 1} \end{cases}$$

$$\Downarrow$$

$$\left.\begin{aligned} \underset{\approx}{T} &= \left(\vec{h}_i \otimes \vec{h}_j\right) \; t^{\,ij} \\ &= \left(\vec{h}^{\,p} \otimes \vec{h}_j\right) \; t_p^{\;\,j} \\ &= \left(\vec{h}_i \otimes \vec{h}^{\,q}\right) \; t^{\,i}_{\;\,q} \\ &= \left(\vec{h}^{\,p} \otimes \vec{h}^{\,q}\right) \; t_{pq} \end{aligned}\right\} \quad \text{donde} \quad \left.\begin{cases} \left\{\vec{h}_i\right\},\ \left\{\vec{h}^{\,p}\right\} & \leftarrow \text{bases naturales: primal y dual} \\ \left\{t^{\,ij}\right\},\ \left\{t_{pq}\right\} & \leftarrow \text{comp.: contra–contra y cova–cova} \\ \left\{t^{\,i}_{\;\,q}\right\},\ \left\{t_p^{\;\,j}\right\} & \leftarrow \text{contra–cova y cova–contra} \\ \underset{\approx}{T} & \leftarrow \textbf{TENSOR DE ORDEN 2} \end{cases}\right\} (*)$$

(*) El producto tensorial es bilineal:

$$\left(a\,\vec{x} + b\,\vec{y}\right) \otimes \left(\alpha\,\vec{\xi} + \beta\,\vec{\eta}\right) = a\,\alpha\left(\vec{x}\otimes\vec{\xi}\right) + a\,\beta\left(\vec{x}\otimes\vec{\eta}\right) + b\,\alpha\left(\vec{y}\otimes\vec{\xi}\right) + b\,\beta\left(\vec{y}\otimes\vec{\eta}\right)$$

CAMBIO DE BASE DE LOS COMPONENTES DE UN TENSOR:

$$\left.\begin{aligned} \underset{\approx}{T} &= (\vec{h}_i \otimes \vec{h}_j)\ t^{\,ij} \\ \vec{h}'_\alpha &= \vec{h}_i \frac{\partial u^i}{\partial v^\alpha} \\ \vec{h}'_\beta &= \vec{h}_j \frac{\partial u^j}{\partial v^\beta} \end{aligned}\right\} \Longrightarrow \underset{\approx}{T} = (\vec{h}'_\alpha \otimes \vec{h}'_\beta)\ t^{\,\prime\alpha\beta} \quad \text{con} \quad \left\{\begin{aligned} t^{\,ij} &= \frac{\partial u^i}{\partial v^\alpha}\frac{\partial u^j}{\partial v^\beta}\ t^{\,\prime\alpha\beta} \\ \boxed{t^{\,\prime\alpha\beta} = \frac{\partial v^\alpha}{\partial u^i}\frac{\partial v^\beta}{\partial u^j}\ t^{\,ij}} & \quad (*) \end{aligned}\right.$$

$$\left.\begin{aligned} \underset{\approx}{T} &= (\vec{h}^{\,p} \otimes \vec{h}^{\,q})\ t_{pq} \\ \vec{h}^{\,\prime\lambda} &= \vec{h}^{\,p} \frac{\partial v^\lambda}{\partial u^p} \\ \vec{h}^{\,\prime\mu} &= \vec{h}^{\,q} \frac{\partial v^\mu}{\partial u^q} \end{aligned}\right\} \Longrightarrow \underset{\approx}{T} = (\vec{h}^{\,\prime\lambda} \otimes \vec{h}^{\,\prime\mu})\ t'_{\lambda\mu} \quad \text{con} \quad \left\{\begin{aligned} t_{pq} &= \frac{\partial v^\lambda}{\partial u^p}\frac{\partial v^\mu}{\partial u^q}\ t'_{\lambda\mu} \\ \boxed{t'_{\lambda\mu} = \frac{\partial u^p}{\partial v^\lambda}\frac{\partial u^q}{\partial v^\mu}\ t_{pq}} & \quad (**) \end{aligned}\right.$$

