ESPACIOS EUCLÍDEOS: NOTACIÓN Y FORMULARIO

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Expresión de un vector en una base (Ia)

EXPRESIÓN GENERAL:

$$\boxed{\vec{u} = \vec{e}_i \, u^i}$$

$$\Updownarrow$$

$$\boxed{\vec{u} = \underset{\sim}{E} \, \bar{u}} \quad \text{con} \quad \underset{\sim}{E} = \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{bmatrix}, \qquad \bar{u} = \begin{Bmatrix} u^1 \\ \vdots \\ u^n \end{Bmatrix}$$

Expresión de un vector en una base (Ib)

EXPRESIÓN EN BASE REFERIDA A LA BASE CANÓNICA:

$$\boxed{\vec{u} = \vec{e}_i \, u^i} \quad \text{con} \quad \vec{e}_i = \bar{e}_i = \begin{Bmatrix} e^1{}_i \\ \vdots \\ e^n{}_i \end{Bmatrix}$$

$$\Updownarrow$$

$$\boxed{\vec{u} = \underset{\sim}{E} \, \bar{u}} \quad \text{con} \quad \underset{\sim}{E} = \begin{bmatrix} e^1{}_1 & \cdots & e^1{}_n \\ \vdots & \ddots & \vdots \\ e^n{}_1 & \cdots & e^n{}_n \end{bmatrix}, \quad \bar{u} = \begin{Bmatrix} u^1 \\ \vdots \\ u^n \end{Bmatrix}$$

Cambio de base (Ia)

CAMBIO DIRECTO:

$$\vec{e}'_{\alpha} = \vec{e}_i \, c^i{}_{\alpha}$$

$$\Updownarrow$$

$$\underset{\sim}{E}' = \underset{\sim}{E} \, \underset{\sim}{C} \quad \text{con} \quad \underset{\sim}{C} = \begin{bmatrix} c^1{}_1 & \cdots & c^1{}_n \\ \vdots & \ddots & \vdots \\ c^n{}_1 & \cdots & c^n{}_n \end{bmatrix}$$

CAMBIO INVERSO:

$$\boxed{\vec{e}_i = \vec{e}\,'_\alpha \, \gamma^{\alpha}{}_{i}} \quad \text{con} \quad \begin{cases} \gamma^{\alpha}{}_{i} \, c^{i}{}_{\beta} = \delta^{\alpha}{}_{\beta} \\ c^{i}{}_{\alpha} \, \gamma^{\alpha}{}_{j} = \delta^{i}{}_{j} \end{cases} \quad (*)$$

$$\Updownarrow$$

$$\boxed{\underset{\sim}{E} = \underset{\sim}{E}\,' \, \underset{\sim}{C}^{-1}} \quad \text{con} \quad \begin{cases} \underset{\sim}{C}^{-1} \, \underset{\sim}{C} = \underset{\sim}{I} \\ \underset{\sim}{C} \, \underset{\sim}{C}^{-1} = \underset{\sim}{I} \end{cases}$$

(*) $\underset{\sim}{C} = \left[c^{i}{}_{j} \right], \quad c = \det(\underset{\sim}{C}) \quad \Longrightarrow \quad \underset{\sim}{C}^{-1} = \left[\gamma^{i}{}_{j} \right]$ donde $\gamma^{i}{}_{j} = \frac{1}{c} \, \mathrm{cofactor}(c^{j}{}_{i})$.

CAMBIO DE BASE DE LOS COMPONENTES DE UN VECTOR:

$$\left.\begin{aligned} \vec{u} &= \vec{e}_i\, u^i \\ \vec{e}\,'_\alpha &= \vec{e}_i\, c^i{}_\alpha \end{aligned}\right\} \Longrightarrow \vec{u} = \vec{e}\,'_\alpha\, u'^\alpha \quad \text{con} \quad \left\{\begin{aligned} u^i &= c^i{}_\alpha\, u'^\alpha \\ u'^\alpha &= \gamma^\alpha{}_i\, u^i \quad (*) \end{aligned}\right.$$

$$\Updownarrow$$

$$\left.\begin{aligned} \vec{u} &= \underset{\sim}{E}\, \bar{u} \\ \underset{\sim}{E}' &= \underset{\sim}{E}\, \underset{\sim}{C} \end{aligned}\right\} \Longrightarrow \vec{u} = \underset{\sim}{E}'\, \bar{u}' \quad \text{con} \quad \left\{\begin{aligned} \bar{u} &= \underset{\sim}{C}\, \bar{u}' \\ \bar{u}' &= \underset{\sim}{C}^{-1}\, \bar{u} \quad (*) \end{aligned}\right.$$

(*) Cambio **CONTRAVARIANTE**

Producto Escalar: Tensor Métrico (Ia)

TENSOR MÉTRICO:

$$\left.\begin{array}{l}\vec{u} = \vec{e}_i\, u^i \\[2em] \vec{v} = \vec{e}_j\, v^j\end{array}\right\} \Longrightarrow \boxed{\vec{u}\cdot\vec{v} = u^i\, g_{ij}\, v^j} \quad \text{con} \quad \boxed{g_{ij} = \vec{e}_i \cdot \vec{e}_j}$$

$$\Updownarrow$$

$$\left.\begin{array}{l}\vec{u} = \underset{\sim}{E}\, \bar{u} \\[2em] \vec{v} = \underset{\sim}{E}\, \bar{v}\end{array}\right\} \Longrightarrow \boxed{\vec{u}\cdot\vec{v} = \bar{u}^{\,T}\, \underset{\sim}{G}\, \bar{v}} \quad \text{con} \quad \boxed{\underset{\sim}{G} = \underset{\sim}{E}^{\,T}\, \underset{\sim}{E}} \qquad (*)$$

(*) $\underset{\sim}{G} = \left[g_{ij}\right]$, $\underset{\sim}{G} = \underset{\sim}{G}^{\,T} \iff \underset{\sim}{G}$ SIM, $\bar{u}^{\,T}\, \underset{\sim}{G}\, \bar{u} > 0 \quad \forall \bar{u} \neq \bar{0} \iff \underset{\sim}{G}$ DEF+.
Si la base es ortonormal: $\underset{\sim}{G} = \underset{\sim}{I}$.

Producto Escalar: Tensor Métrico (Ib)

INVERSA DEL TENSOR MÉTRICO:

$$\begin{cases} g^{\alpha i}\, g_{i\beta} = \delta^{\alpha}{}_{\beta} \\ g_{i\alpha}\, g^{\alpha j} = \delta_i{}^{j} \end{cases} \quad (*)$$

$$\Updownarrow$$

$$\begin{cases} \underset{\sim}{G}^{-1}\, \underset{\sim}{G} = \underset{\sim}{I} \\ \underset{\sim}{G}\, \underset{\sim}{G}^{-1} = \underset{\sim}{I} \end{cases}$$

(*) $\underset{\sim}{G} = \left[g_{ij} \right]$, $g = \det(\underset{\sim}{G}) \implies \underset{\sim}{G}^{-1} = \left[g^{ij} \right]$ donde $g^{ij} = \frac{1}{g}\ \mathrm{cofactor}(g_{ji})$.

Producto Escalar: Tensor Métrico (II)

CAMBIO DE BASE DE LOS COMPONENTES DEL TENSOR MÉTRICO:

$$\left.\begin{array}{l} g_{ij} = \vec{e}_i \cdot \vec{e}_j \\ \vec{e}\,'_\alpha = \vec{e}_i \, c^i{}_\alpha \end{array}\right\} \Longrightarrow \quad g'_{\alpha\beta} = \vec{e}\,'_\alpha \cdot \vec{e}\,'_\beta \quad \text{con} \quad \left\{\begin{array}{ll} \boxed{g'_{\alpha\beta} = c^i{}_\alpha \, g_{ij} \, c^j{}_\beta} & (*) \\ g_{ij} = \gamma^\alpha{}_i \, g'_{\alpha\beta} \, \gamma^\beta{}_j & \end{array}\right.$$

$$\Updownarrow$$

$$\left.\begin{array}{l} \underset{\sim}{G} = \underset{\sim}{E}^T \underset{\sim}{E} \\ \underset{\sim}{E}' = \underset{\sim}{E} \, \underset{\sim}{C} \end{array}\right\} \Longrightarrow \quad \underset{\sim}{G}' = (\underset{\sim}{E}')^T \underset{\sim}{E}' \quad \text{con} \quad \left\{\begin{array}{ll} \boxed{\underset{\sim}{G}' = \underset{\sim}{C}^T \underset{\sim}{G} \, \underset{\sim}{C}} & (*) \\ \underset{\sim}{G} = \underset{\sim}{C}^{-T} \underset{\sim}{G}' \, \underset{\sim}{C}^{-1} & \end{array}\right.$$

(*) Cambio (doblemente) **COVARIANTE**

HIPER-VOLUMEN DEL HIPER-PARALELEPÍPEDO $\mathcal{P}_\nu$: (*)

$$\left.\begin{array}{ll} \{\vec{h}_k\}_{k=1,\nu} & \textbf{L.I.} \\ \vec{h}_k \in \mathbb{R}^n, & \nu \leq n \end{array}\right\} \longrightarrow \mathcal{P}_\nu \equiv \left\{ \vec{r} = \sum_{k=1}^{\nu} \vec{h}_k \, r^k, \quad 0 \leq r^k \leq 1 \right\}$$

$$\Downarrow$$

$$\boxed{\text{Vol}(\mathcal{P}_\nu) = \sqrt{\det(\underset{\sim}{G}_\nu)}, \quad \text{con} \ \underset{\sim}{G}_\nu = \begin{bmatrix} g_{11} & \cdots & g_{1\nu} \\ \vdots & \ddots & \vdots \\ g_{\nu 1} & \cdots & g_{\nu\nu} \end{bmatrix}, \quad g_{ij} = \vec{h}_i \cdot \vec{h}_j}$$

(*) También llamado **PARALELOTOPO**

OBTENCIÓN DE LAS COMPONENTES DE UN VECTOR:

$$u_p = \vec{u}\cdot\vec{e}_p \quad\Longrightarrow\quad \vec{u} = \vec{e}_i\, u^i \quad \text{con} \quad \begin{cases} u_p = \; g_{pi}\, u^i \\ \boxed{u^i = \; g^{\,ip}\, u_p} \end{cases} \qquad (*)$$

$$\Updownarrow$$

$$\underline{u} = \begin{Bmatrix} u_1 \\ \vdots \\ u_n \end{Bmatrix} = \begin{Bmatrix} \bar{u}\cdot\vec{e}_1 \\ \vdots \\ \bar{u}\cdot\vec{e}_n \end{Bmatrix} \quad\Longrightarrow\quad \vec{u} = \underset{\sim}{E}\,\bar{u} \quad \text{con} \quad \begin{cases} \underline{u} = \; \underset{\sim}{G} \quad \bar{u} \\ \boxed{\bar{u} = \; \underset{\sim}{G}^{-1}\,\underline{u}} \end{cases} \qquad (*)$$

(*) Si la base es ortonormal: $\left[g_{pi} = \delta_{pi} \quad\Longrightarrow\quad u_p = u^p\right] \quad\Longleftrightarrow\quad \left[\underset{\sim}{G} = \underset{\sim}{I} \quad\Longrightarrow\quad \underline{u} = \bar{u}\right]$,

(luego las componentes y las proyecciones coinciden).

BASES PRIMAL Y DUAL. COMPONENTES CONTRA Y COVARIANTES:

$$\left.\begin{aligned} \vec{u} &= \vec{e}_i\, u^i \\ \vec{e}^{\,p} &= \vec{e}_i\, g^{\,ip} \end{aligned}\right\} \Longrightarrow \vec{u} = \vec{e}^{\,p}\, u_p \quad \text{con} \quad \left\{\begin{aligned} u^i &= g^{\,ip}\, u_p \\ u_p &= g_{pi}\, u^i \end{aligned}\right. \qquad (*)$$

$$\Updownarrow$$

$$\left.\begin{aligned} \vec{u} &= \underset{\sim}{E}\, \bar{u} \\ \widetilde{E} &= \underset{\sim}{E}\, \underset{\sim}{G}^{-1} \end{aligned}\right\} \Longrightarrow \vec{u} = \widetilde{E}\, \underline{u} \quad \text{con} \quad \left\{\begin{aligned} \bar{u} &= \underset{\sim}{G}^{-1}\, \underline{u} \\ \underline{u} &= \underset{\sim}{G}\, \bar{u} \end{aligned}\right. \qquad (*)$$

(*) El mismo vector $\vec{u}$ se puede expresar en dos bases diferentes, de forma que $\vec{u} = \vec{e}_i\, u^i = \vec{e}^{\,p}\, u_p$, donde

$$\left\{\begin{array}{ll} \{\vec{e}_i\},\ \underset{\sim}{E} \equiv \textbf{BASE PRIMAL} & \longrightarrow \{u^i\},\ \bar{u} \equiv \textbf{COMPONENTES CONTRAVARIANTES} \\ \{\vec{e}^{\,p}\},\ \widetilde{E} \equiv \textbf{BASE DUAL} \quad \textbf{(CONTRAVARIANTE)} & \longrightarrow \{u_p\},\ \underline{u} \equiv \textbf{COMPONENTES COVARIANTES} \end{array}\right.$$

Componentes contra y covariantes. Base Dual (III)

CAMBIO DE BASE DUAL Y COMPONENTES COVARIANTES:

$$\left.\begin{aligned} \vec{u} &= \vec{e}^{\,p}\, u_p \\ \vec{e}\,'_{\alpha} &= \vec{e}_i\, c^{i}{}_{\alpha} \end{aligned}\right\} \Longrightarrow \vec{u} = \vec{e}\,'^{\pi}\, u'_{\pi} \quad \text{con} \quad \left\{\begin{aligned} &\boxed{\vec{e}\,'^{\pi} = \vec{e}^{\,p}\, \gamma^{\pi}{}_{p}} && (*) \\ &\boxed{u'_{\pi} = c^{p}{}_{\pi}\, u_p} && (**) \end{aligned}\right.$$

$$\Updownarrow$$

$$\left.\begin{aligned} \vec{u} &= \widetilde{E}\, \underset{\sim}{u} \\ \underset{\sim}{E}\,' &= \underset{\sim}{E}\, \underset{\sim}{C} \end{aligned}\right\} \Longrightarrow \vec{u} = \widetilde{E}\,'\, \underset{\sim}{u}\,' \quad \text{con} \quad \left\{\begin{aligned} &\boxed{\widetilde{E}\,' = \widetilde{E}\, \underset{\sim}{C}^{-T}} && (*) \\ &\boxed{\underset{\sim}{u}\,' = \underset{\sim}{C}^{T}\, \underset{\sim}{u}} && (**) \end{aligned}\right.$$

(*) Cambio **CONTRAVARIANTE**
(**) Cambio **COVARIANTE**

Componentes contra y covariantes. Base Dual (VI)

INTERPRETACIÓN DE LA BASE DUAL:

$$\boxed{\begin{array}{ll} \vec{e}_i\cdot \vec{e}_j = g_{ij}, & \vec{e}_i \cdot \vec{e}^{\,q} = \delta_i^{\;q} \\ \vec{e}^{\,p}\cdot \vec{e}_j = \delta^{\,p}_{\;j}, & \vec{e}^{\,p}\cdot \vec{e}^{\,q} = g^{\,pq} \end{array}}$$

$$\Updownarrow \qquad \text{(*)}$$

$$\boxed{\begin{array}{ll} \underset{\sim}{E}^{\,T}\,\underset{\sim}{E} = \underset{\sim}{G}, & \underset{\sim}{E}^{\,T}\,\widetilde{E} = \underset{\sim}{I} \\ \widetilde{E}^{\,T}\,\underset{\sim}{E} = \underset{\sim}{I}, & \widetilde{E}^{\,T}\,\widetilde{E} = \underset{\sim}{G}^{-1} \end{array}}$$

(*) Si la base es ortonormal, $g_{ij} = \delta_{ij} \;\Rightarrow\; \vec{e}^{\,p} = \vec{e}_p \;\Longleftrightarrow\; \underset{\sim}{G} = \underset{\sim}{I} \;\Rightarrow\; \widetilde{E} = \underset{\sim}{E}$,

(luego la base dual y la base primal coinciden).