(*) Cambio **DOBLEMENTE CONTRAVARIANTE**
(**) Cambio **DOBLEMENTE COVARIANTE**

Tensores homogéneos de orden superior (IIb)

CAMBIO DE BASE DE LOS COMPONENTES DE UN TENSOR:

$$\left.\begin{aligned} \underset{\approx}{T} &= (\vec{h}_i \otimes \vec{h}^{\,q})\ t^{\,i}_{\ q} \\ \vec{h}'_\alpha &= \vec{h}_i \frac{\partial u^i}{\partial v^\alpha} \\ \vec{h}'^{\mu} &= \vec{h}^{\,q} \frac{\partial v^\mu}{\partial u^q} \end{aligned}\right\} \Longrightarrow \underset{\approx}{T} = (\vec{h}'_\alpha \otimes \vec{h}'^{\mu})\ t'^{\alpha}_{\ \mu} \quad \text{con} \quad \left\{\begin{aligned} t^{\,i}_{\ q} &= \frac{\partial u^i}{\partial v^\alpha}\frac{\partial v^\mu}{\partial u^q}\ t'^{\alpha}_{\ \mu} \\ \boxed{t'^{\alpha}_{\ \mu} = \frac{\partial v^\alpha}{\partial u^i}\frac{\partial u^q}{\partial v^\mu}\ t^{\,i}_{\ q}} & \quad (*) \end{aligned}\right.$$

$$\left.\begin{aligned} \underset{\approx}{T} &= (\vec{h}^{\,p} \otimes \vec{h}_j)\ t_p^{\ j} \\ \vec{h}'^{\lambda} &= \vec{h}^{\,p} \frac{\partial v^\lambda}{\partial u^p} \\ \vec{h}'_\beta &= \vec{h}_j \frac{\partial u^j}{\partial v^\beta} \end{aligned}\right\} \Longrightarrow \underset{\approx}{T} = (\vec{h}'^{\lambda} \otimes \vec{h}'_\beta)\ t'_{\lambda}{}^{\beta} \quad \text{con} \quad \left\{\begin{aligned} t_p^{\ j} &= \frac{\partial v^\lambda}{\partial u^p}\frac{\partial u^j}{\partial v^\beta}\ t'_{\lambda}{}^{\beta} \\ \boxed{t'_{\lambda}{}^{\beta} = \frac{\partial u^p}{\partial v^\lambda}\frac{\partial v^\beta}{\partial u^j}\ t_p^{\ j}} & \quad (**) \end{aligned}\right.$$

(*) Cambio **CONTRA–COVA**
(**) Cambio **COVA-CONTRA**

Tensores homogéneos de orden superior (III)

OPERACIONES DE SUBIDA Y BAJADA DE ÍNDICES:

$$
\left.
\begin{array}{l}
\underset{\approx}{T} = (\vec{h}_i \otimes \vec{h}_j)\; t^{\,ij} \\[2ex]
\vec{h}^{\,p} = \vec{h}_i \; g^{\,ip} \\[1ex]
\vec{h}^{\,q} = \vec{h}_j \; g^{\,jq}
\end{array}
\right\}
\Longrightarrow
\begin{cases}
\underset{\approx}{T} = (\vec{h}^{\,p} \otimes \vec{h}_j)\; t_p{}^j & \text{con } \begin{cases} t^{\,ij} = g^{\,ip}\, t_p{}^j \\[2ex] \boxed{t_p{}^j = g_{pi}\, t^{\,ij}} \end{cases} \\[6ex]
\underset{\approx}{T} = (\vec{h}_i \otimes \vec{h}^{\,q})\; t^i{}_q & \text{con } \begin{cases} t^{\,ij} = g^{\,jq}\, t^i{}_q \\[2ex] \boxed{t^i{}_q = g_{qj}\, t^{\,ij}} \end{cases} \\[6ex]
\underset{\approx}{T} = (\vec{h}^{\,p} \otimes \vec{h}^{\,q})\; t_{pq} & \text{con } \begin{cases} t^{\,ij} = g^{\,ip}\, g^{\,jq}\, t_{pq} \\[2ex] \boxed{t_{pq} = g_{pi}\, g_{qj}\, t^{\,ij}} \end{cases}
\end{cases}
\quad (*)
$$

(*) El mismo tensor $\underset{\approx}{T}$ se puede expresar en 2^n bases tensoriales diferentes, donde n es el orden del tensor.