PRODUCTO TENSORIAL:

$$\left.\begin{aligned}\vec{u} &= \vec{e}_i \; u^{\,i} \\ &= \vec{e}^{\,p} \; u_p\end{aligned}\right\} \text{ donde } \left\{\begin{array}{lll} \{\vec{e}_i\},\ \{\vec{e}^{\,p}\} & \leftarrow & \text{bases primal y dual} \\ \{u^{\,i}\},\ \{u_p\} & \leftarrow & \text{comp. contravariantes y covariantes} \\ \vec{u} & \leftarrow & \textbf{vector} \equiv \textbf{TENSOR DE ORDEN 1} \end{array}\right.$$

$$\Downarrow$$

$$\left.\begin{aligned}\underset{\approx}{T} &= (\vec{e}_i \otimes \vec{e}_j) \;\; t^{\,ij} \\ &= (\vec{e}^{\,p} \otimes \vec{e}_j) \;\; t_p{}^{\,j} \\ &= (\vec{e}_i \otimes \vec{e}^{\,q}) \;\; t^{\,i}{}_q \\ &= (\vec{e}^{\,p} \otimes \vec{e}^{\,q}) \;\; t_{pq}\end{aligned}\right\} \text{ donde } \left\{\begin{array}{lll} \{\vec{e}_i\},\ \{\vec{e}^{\,p}\} & \leftarrow & \text{bases primal y dual} \\ \{t^{\,ij}\},\ \{t_{pq}\} & \leftarrow & \text{comp.: contra–contra y cova–cova} \\ \{t^{\,i}{}_q\},\ \{t_p{}^{\,j}\} & \leftarrow & \text{contra–cova y cova–contra} \\ \underset{\approx}{T} & \leftarrow & \textbf{TENSOR DE ORDEN 2} \end{array}\right\} \ (*)$$

(*) El producto tensorial es bilineal:

$$\left(a\,\vec{x} + b\,\vec{y}\right) \otimes \left(\alpha\,\vec{\xi} + \beta\,\vec{\eta}\right) = a\,\alpha\left(\vec{x}\otimes\vec{\xi}\right) + a\,\beta\left(\vec{x}\otimes\vec{\eta}\right) + b\,\alpha\left(\vec{y}\otimes\vec{\xi}\right) + b\,\beta\left(\vec{y}\otimes\vec{\eta}\right)$$

Tensores homogéneos de orden superior (IIa)

CAMBIO DE BASE DE LOS COMPONENTES DE UN TENSOR:

$$\left.\begin{aligned} \underset{\approx}{T} &= (\vec{e}_i \otimes \vec{e}_j)\ t^{\,ij} \\ \vec{e}\,'_\alpha &= \vec{e}_i\ c^{\,i}_{\ \alpha} \\ \vec{e}\,'_\beta &= \vec{e}_j\ c^{\,j}_{\ \beta} \end{aligned}\right\} \Longrightarrow \underset{\approx}{T} = (\vec{e}\,'_\alpha \otimes \vec{e}\,'_\beta)\ t^{\,\prime\alpha\beta} \text{ con } \left\{\begin{aligned} t^{\,ij} &= c^{\,i}_{\ \alpha}\, c^{\,j}_{\ \beta}\ t^{\,\prime\alpha\beta} \\ t^{\,\prime\alpha\beta} &= \gamma^{\,\alpha}_{\ i}\, \gamma^{\,\beta}_{\ j}\ t^{\,ij} \quad (*) \end{aligned}\right.$$

$$\left.\begin{aligned} \underset{\approx}{T} &= (\vec{e}^{\,p} \otimes \vec{e}^{\,q})\ t_{pq} \\ \vec{e}^{\,\prime\lambda} &= \vec{e}^{\,p}\ \gamma^{\,\lambda}_{\ p} \\ \vec{e}^{\,\prime\mu} &= \vec{e}^{\,q}\ \gamma^{\,\mu}_{\ q} \end{aligned}\right\} \Longrightarrow \underset{\approx}{T} = (\vec{e}^{\,\prime\lambda} \otimes \vec{e}^{\,\prime\mu})\ t'_{\lambda\mu} \text{ con } \left\{\begin{aligned} t_{pq} &= \gamma^{\,\lambda}_{\ p}\, \gamma^{\,\mu}_{\ q}\ t'_{\lambda\mu} \\ t'_{\lambda\mu} &= c^{\,p}_{\ \lambda}\, c^{\,q}_{\ \mu}\ t_{pq} \quad (**) \end{aligned}\right.$$

(*) Cambio **DOBLEMENTE CONTRAVARIANTE**
(**) Cambio **DOBLEMENTE COVARIANTE**

Tensores homogéneos de orden superior (IIb)

CAMBIO DE BASE DE LOS COMPONENTES DE UN TENSOR:

$$\left.\begin{aligned} \underset{\approx}{T} &= (\vec{e}_i \otimes \vec{e}^{\,q})\ t^{i}{}_{q} \\ \vec{e}\,'_{\alpha} &= \vec{e}_i\, c^{i}{}_{\alpha} \\ \vec{e}\,'^{\mu} &= \vec{e}^{\,q}\, \gamma^{\mu}{}_{q} \end{aligned}\right\} \Longrightarrow \underset{\approx}{T} = (\vec{e}\,'_{\alpha} \otimes \vec{e}\,'^{\mu})\ t'^{\alpha}{}_{\mu} \ \text{con} \ \left\{\begin{aligned} t^{i}{}_{q} &= c^{i}{}_{\alpha}\, \gamma^{\mu}{}_{q}\ t'^{\alpha}{}_{\mu} \\ t'^{\alpha}{}_{\mu} &= \gamma^{\alpha}{}_{i}\, c^{q}{}_{\mu}\ t^{i}{}_{q} \quad (*) \end{aligned}\right.$$

$$\left.\begin{aligned} \underset{\approx}{T} &= (\vec{e}^{\,p} \otimes \vec{e}_j)\ t_{p}{}^{j} \\ \vec{e}\,'^{\lambda} &= \vec{e}^{\,p}\, \gamma^{\lambda}{}_{p} \\ \vec{e}\,'_{\beta} &= \vec{e}_j\, c^{j}{}_{\beta} \end{aligned}\right\} \Longrightarrow \underset{\approx}{T} = (\vec{e}\,'^{\lambda} \otimes \vec{e}\,'_{\beta})\ t'_{\lambda}{}^{\beta} \ \text{con} \ \left\{\begin{aligned} t_{p}{}^{j} &= \gamma^{\lambda}{}_{p}\, c^{j}{}_{\beta}\ t'_{\lambda}{}^{\beta} \\ t'_{\lambda}{}^{\beta} &= c^{p}{}_{\lambda}\, \gamma^{\beta}{}_{j}\ t_{p}{}^{j} \quad (**) \end{aligned}\right.$$

(*) Cambio **CONTRA–COVA**
(**) Cambio **COVA-CONTRA**

OPERACIONES DE SUBIDA Y BAJADA DE ÍNDICES:

$$\left.\begin{array}{l} \underset{\approx}{T} = (\vec{e}_i \otimes \vec{e}_j)\ t^{\,ij} \\ \vec{e}^{\,p} = \vec{e}_i\ g^{\,ip} \\ \vec{e}^{\,q} = \vec{e}_j\ g^{\,jq} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \underset{\approx}{T} = (\vec{e}^{\,p} \otimes \vec{e}_j)\ t_p{}^j \quad \text{con} \left\{\begin{array}{l} t^{\,ij} = g^{\,ip}\, t_p{}^j \\ \boxed{t_p{}^j = g_{pi}\, t^{\,ij}} \end{array}\right. \\ \underset{\approx}{T} = (\vec{e}_i \otimes \vec{e}^{\,q})\ t^i{}_q \quad \text{con} \left\{\begin{array}{l} t^{\,ij} = g^{\,jq}\, t^i{}_q \\ \boxed{t^i{}_q = g_{qj}\, t^{\,ij}} \end{array}\right. \\ \underset{\approx}{T} = (\vec{e}^{\,p} \otimes \vec{e}^{\,q})\ t_{pq} \quad \text{con} \left\{\begin{array}{l} t^{\,ij} = g^{\,ip}\, g^{\,jq}\, t_{pq} \\ \boxed{t_{pq} = g_{pi}\, g_{qj}\, t^{\,ij}} \end{array}\right. \end{array}\right. \quad (*)$$

(*) El mismo tensor $\underset{\approx}{T}$ se puede expresar en 2^n bases tensoriales diferentes, donde n es el orden del tensor.

Producto Vectorial en $\mathbb{R}^3$ (I)

EXPRESIÓN DEL PRODUCTO MIXTO (O TRIPLE) EN UNA BASE:

$$\left.\begin{array}{l} \vec{u} = \vec{e}_i\, u^i \\ \vec{v} = \vec{e}_j\, v^j \\ \vec{w} = \vec{e}_k\, w^k \end{array}\right\} \Longrightarrow [\vec{u},\vec{v},\vec{w}] = \left\{\begin{array}{ll} = (\vec{u}\wedge\vec{v})\cdot\vec{w} & \text{(*)} \\ = [\vec{e}_1,\vec{e}_2,\vec{e}_3]\ \det\left[\bar{u} \,\vdots\, \bar{v} \,\vdots\, \bar{w}\right] & \end{array}\right.$$

$$\left.\begin{array}{r} [\vec{e}_1,\vec{e}_2,\vec{e}_3] = \pm\sqrt{g} \\ \text{con}\ \ g = \det(\underset{\sim}{G}) \end{array}\right\} \Longrightarrow \boxed{[\vec{u},\vec{v},\vec{w}] = \pm\sqrt{g}\ \det\left[\bar{u} \,\vdots\, \bar{v} \,\vdots\, \bar{w}\right]} \quad \text{(**)}$$

(*) $\Longrightarrow \left\{\begin{array}{rcrcr} [\vec{u},\vec{v},\vec{w}] &=& [\vec{v},\vec{w},\vec{u}] &=& [\vec{w},\vec{u},\vec{v}] \\ -[\vec{v},\vec{u},\vec{w}] &=& -[\vec{w},\vec{v},\vec{u}] &=& -[\vec{u},\vec{w},\vec{v}] \end{array}\right.$

(**) El signo depende de la **quiralidad** del triedro $\vec{e}_1, \vec{e}_2, \vec{e}_3$. También: $\det\left[\bar{u} \,\vdots\, \bar{v} \,\vdots\, \bar{w}\right] = \det\left\{\begin{array}{c} \bar{u}^{\,T} \\ \cdots \\ \bar{v}^{\,T} \\ \cdots \\ \bar{w}^{\,T} \end{array}\right\}$.

EXPRESIÓN DEL PRODUCTO VECTORIAL EN UNA BASE:

$$\left.\begin{array}{l} \vec{u} = \vec{e}_i\, u^i \\ \vec{v} = \vec{e}_j\, v^j \end{array}\right\} \Longrightarrow \boxed{\vec{u} \wedge \vec{v} = \pm \sqrt{g}\ \det \begin{bmatrix} \vec{e}^{\,1} & \vec{e}^{\,2} & \vec{e}^{\,3} \\ u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \end{bmatrix}} \quad (*)$$

$$\left.\begin{array}{l} \vec{u} = \vec{e}^{\,i}\, u_i \\ \vec{v} = \vec{e}^{\,j}\, v_j \end{array}\right\} \Longrightarrow \boxed{\vec{u} \wedge \vec{v} = \pm \frac{1}{\sqrt{g}}\ \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}} \quad (**)$$

(*) Datos en **CONTRAVARIANTES** $\Longrightarrow$ resultado en **COVARIANTES** (**BASE DUAL**).

(**) Datos en **COVARIANTES** $\Longrightarrow$ resultado en **CONTRAVARIANTES** (**BASE PRIMAL**).

Producto Vectorial en $\mathbb{R}^3$ (III)

FÓRMULAS DE INTERÉS:

1) Módulo del producto vectorial:

$$|\vec{u} \wedge \vec{v}| = \begin{vmatrix} \vec{u} \cdot \vec{u} & \vec{u} \cdot \vec{v} \\ \vec{v} \cdot \vec{u} & \vec{v} \cdot \vec{v} \end{vmatrix}^{1/2}$$

2) Fórmula de expulsión:

$$\vec{a} \wedge (\vec{b} \wedge \vec{c}) = \begin{vmatrix} \vec{b} & \vec{c} \\ \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{c} \end{vmatrix}$$

3) Producto escalar de productos vectoriales:

$$(\vec{a} \wedge \vec{b}) \cdot (\vec{c} \wedge \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

Transformaciones lineales (I)

EXPRESIÓN GENERAL:

$$\left.\begin{aligned} \vec{u} &= \vec{e}_i\, u^i \\ T(\vec{e}_i) &= \vec{\varphi}_\ell\, t^\ell{}_i \end{aligned}\right\} \Longrightarrow T(\vec{u}) = \vec{\xi} = \vec{\varphi}_\ell\, \xi^\ell \quad \text{con} \quad \boxed{\xi^\ell = t^\ell{}_i\, u^i}$$

$$\Updownarrow$$

$$\left.\begin{aligned} \vec{u} &= \underset{\sim}{E}\, \bar{u} \\ T(\underset{\sim}{E}) &= \underset{\sim}{\Phi}\, \underset{\sim}{T} \end{aligned}\right\} \Longrightarrow T(\vec{u}) = \vec{\xi} = \underset{\sim}{\Phi}\, \bar{\xi} \quad \text{con} \quad \boxed{\bar{\xi} = \underset{\sim}{T}\, \bar{u}}$$

Transformaciones lineales (II)

CAMBIO DE BASE DEL TENSOR DE UNA TRANSFORMACIÓN:

$$\left.\begin{aligned} \xi^{\ell} &= t^{\ell}{}_{i}\, u^{i} \\ u'^{\alpha} &= \gamma^{\alpha}{}_{i}\, u^{i} \\ \xi'^{\lambda} &= \mu^{\lambda}{}_{\ell}\, \xi^{\ell} \end{aligned}\right\} \Longrightarrow \quad \xi'^{\lambda} = t'^{\lambda}{}_{\alpha}\, u'^{\alpha} \quad \text{con} \quad \left\{\begin{aligned} t^{\ell}{}_{i} &= m^{\ell}{}_{\lambda}\, t'^{\lambda}{}_{\alpha}\, \gamma^{\alpha}{}_{i} \\ t'^{\lambda}{}_{\alpha} &= \mu^{\lambda}{}_{\ell}\, t^{\ell}{}_{i}\, c^{i}{}_{\alpha} \quad (*) \end{aligned}\right.$$

$$\Updownarrow$$

$$\left.\begin{aligned} \bar{\xi} &= \underset{\sim}{T}\, \bar{u} \\ \bar{u}' &= \underset{\sim}{C}^{-1}\, \bar{u} \\ \bar{\xi}' &= \underset{\sim}{M}^{-1}\, \bar{\xi} \end{aligned}\right\} \Longrightarrow \quad \bar{\xi}' = \underset{\sim}{T}'\, \bar{u}' \quad \text{con} \quad \left\{\begin{aligned} \underset{\sim}{T} &= \underset{\sim}{M}\, \underset{\sim}{T}'\, \underset{\sim}{C}^{-1} \\ \underset{\sim}{T}' &= \underset{\sim}{M}^{-1}\, \underset{\sim}{T}\, \underset{\sim}{C} \quad (*) \end{aligned}\right.$$

(*) Cambio **CONTRA-COVARIANTE** (carácter tensorial),

donde: $\mu^{\lambda}{}_{\ell}\, m^{\ell}{}_{\pi} = \delta^{\lambda}{}_{\pi}$; $\quad m^{\ell}{}_{\lambda}\, \mu^{\lambda}{}_{p} = \delta^{\ell}{}_{p}$.

ENDOMORFISMO:

$$\left.\begin{aligned} \vec{u} &= \vec{e}_i\, u^i \\ T(\vec{e}_i) &= \vec{e}_\ell\, t^{\ell}{}_{i} \end{aligned}\right\} \Longrightarrow \quad T(\vec{u}) = \vec{\xi} = \vec{e}_\ell\, \xi^\ell \quad \text{con} \quad \boxed{\xi^\ell = t^{\ell}{}_{i}\, u^i}$$

$$\Updownarrow$$

$$\left.\begin{aligned} \vec{u} &= \underset{\sim}{E}\, \bar{u} \\ T(\underset{\sim}{E}) &= \underset{\sim}{E}\, \underset{\sim}{T} \end{aligned}\right\} \Longrightarrow \quad T(\vec{u}) = \vec{\xi} = \underset{\sim}{E}\, \bar{\xi} \quad \text{con} \quad \boxed{\bar{\xi} = \underset{\sim}{T}\, \bar{u}}$$