EXPRESIÓN GENERAL:

$$\frac{\partial \vec{h}_i}{\partial u^j} = \vec{h}_k \;\, \Gamma^k{}_{ij} \qquad (*)$$

$$\Downarrow$$

$$\Gamma^k{}_{ij} = \frac{1}{2}\, g^{k\ell} \,\left(\frac{\partial g_{i\ell}}{\partial u^j} + \frac{\partial g_{j\ell}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^\ell} \right) \qquad (**)$$

(*) Son las componentes de las derivadas de los vectores naturales (respecto a las curvilíneas) expresadas en la propia base de vectores naturales.
Quedan totalmente definidos por el tensor métrico y sus derivadas.

(**) **Los símbolos de Christoffel NO SON TENSORES** (a pesar de que la notación parece indicarlo).

Símbolos de Christoffel (II)

PROPIEDADES ÚTILES:

1) $$\Gamma^{k}{}_{ij} = \Gamma^{k}{}_{ji}$$

2) $$\frac{\partial g_{ij}}{\partial u^{\ell}} = g_{jm}\ \Gamma^{m}{}_{i\ell} + g_{im}\ \Gamma^{m}{}_{j\ell}$$

3) $$\Gamma^{i}{}_{i\ell} = \frac{1}{2g}\ \frac{\partial g}{\partial u^{\ell}}$$ donde $g = \det(\underset{\sim}{G})$

4) $$\frac{\partial \vec{h}^{\,p}}{\partial u^{j}} = -\vec{h}^{\,r}\ \Gamma^{p}{}_{rj}$$

DIFERENCIAL Y DERIVADA DIRECCIONAL DE UN CAMPO ESCALAR:

Diferencial:

$$f(\bar{u}) \quad \longrightarrow \quad \boxed{df = \frac{\partial f}{\partial u^j}\, du^j}$$

Derivada Direccional:

$$f(\bar{u})\Big|_{\bar{u}=\bar{u}_\mathrm{o}+\lambda\bar{s}} = \Phi(\lambda) \quad \longrightarrow \quad \Phi'(0) = \lim_{\lambda\to 0} \frac{f(\bar{u}_\mathrm{o}+\lambda\bar{s}) - f(\bar{u}_\mathrm{o})}{\lambda} = \frac{\partial f}{\partial u^j}(\bar{u}_\mathrm{o})\; s^j$$

$$\Downarrow$$

$$\boxed{D_s f\,(\bar{u}_\mathrm{o}) = \frac{\partial f}{\partial u^j}\,(\bar{u}_\mathrm{o})\; s^j}$$

Derivación tensorial (Ib)

DERIVADA TENSORIAL (O COVARIANTE) DE UN CAMPO ESCALAR:

$$f(\bar{u}) \longrightarrow \boxed{f_{,j} = \frac{\partial f}{\partial u^j}}$$

Cambio de coordenadas curvilíneas:

$$\left.\begin{array}{l} f(\bar{u}) \\ \bar{u} = \bar{\phi}(\bar{v}) \end{array}\right\} \longrightarrow \varphi(\bar{v}) = f(\bar{u})\Big|_{\bar{u}=\bar{\phi}(\bar{v})}$$

$$\Downarrow$$

$$\varphi_{,\beta} = \frac{\partial \varphi}{\partial v^\beta} \quad \text{donde} \quad \varphi_{,\beta} = f_{,j}\left(\frac{\partial u^j}{\partial v^\beta}\right) \quad (*)$$

(*) Cambio tensorial **COVARIANTE** $\Longrightarrow$ $f_{,j}$ son las comp. covariantes de un vector (**GRADIENTE**).

DIFERENCIAL Y DERIVADA DIRECCIONAL DE UN CAMPO VECTORIAL:

Diferencial:

$$\vec{f}(\bar{u}) = \vec{h}_i \, f^{\,i}(\bar{u}) = \vec{h}^{\,p} \, f_p(\bar{u}) \quad \longrightarrow \quad \boxed{d\vec{f} = \frac{\partial \vec{f}}{\partial u^j} \, du^j}$$

Derivada Direccional:

$$\vec{f}(\bar{u})\Big|_{\bar{u}=\bar{u}_{\mathrm{o}}+\lambda\bar{s}} = \vec{\Phi}(\lambda) \quad \longrightarrow \quad \vec{\Phi}'(0) = \lim_{\lambda\to 0} \frac{\vec{f}(\bar{u}_{\mathrm{o}}+\lambda\bar{s}) - \vec{f}(\bar{u}_{\mathrm{o}})}{\lambda} = \frac{\partial \vec{f}}{\partial u^j}(\bar{u}_{\mathrm{o}}) \; s^j$$

$$\Downarrow$$

$$\boxed{D_s\vec{f}\,(\bar{u}_{\mathrm{o}}) = \frac{\partial \vec{f}}{\partial u^j}\,(\bar{u}_{\mathrm{o}}) \; s^j}$$

Derivación tensorial (IIb)

DERIVADA TENSORIAL (O COVARIANTE) DE UN CAMPO VECTORIAL:

campo en componentes contravariantes $\longrightarrow \quad \vec{f}(\bar{u}) = \vec{h}_i \, f^{\,i}(\bar{u})$

$$\Downarrow$$

$$\frac{\partial \vec{f}}{\partial u^j} = \frac{\partial \vec{h}_i}{\partial u^j} \qquad f^{\,i} + \vec{h}_i \, \frac{\partial f^{\,i}}{\partial u^j} = \left(\vec{h}_k \, \Gamma^k{}_{ij}\right) f^{\,i} + \vec{h}_i \, \frac{\partial f^{\,i}}{\partial u^j}$$

$$= \left(\vec{h}_i \, \Gamma^i{}_{kj}\right) f^{\,k} + \vec{h}_i \, \frac{\partial f^{\,i}}{\partial u^j} = \vec{h}_i \left(\frac{\partial f^{\,i}}{\partial u^j} \qquad + \Gamma^i{}_{kj} \, f^{\,k}\right)$$

$$\Downarrow$$

$$\frac{\partial \vec{f}}{\partial u^j} = \vec{h}_i \, f^{\,i}{}_{,\,j}$$

con

$$\boxed{f^{\,i}{}_{,\,j} = \frac{\partial f^{\,i}}{\partial u^j} + \Gamma^i{}_{kj} \, f^{\,k}}$$

DERIVADA TENSORIAL (O COVARIANTE) DE UN CAMPO VECTORIAL:

campo en componentes covariantes $\longrightarrow$ $\vec{f}(\bar{u}) = \vec{h}^{\,p}\, f_p(\bar{u})$

$$\Downarrow$$

$$\frac{\partial \vec{f}}{\partial u^j} = \frac{\partial \vec{h}^{\,p}}{\partial u^j} f_p + \vec{h}^{\,p}\, \frac{\partial f_p}{\partial u^j} = \left(-\vec{h}^{\,r}\, \Gamma^p{}_{rj}\right) f_p + \vec{h}_p\, \frac{\partial f_p}{\partial u^j}$$

$$= \left(-\vec{h}^{\,p}\, \Gamma^r{}_{pj}\right) f_r + \vec{h}^{\,p}\, \frac{\partial f_p}{\partial u^j} = \vec{h}^{\,p} \left(\frac{\partial f_p}{\partial u^j} - \Gamma^r{}_{pj}\, f_r\right)$$

$$\Downarrow$$

$$\frac{\partial \vec{f}}{\partial u^j} = \vec{h}^{\,p}\, f_{p,\,j}$$ con $$\boxed{f_{p,\,j} = \frac{\partial f_p}{\partial u^j} - \Gamma^r{}_{pj}\, f_r}$$