CAMBIO DE BASE DEL TENSOR DE UN ENDOMORFISMO:

$$\left.\begin{aligned} \xi^{\ell} &= t^{\ell}{}_{i}\, u^{i} \\ u^{\prime\alpha} &= \gamma^{\alpha}{}_{i}\, u^{i} \\ \xi^{\prime\lambda} &= \gamma^{\lambda}{}_{\ell}\, \xi^{\ell} \end{aligned}\right\} \Longrightarrow \quad \xi^{\prime\lambda} = t^{\prime\lambda}{}_{\alpha}\, u^{\prime\alpha} \quad \text{con} \quad \left\{\begin{aligned} t^{\ell}{}_{i} &= c^{\ell}{}_{\lambda}\, t^{\prime\lambda}{}_{\alpha}\, \gamma^{\alpha}{}_{i} \\ t^{\prime\lambda}{}_{\alpha} &= \gamma^{\lambda}{}_{\ell}\, t^{\ell}{}_{i}\, c^{i}{}_{\alpha} \quad (*) \end{aligned}\right.$$

$$\Updownarrow$$

$$\left.\begin{aligned} \bar{\xi} &= \underset{\sim}{T}\, \bar{u} \\ \bar{u}^{\prime} &= \underset{\sim}{C}^{-1}\, \bar{u} \\ \bar{\xi}^{\prime} &= \underset{\sim}{C}^{-1}\, \bar{\xi} \end{aligned}\right\} \Longrightarrow \quad \bar{\xi}^{\prime} = \underset{\sim}{T}^{\prime}\, \bar{u}^{\prime} \quad \text{con} \quad \left\{\begin{aligned} \underset{\sim}{T} &= \underset{\sim}{C}\, \underset{\sim}{T}^{\prime}\, \underset{\sim}{C}^{-1} \\ \underset{\sim}{T}^{\prime} &= \underset{\sim}{C}^{-1}\, \underset{\sim}{T}\, \underset{\sim}{C} \quad (*) \end{aligned}\right.$$

(*) Cambio **CONTRA-COVARIANTE** (carácter tensorial)

PRODUCTO DE ENDOMORFISMOS:

$$\left.\begin{aligned} \xi^{\ell} &= t_{\mathsf{A}}{}^{\ell}{}_{i}\ u^{i} \\ \chi^{p} &= t_{\mathsf{B}}{}^{p}{}_{\ell}\ \xi^{\ell} \end{aligned}\right\} \Longrightarrow \quad \chi^{p} = t^{p}{}_{i}\ u^{i} \quad \text{con} \quad \boxed{t^{p}{}_{i} = t_{\mathsf{B}}{}^{p}{}_{\ell}\ t_{\mathsf{A}}{}^{\ell}{}_{i}}$$

$$\Updownarrow$$

$$\left.\begin{aligned} \bar{\xi} &= \underset{\sim}{T}_{\mathsf{A}}\ \bar{u} \\ \bar{\chi} &= \underset{\sim}{T}_{\mathsf{B}}\ \bar{\xi} \end{aligned}\right\} \Longrightarrow \quad \bar{\chi} = \underset{\sim}{T}\ \bar{u} \quad \text{con} \quad \boxed{\underset{\sim}{T} = \underset{\sim}{T}_{\mathsf{B}}\ \underset{\sim}{T}_{\mathsf{A}}}$$

Transformaciones geométricas (Ia)

EXPRESIÓN GENERAL:

$$\delta\bar{r} = \underset{\sim}{\mathcal{F}}\, \delta\bar{r}_{\mathrm{o}}, \quad \text{con} \quad \underset{\sim}{\mathcal{F}} = \underset{\sim}{I} + \underset{\sim}{\mathcal{J}} \qquad (*)$$

(*) Se exige que $\mathcal{F} = \det\left(\underset{\sim}{\mathcal{F}}\right) > 0$ para que la transformación tenga sentido físico.

Transformaciones geométricas (Ib)

SIGNIFICADO FÍSICO:

$$\text{Sean} \begin{cases} \delta\vec{r}_\mathrm{o} & \equiv \text{vector material } \textbf{antes} \text{ de la transformación} \\ \delta\vec{r} = \mathcal{F}(\delta\vec{r}_\mathrm{o}) & \equiv \text{vector material } \textbf{después} \text{ de la transformación} \end{cases} \quad (*)$$

$$\left.\begin{aligned} \delta\vec{r}_\mathrm{o} &= \underset{\sim}{E}\ \delta\bar{r}_\mathrm{o} \\ \mathcal{F}(\underset{\sim}{E}) &= \underset{\sim}{E}\ \underset{\sim}{\mathcal{F}} \end{aligned}\right\} \Longrightarrow \quad \delta\vec{r} = \underset{\sim}{E}\ \delta\bar{r} \quad \text{con} \quad \boxed{\delta\bar{r} = \underset{\sim}{\mathcal{F}}\ \delta\bar{r}_\mathrm{o}}$$

(*) Vector material = vector que une dos puntos del espacio.

TRANSFORMACIÓN DE DISTANCIAS, ÁNGULOS Y VOLÚMENES:

Sean $\begin{cases} \Omega_o & \equiv \text{dominio material } \textbf{antes} \text{ de la transformación} \\ \Omega = \mathcal{F}(\Omega_o) & \equiv \text{dominio material } \textbf{después} \text{ de la transformación} \end{cases}$ (*)

$$\left.\begin{array}{l} \delta\bar{r} = \underset{\sim}{\mathcal{F}}\ \delta\bar{r}_o \\ \delta\bar{s} = \underset{\sim}{\mathcal{F}}\ \delta\bar{s}_o \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \delta\vec{r}_o \cdot \delta\vec{s}_o = \delta\bar{r}_o^{\,T}\ \underset{\sim}{G}\ \delta\bar{s}_o \\ \delta\vec{r} \cdot \delta\vec{s} = \delta\bar{r}_o^{\,T}\ \underset{\sim}{\mathcal{C}}\ \delta\bar{s}_o \end{array}\right\} \quad \text{con} \quad \boxed{\underset{\sim}{\mathcal{C}} = \underset{\sim}{\mathcal{F}}^{T}\ \underset{\sim}{G}\ \underset{\sim}{\mathcal{F}}} \quad (**)$$

$$\Downarrow$$

$$\text{Vol}(d\,\Omega) = \eta\ \ \text{Vol}(d\,\Omega_o) \quad \text{con} \quad \boxed{\eta = \frac{\sqrt{\det(\underset{\sim}{\mathcal{C}})}}{\sqrt{\det(\underset{\sim}{G})}} = \det\left(\underset{\sim}{\mathcal{F}}\right) = \mathcal{F} > 0} \quad (*)$$

(*) Dominio material = parte del espacio ocupada por un cuerpo; $\eta \equiv$ factor de dilatación volumétrica.

(**) $\underset{\sim}{\mathcal{C}}$ = **TENSOR (DERECHO) de CAUCHY-GREEN** (juega un papel similar al del tensor métrico).

CASOS RELEVANTES:

$$\begin{cases} 1) \quad \textbf{DEFORMACIÓN} \\ \\ 2) \quad \textbf{ROTACIÓN} \end{cases}$$

Nota: **EN LO SUCESIVO SUPONDREMOS QUE LA BASE ES ORTONORMAL** $\Longrightarrow \quad \underset{\sim}{G} = \underset{\sim}{I}.$

1) DEFORMACIÓN:

$$\boxed{\underset{\sim}{\mathcal{F}} = \underset{\sim}{\mathcal{S}} = \underset{\sim}{I} + \underset{\sim}{\mathcal{E}}, \text{ con } \underset{\sim}{\mathcal{S}}^T = \underset{\sim}{\mathcal{S}}, \ \underset{\sim}{\mathcal{S}} \text{ Def. } +} \Longrightarrow \left\{ \begin{array}{l} \underset{\sim}{\mathcal{S}} \quad \underset{\sim}{U} = \underset{\sim}{U}\, \underset{\sim}{\Lambda} \\ \underset{\sim}{U}^T \underset{\sim}{U} = \underset{\sim}{I} \end{array} \right. \qquad (*)$$

$$\Downarrow$$

$$\left. \begin{array}{l} \delta\bar{r} = \underset{\sim}{\mathcal{F}}\, \delta\bar{r}_{\mathrm{o}} \\ \underset{\sim}{E}' = \underset{\sim}{E}\, \underset{\sim}{U} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} \delta\bar{r}\,' = \mathcal{F}'\, \delta\bar{r}_{\mathrm{o}}' \\ \mathcal{F}' = \underset{\sim}{\Lambda} \end{array} \right\} \Longrightarrow \boxed{\delta\bar{r}\,' = \underset{\sim}{\Lambda}\, \delta\bar{r}_{\mathrm{o}}'} \qquad (**)$$

(*) $\underset{\sim}{\mathcal{E}} \equiv$ **TENSOR DE DEFORMACIÓN DE BIOT**

$$\underset{\sim}{U} = \left[\bar{u}_1 \vdots \bar{u}_2 \vdots \cdots \vdots \bar{u}_n\right], \quad \underset{\sim}{\Lambda} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \quad \boxed{\underset{\sim}{\mathcal{S}}\, \bar{u}_i = \lambda_i\, \bar{u}_i, \text{ con } \lambda_i > 0 \ \forall i}$$

(**) Luego $\boxed{\delta r'^{\alpha} - \delta r_{\mathrm{o}}'^{\alpha} = \varepsilon_\alpha\ \delta r_{\mathrm{o}}'^{\alpha}, \text{ con } \varepsilon_\alpha = \lambda_\alpha - 1 > -1 \ \forall \alpha}$ $\Longrightarrow$ **EJES PRINCIPALES**.