VARIEDAD DE RIEMANN:

$$\Rightarrow \left\{ \begin{array}{lll} \boxed{\bar{u}} & \equiv & \textbf{coordenadas cuvilíneas}, \quad \bar{u} = \left\{ \begin{array}{c} u^1 \\ \vdots \\ u^\nu \end{array} \right\} \\ \\ \boxed{\underset{\sim}{G}(\bar{u})} & \equiv & \textbf{tensor métrico}, \quad \underset{\sim}{G}(\bar{u}) \ \text{es SIM. y DEF+} \end{array} \right\} \ (*)$$

(*) Luego se tienen únicamente en cuenta las propiedades métricas de la variedad.
$\Longrightarrow$ ¡El espacio en el que se halla inmersa la variedad es irrelevante!.

TENSOR DE RIEMANN:

$$R^{i}{}_{jk\ell} = \begin{vmatrix} \dfrac{\partial}{\partial u^k} & \dfrac{\partial}{\partial u^\ell} \\ \Gamma^i{}_{jk} & \Gamma^i{}_{j\ell} \end{vmatrix} + \begin{vmatrix} \Gamma^i{}_{\alpha k} & \Gamma^i{}_{\alpha \ell} \\ \Gamma^\alpha{}_{jk} & \Gamma^\alpha{}_{j\ell} \end{vmatrix} \qquad (*)$$

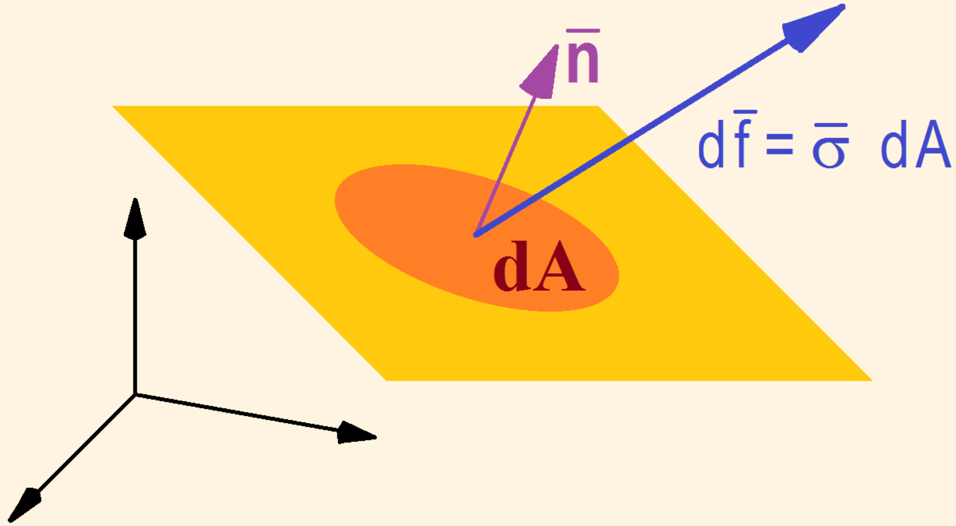
TENSOR DE RICCI:

$$R_{ij} = R^{\alpha}{}_{i\alpha j} \quad \longrightarrow \quad R_{ij} = \rho\, g_{ij} \qquad (**)$$

(*) Contiene toda la información sobre la curvatura de la variedad.
Queda totalmente definido por el tensor métrico y sus derivadas.

(**) Ejemplo de aplicación en Teoría de la Relatividad: ecuación de la gravitación de Einstein (en puntos donde hay materia).

TENSOR DE TENSIONES $\left(\underset{\sim}{\sigma}\right)$:

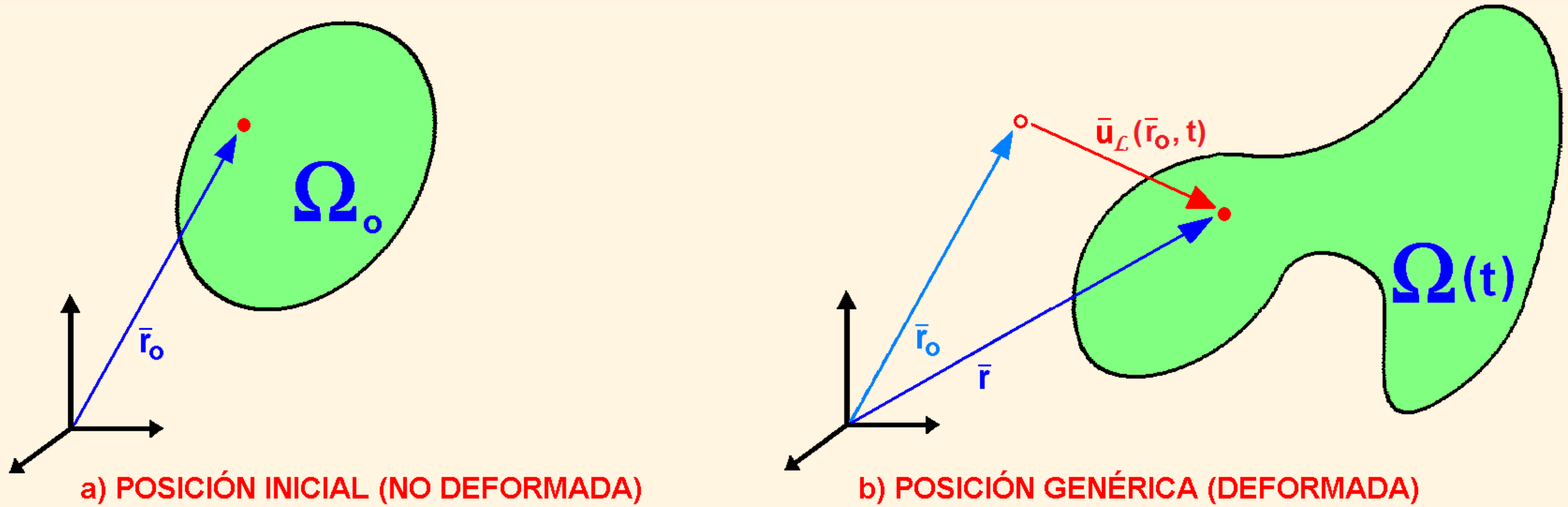


$$\left.\begin{aligned} \vec{n} &= \vec{h}_i\, n^i \\ S(\vec{h}_i) &= \vec{h}_\ell\, \sigma^{\ell}{}_{i} \end{aligned}\right\} \Longrightarrow S(\vec{n}) = \vec{\sigma} = \vec{h}_\ell\, \sigma^\ell \quad \text{con} \quad \boxed{\sigma^\ell = \sigma^{\ell}{}_{i}\, n^i}$$