¡OJO!: Conviene elegir la orientación de los $\{\bar{u}_i\}_{i=1,n}$ para que $\underset{\sim}{U}$ represente una rotación propia.

Casos particulares:

1.1) **Inflación** (DEFORMACIÓN ISÓTROPA)

$$\underset{\sim}{\mathcal{F}} = \underset{\sim}{\mathcal{S}}, \quad \text{con } \underset{\sim}{\mathcal{S}} = \eta^{1/n}\, \underset{\sim}{I}, \quad \eta > 0 \qquad (*)$$

1.2) **Distorsión** (DEFORMACIÓN ISOCÓRICA)

$$\underset{\sim}{\mathcal{F}} = \underset{\sim}{\mathcal{S}}, \quad \text{con } \underset{\sim}{\mathcal{S}}^{T} = \underset{\sim}{\mathcal{S}}, \quad \underset{\sim}{\mathcal{S}} \text{ Def. +}, \quad \det(\underset{\sim}{\mathcal{S}}) = 1 \qquad (**)$$

(*) $\eta \equiv$ factor de dilatación volumétrica.

(**) Por tanto, el volumen se mantiene constante.

2) **ROTACIÓN:**

$$\boxed{\underset{\sim}{\mathcal{F}} = \underset{\sim}{\mathcal{R}}, \quad \text{con } \underset{\sim}{\mathcal{R}}^T = \underset{\sim}{\mathcal{R}}^{-1}} \Longrightarrow \begin{cases} \underset{\sim}{\mathcal{R}}^T \underset{\sim}{\mathcal{R}} = \underset{\sim}{I} \\ \underset{\sim}{\mathcal{R}} \ \underset{\sim}{\mathcal{R}}^T = \underset{\sim}{I} \end{cases} \quad (*)$$

$$\Downarrow$$

$$\boxed{\underset{\sim}{\mathcal{C}} = \underset{\sim}{\mathcal{F}}^T \underset{\sim}{I} \ \underset{\sim}{\mathcal{F}} = \underset{\sim}{I}} \quad (**)$$

(*) $\eta = |\underset{\sim}{\mathcal{F}}| = |\det(\underset{\sim}{\mathcal{R}})| = 1 \quad \Longrightarrow$ se conserva el volumen.

(**) Se conservan las distancias y los ángulos (y —en consecuencia— también el volumen).

En $\mathbb{R}^3 \rightarrow \begin{cases} \det(\underset{\sim}{\mathcal{R}}) = 1 \implies \textbf{rotación}. \\ \det(\underset{\sim}{\mathcal{R}}) = -1 \implies \textbf{reflexión}, \text{ o } \textbf{rotación+reflexión} \text{ (rotación impropia), } \underline{\text{sin sentido físico}}. \end{cases}$

DESCOMPOSICIÓN POLAR:

$$\forall \underset{\sim}{\mathcal{F}} \,/\, \mathcal{F} \neq 0 \;\rightarrow\; \boxed{\underset{\sim}{\mathcal{C}} = \left(\underset{\sim}{\mathcal{F}}^{T}\, \underset{\sim}{I}\, \underset{\sim}{\mathcal{F}}\right) \text{ SIM y DEF+}} \;\Longrightarrow\; \begin{cases} \underset{\sim}{\mathcal{C}}\;\; \underset{\sim}{U} = \underset{\sim}{U}\, \underset{\sim}{\Lambda}^{2} \\ \underset{\sim}{U}^{T}\, \underset{\sim}{U} = \underset{\sim}{I} \end{cases} \quad (*)$$

$$\Downarrow$$

$$\boxed{\underset{\sim}{\mathcal{F}} = \overbrace{\underset{\sim}{\mathcal{R}}}^{\text{rot.}}\, \overbrace{\left(\underset{\sim}{I} + \underset{\sim}{\mathcal{E}}\right)}^{\text{def.}}} \quad \text{con} \quad \left\{ \begin{array}{l} \underset{\sim}{\mathcal{E}} = \underset{\sim}{U}\, \left(\underset{\sim}{\Lambda} - \underset{\sim}{I}\right)\, \underset{\sim}{U}^{T} \\ \underset{\sim}{\mathcal{R}} = \underset{\sim}{\mathcal{F}}\, \underset{\sim}{U}\, \underset{\sim}{\Lambda}^{-1}\, \underset{\sim}{U}^{T} \end{array} \right\} \quad (**)$$

(*) Con $\underset{\sim}{U} = \left[\bar{u}_1 \,\vdots\, \bar{u}_2 \,\vdots\, \cdots \,\vdots\, \bar{u}_n\right]$, $\underset{\sim}{\Lambda}^2 = \begin{bmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{bmatrix}$, $\boxed{\underset{\sim}{\mathcal{C}}\, \bar{u}_i = \left(\lambda_i\right)^2 \bar{u}_i \;\; \forall i}$

(**) **¡OJO!:** Conviene elegir la orientación de los $\{\bar{u}_i\}_{i=1,n}$ para que $\underset{\sim}{U}$ represente una rotación propia, pero no es estrictamente necesario porque la descomposición se obtiene correctamente en cualquier caso.

Muy importante:

♡ Cualquier transformación geométrica con sentido físico $\left(\Longleftrightarrow \det(\underset{\sim}{\mathcal{F}}) > 0\right)$ se puede descomponer en las formas siguientes:

$$\underset{\sim}{\mathcal{F}} = \overbrace{\underset{\sim}{\mathcal{R}}}^{\text{rot.}} \overbrace{(\underset{\sim}{I} + \underset{\sim}{\mathcal{E}})}^{\text{deformación}} = \overbrace{\underset{\sim}{\mathcal{R}}}^{\text{rot.}} \overbrace{(\underset{\sim}{I} + \underset{\sim}{\mathcal{E}}_d)}^{\text{distorsión}} \overbrace{(\mathcal{F}^{1/n}\, \underset{\sim}{I})}^{\text{inflación}} \qquad (*)$$

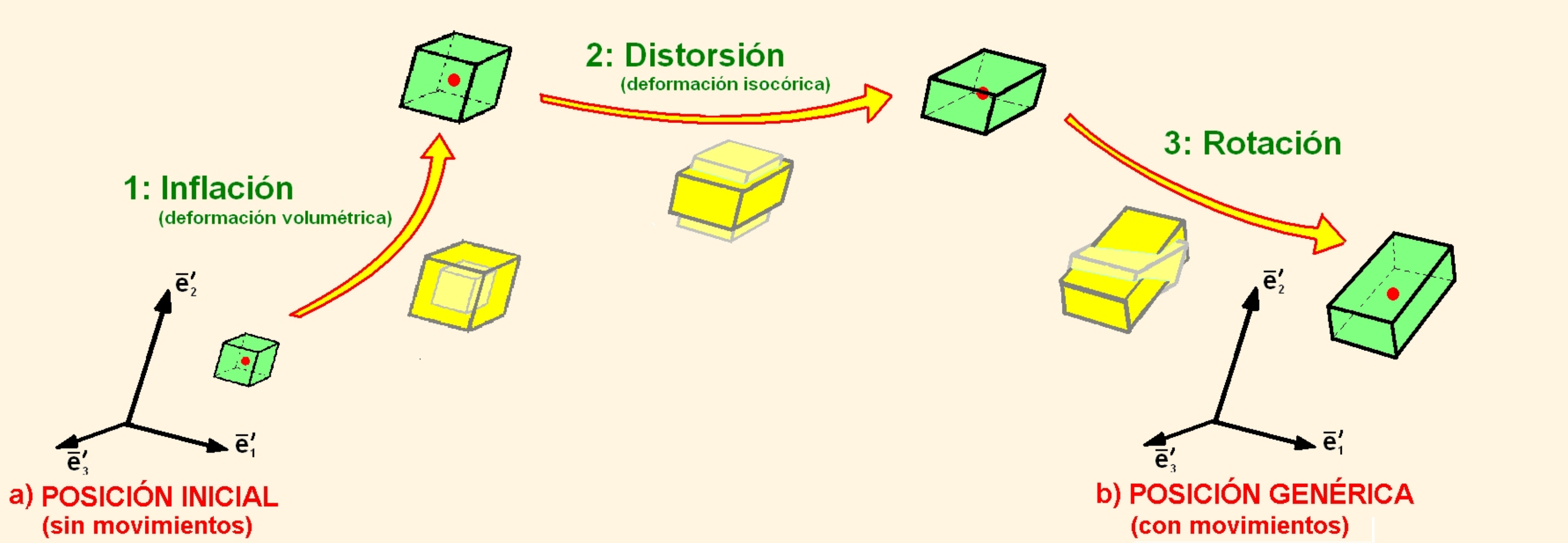
donde

$$\underset{\sim}{\mathcal{E}}_d = \mathcal{F}^{-1/n} \left(\underset{\sim}{\mathcal{E}} + \left(1 - \mathcal{F}^{1/n}\right) \underset{\sim}{I}\right) \qquad (**)$$

(*) El tensor $\underset{\sim}{\mathcal{E}}$ se denomina **tensor de deformación de Biot**.

(**) El tensor $\underset{\sim}{\mathcal{E}}_d$ se denomina **tensor de distorsión (o de deformación desviadora)**.

Interpretación:



Nota: REPRESENTACIÓN EN EJES PRINCIPALES

EXPRESIÓN GENERAL:

$$\delta\bar{r} = \underset{\sim}{F}\, \delta\bar{r}_{\rm o}, \quad \text{con} \quad \underset{\sim}{F} = \underset{\sim}{I} + \underset{\sim}{J}, \quad ||\underset{\sim}{J}|| << ||\underset{\sim}{I}||. \qquad (*)$$

(*) O lo que es lo mismo: $||\underset{\sim}{J}|| = \max\limits_{\delta\bar{r}_{\rm o}\neq 0} \dfrac{||\underset{\sim}{J}\,\delta\bar{r}_{\rm o}||}{||\delta\bar{r}_{\rm o}||} \;\Rightarrow\; ||\underset{\sim}{J}\,\delta\bar{r}_{\rm o}|| << ||\delta\bar{r}_{\rm o}|| \quad \forall\delta\bar{r}_{\rm o} \neq \bar{0}$.

CASOS RELEVANTES:

$$\left\{ \begin{array}{l} 1) \quad \underline{\textbf{DEFORMACIÓN INFINITESIMAL}} \\ \\ 2) \quad \underline{\textbf{ROTACIÓN INFINITESIMAL}} \end{array} \right.$$

1) DEFORMACIÓN INFINITESIMAL:

$$\underset{\sim}{\mathcal{F}} = \underset{\sim}{\mathcal{S}} = \underset{\sim}{I} + \underset{\sim}{\mathcal{E}}, \quad \text{con} \quad \underset{\sim}{\mathcal{E}}^{T} = \underset{\sim}{\mathcal{E}}, \quad ||\underset{\sim}{\mathcal{E}}|| << ||\underset{\sim}{I}|| \qquad (*)$$

(*) Se comprueba fácilmente que $\underset{\sim}{\mathcal{S}} = \left(\underset{\sim}{I} + \underset{\sim}{\mathcal{E}}\right)$ es simétrica y definida positiva.

2) ROTACIÓN INFINITESIMAL:

$$\underset{\sim}{F} = \underset{\sim}{R} = \underset{\sim}{I} + \underset{\sim}{\Theta}, \quad \text{con} \quad \underset{\sim}{\Theta}^T = -\underset{\sim}{\Theta}, \quad ||\underset{\sim}{\Theta}|| << ||\underset{\sim}{I}|| \qquad (*)$$

$$\Downarrow$$

$$\left.\begin{array}{l} \delta\bar{r} = \delta\bar{r}_o + \underset{\sim}{\Theta}\,\delta\bar{r}_o \\ \text{con} \quad \underset{\sim}{\Theta} = \begin{bmatrix} 0 & -\theta^3 & \theta^2 \\ \theta^3 & 0 & -\theta^1 \\ -\theta^2 & \theta^1 & 0 \end{bmatrix} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \delta\vec{r} = \delta\vec{r}_o + \vec{\theta} \wedge \delta\vec{r}_o \\ \text{con} \quad \vec{\theta} = \underset{\sim}{E}\,\bar{\theta}, \quad \bar{\theta} = \begin{Bmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{Bmatrix} \end{array}\right.$$

(*) Se comprueba fácilmente que $\left\{\begin{array}{l} \underset{\sim}{R}^T \underset{\sim}{R} = \underset{\sim}{I} + \mathcal{O}\left(||\underset{\sim}{\Theta}||^2\right) \\ \underset{\sim}{R}\,\underset{\sim}{R}^T = \underset{\sim}{I} + \mathcal{O}\left(||\underset{\sim}{\Theta}||^2\right) \end{array}\right.$.

DESCOMPOSICIÓN POLAR INFINITESIMAL:

$$\underset{\sim}{\mathcal{F}} = \underset{\sim}{I} + \underset{\sim}{\mathcal{J}}, \quad \text{con} \quad ||\underset{\sim}{\mathcal{J}}|| << ||\underset{\sim}{I}||$$

$$\Downarrow$$

$$\boxed{\underset{\sim}{I} + \underset{\sim}{\mathcal{J}} \approx \overbrace{\left(\underset{\sim}{I} + \underset{\sim}{\widehat{\Theta}}\right)}^{\text{rotación}} \overbrace{\left(\underset{\sim}{I} + \underset{\sim}{\widehat{\mathcal{E}}}\right)}^{\text{deformación}}} \quad \text{con} \quad \left\{ \boxed{\begin{aligned} \underset{\sim}{\widehat{\mathcal{E}}} &= \frac{1}{2}\left(\underset{\sim}{\mathcal{J}} + \underset{\sim}{\mathcal{J}}^T\right) \\ \underset{\sim}{\widehat{\Theta}} &= \frac{1}{2}\left(\underset{\sim}{\mathcal{J}} - \underset{\sim}{\mathcal{J}}^T\right) \end{aligned}} \right. \quad (*)$$

(*) Se comprueba fácilmente que $\left(\underset{\sim}{I} + \underset{\sim}{\widehat{\mathcal{E}}}\right)$ es simétrica y definida positiva y que además se cumple

$$\left(\underset{\sim}{I} + \underset{\sim}{\widehat{\Theta}}\right)^T \left(\underset{\sim}{I} + \underset{\sim}{\widehat{\Theta}}\right) = \underset{\sim}{I} + \mathcal{O}\left(||\underset{\sim}{\mathcal{J}}||^2\right), \quad \left(\underset{\sim}{I} + \underset{\sim}{\widehat{\Theta}}\right)\left(\underset{\sim}{I} + \underset{\sim}{\widehat{\mathcal{E}}}\right) = \underset{\sim}{\mathcal{F}} + \mathcal{O}\left(||\underset{\sim}{\mathcal{J}}||^2\right).$$

Además,

$$\underset{\sim}{\mathcal{F}} = \underset{\sim}{I} + \underset{\sim}{\mathcal{J}}, \quad \text{con} \quad ||\underset{\sim}{\mathcal{J}}|| << ||\underset{\sim}{I}||$$

$$\Downarrow$$

$$\boxed{\mathcal{F} = \det\big(\underset{\sim}{I} + \underset{\sim}{\mathcal{J}}\big) \approx 1 + \mathrm{tr}(\underset{\sim}{\mathcal{J}})} \quad \Longrightarrow \quad \begin{cases} \boxed{\mathcal{F}^{\,1/n} \approx 1 + \dfrac{\mathrm{tr}(\underset{\sim}{\mathcal{J}})}{n}} \\[2ex] \boxed{\mathcal{F}^{-1/n} \approx 1 - \dfrac{\mathrm{tr}(\underset{\sim}{\mathcal{J}})}{n}} \end{cases} \quad (*)$$

(*) Se comprueba fácilmente que $\mathrm{tr}(\underset{\sim}{\mathcal{J}}) = \mathrm{tr}(\widehat{\underset{\sim}{\mathcal{E}}}) << 1$.