$$\Updownarrow$$

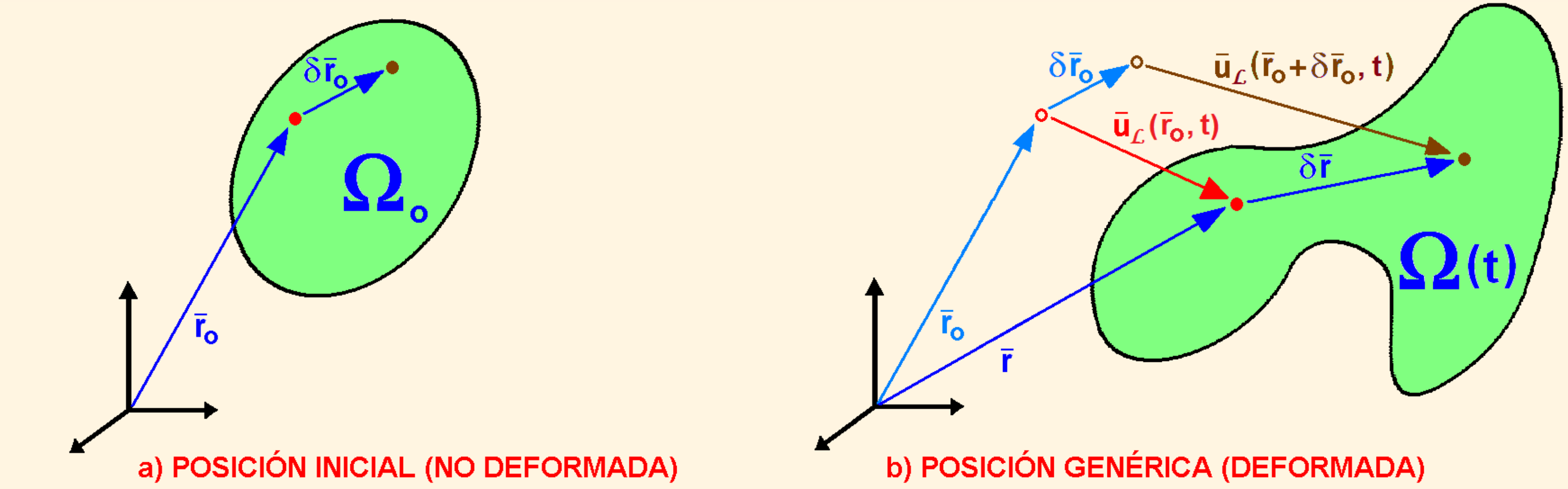
$$\left.\begin{aligned} \vec{n} &= \underset{\sim}{H}\, \bar{n} \\ S(\underset{\sim}{H}) &= \underset{\sim}{H}\, \underset{\sim}{\sigma} \end{aligned}\right\} \Longrightarrow S(\vec{n}) = \vec{\sigma} = \underset{\sim}{H}\, \bar{\sigma} \quad \text{con} \quad \boxed{\bar{\sigma} = \underset{\sim}{\sigma}\, \bar{n}}$$

TENSOR GRADIENTE DE MOVIMIENTOS $\left(\underset{\sim}{\mathcal{F}}_{\mathcal{L}}\right)$:



$$\begin{cases} \bar{r} = \bar{r}_{\mathcal{L}}(\bar{r}_0, t), \qquad \underbrace{\bar{r}_{\mathcal{L}}(\bar{r}_0, t)}_{\text{MOVIMIENTO}} = \bar{r}_0 + \underbrace{\bar{u}_{\mathcal{L}}(\bar{r}_0, t)}_{\text{DESPLAZAMIENTO}}, \\ \Omega(t) = \bar{r}_{\mathcal{L}}(\Omega_0, t). \longleftarrow \text{DOMINIO MATERIAL} \end{cases}$$

Movimientos relativos del medio en el entorno de un punto:



$$\delta\bar{r} = \bar{r}_{\mathcal{L}}(\bar{r}_0 + \delta\bar{r}_0, t) - \bar{r}_{\mathcal{L}}(\bar{r}_0, t) = \left[\frac{\partial\bar{r}_{\mathcal{L}}}{\partial\bar{r}_0}\right] \delta\bar{r}_0 + \mathcal{O}(\|\delta\bar{r}_0\|^2),$$

Luego,

$$\boxed{\delta\bar{r} = \underset{\sim}{\mathcal{F}}_{\mathcal{L}}\, \delta\bar{r}_0 + \mathcal{O}(\|\delta\bar{r}_0\|^2), \quad \text{con } \underset{\sim}{\mathcal{F}}_{\mathcal{L}} = \left[\frac{\partial\bar{r}_{\mathcal{L}}}{\partial\bar{r}_0}\right] = \underset{\sim}{I} + \underset{\sim}{\mathcal{J}}_{\mathcal{L}}, \quad \underset{\sim}{\mathcal{J}}_{\mathcal{L}} = \left[\frac{\partial\bar{u}_{\mathcal{L}}}{\partial\bar{r}_0}\right].}$$