Muy importante:

♡ Cualquier transformación geométrica infinitesimal $\left(\Longleftrightarrow \; ||\underset{\sim}{\mathcal{J}}|| << ||\underset{\sim}{I}|| \right)$ se puede descomponer **con errores de segundo orden** en las formas siguientes:

$$\underset{\sim}{I} + \underset{\sim}{\mathcal{J}} \approx \overbrace{\left(\underset{\sim}{I} + \underset{\sim}{\widehat{\Theta}} \right)}^{\text{rotación}} \overbrace{\left(\underset{\sim}{I} + \underset{\sim}{\widehat{\mathcal{E}}} \right)}^{\text{deformación}} \approx \overbrace{\left(\underset{\sim}{I} + \underset{\sim}{\widehat{\Theta}} \right)}^{\text{rotación}} \overbrace{\left(\underset{\sim}{I} + \underset{\sim}{\widehat{\mathcal{E}}}_d \right)}^{\text{distorsión}} \overbrace{\left(\left(1 + \frac{\mathrm{tr}(\underset{\sim}{\widehat{\mathcal{E}}})}{n} \right) \underset{\sim}{I} \right)}^{\text{inflación}} \quad (*)$$

donde

$$\underset{\sim}{\widehat{\mathcal{E}}}_d = \underset{\sim}{\widehat{\mathcal{E}}} - \left(\frac{\mathrm{tr}(\underset{\sim}{\widehat{\mathcal{E}}})}{n} \right) \underset{\sim}{I} \quad (**)$$

(*) El tensor $\underset{\sim}{\widehat{\mathcal{E}}}$ se denomina **tensor infinitesimal de deformación**.

(**) El tensor $\underset{\sim}{\widehat{\mathcal{E}}}_d$ se denomina **tensor infinitesimal de distorsión (o de deformación desviadora)**.

EXPRESIÓN GENERAL:

$$\delta\bar{r}\big|_{t+dt} = \left(\underset{\sim}{I} + \underset{\sim}{l}\, dt\right)\, \delta\bar{r}\big|_{t}. \qquad (*)$$

(*) Es obvio que $||\underset{\sim}{l}\; dt||$ es infinitamente pequeño en comparación con $||\underset{\sim}{I}||$, por tratarse de un infinitésimo.

Por tanto, pueden utilizarse las aproximaciones infinitesimales, donde $\left(\underset{\sim}{l}\; dt\right)$ toma el lugar de $\underset{\sim}{J}$.

En este caso, las aproximaciones infinitesimales se convierten en EXACTAS.

CASOS RELEVANTES:

1) <u>**DEFORMACIÓN DIFERENCIAL**</u>

2) <u>**ROTACIÓN DIFERENCIAL**</u>

1) **DEFORMACIÓN DIFERENCIAL:**

$$\underset{\sim}{I} + \underset{\sim}{l}\, dt = \underset{\sim}{I} + \underset{\sim}{e}\, dt, \quad \text{con} \quad \underset{\sim}{e}^T = \quad \underset{\sim}{e}. \qquad (*)$$

(*) Es obvio que $\left(\underset{\sim}{I} + \underset{\sim}{e}\, dt\right)$ es simétrica y definida positiva.

2) **ROTACIÓN DIFERENCIAL:**

$$\underset{\sim}{I} + \underset{\sim}{l}\, dt = \underset{\sim}{I} + \underset{\sim}{\omega}\, dt, \quad \text{con} \quad \underset{\sim}{\omega}^{T} = -\underset{\sim}{\omega}. \qquad (*)$$

$$\Downarrow$$

$$\left.\begin{array}{l} \delta\bar{r}\big|_{t+dt} = \delta\bar{r}\big|_{t} + \left(\underset{\sim}{\omega}\, dt\right) \delta\bar{r}\big|_{t} \\ \text{con} \quad \underset{\sim}{\omega} = \begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \delta\vec{r}\big|_{t+dt} = \delta\vec{r}\big|_{t} + \left(\vec{\omega}\, dt\right) \wedge \delta\vec{r}\big|_{t} \\ \text{con} \quad \vec{\omega} = \underset{\sim}{E}\, \bar{\omega}, \quad \bar{\omega} = \begin{Bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{Bmatrix} \end{array}\right.$$

(*) Se comprueba fácilmente que $\left(\underset{\sim}{I} + \underset{\sim}{\omega}\, dt\right)^{T} \left(\underset{\sim}{I} + \underset{\sim}{\omega}\, dt\right) = \left(\underset{\sim}{I} + \underset{\sim}{\omega}\, dt\right) \left(\underset{\sim}{I} + \underset{\sim}{\omega}\, dt\right)^{T} = \underset{\sim}{I}$.

DESCOMPOSICIÓN POLAR DIFERENCIAL:

$$\delta \bar{r}\Big|_{t+dt} = \left(\underset{\sim}{I} + \underset{\sim}{l}\, dt\right) \delta \bar{r}\Big|_{t}$$

$$\Downarrow$$

$$\underset{\sim}{I} + \underset{\sim}{l}\, dt = \overbrace{\left(\underset{\sim}{I} + \underset{\sim}{\omega}\, dt\right)}^{\text{rotación}} \overbrace{\left(\underset{\sim}{I} + \underset{\sim}{e}\, dt\right)}^{\text{deformación}} \quad \text{con} \quad \begin{cases} \underset{\sim}{e} = \dfrac{1}{2}\left(\underset{\sim}{l} + \underset{\sim}{l}^{T}\right) \\ \underset{\sim}{\omega} = \dfrac{1}{2}\left(\underset{\sim}{l} - \underset{\sim}{l}^{T}\right) \end{cases} \quad (*)$$

(*) Se comprueba fácilmente que $\left(\underset{\sim}{I} + \underset{\sim}{e}\, dt\right)$ es simétrica y definida positiva y que además se cumple

$$\left(\underset{\sim}{I} + \underset{\sim}{\omega}\, dt\right)^{T} \left(\underset{\sim}{I} + \underset{\sim}{\omega}\, dt\right) = \underset{\sim}{I}, \quad \left(\underset{\sim}{I} + \underset{\sim}{\omega}\, dt\right) \left(\underset{\sim}{I} + \underset{\sim}{e}\, dt\right) = \left(\underset{\sim}{I} + \underset{\sim}{l}\, dt\right).$$

Transformaciones geométricas diferenciales (Va2)

Además,

$$\delta\bar{r}\Big|_{t+dt} = \Big(\underset{\sim}{I} + \underset{\sim}{l}\,dt\Big)\;\delta\bar{r}\Big|_{t}$$

$$\Downarrow$$

$$\boxed{\det\big(\underset{\sim}{I} + \underset{\sim}{l}\,dt\big) = 1 + \mathrm{tr}(\underset{\sim}{l})\;dt} \implies \left\{ \begin{aligned} \Big(\det\big(\underset{\sim}{I} + \underset{\sim}{l}\,dt\big)\Big)^{1/n} &= 1 + \frac{\mathrm{tr}(\underset{\sim}{l})}{n}\,dt \\ \Big(\det\big(\underset{\sim}{I} + \underset{\sim}{l}\,dt\big)\Big)^{-1/n} &= 1 - \frac{\mathrm{tr}(\underset{\sim}{l})}{n}\,dt \end{aligned} \right. \quad (*)$$

(*) Se comprueba fácilmente que $\mathrm{tr}(\underset{\sim}{l}) = \mathrm{tr}(\underset{\sim}{e})$.

Muy importante:

♡ Cualquier transformación geométrica diferencial
se puede descomponer **exactamente** en las formas siguientes:

$$\underset{\sim}{I} + \underset{\sim}{l}\, dt = \overbrace{\left(\underset{\sim}{I} + \underset{\sim}{\omega}\, dt\right)}^{\text{rotación}} \overbrace{\left(\underset{\sim}{I} + \underset{\sim}{e}\, dt\right)}^{\text{deformación}} = \overbrace{\left(\underset{\sim}{I} + \underset{\sim}{\omega}\, dt\right)}^{\text{rotación}} \overbrace{\left(\underset{\sim}{I} + \underset{\sim}{e}_d\, dt\right)}^{\text{distorsión}} \overbrace{\left(\left(1 + \frac{\mathrm{tr}(\underset{\sim}{e})}{n}\, dt\right) \underset{\sim}{I}\right)}^{\text{inflación}} \quad (*)$$

donde

$$\underset{\sim}{e}_d = \underset{\sim}{e} - \left(\frac{\mathrm{tr}(\underset{\sim}{e})}{n}\right) \underset{\sim}{I} \quad (**)$$

(*) Los tensores $\underset{\sim}{e}$ y $\underset{\sim}{\omega}$ se denominan **tensor velocidad de deformación** y **tensor spin**, respectivamente.

(**) El tensor $\underset{\sim}{e}_d$ se denomina **tensor velocidad de distorsión (o de deformación desviadora)**.