$$\boxed{\iiint_{\Omega} f(\bar{x})\,d\Omega}$$

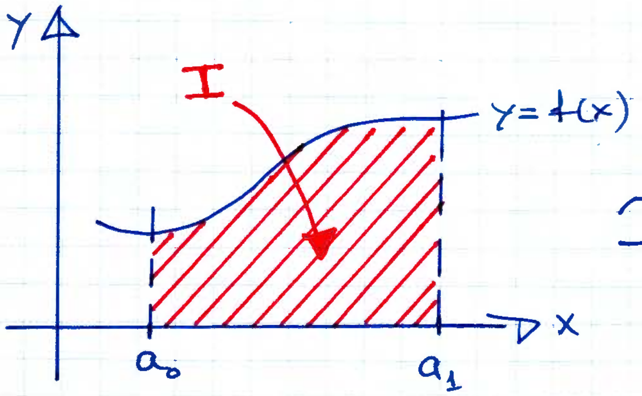
Integrales Multiples

CÁLCULO DE PRIMITIVAS (ANTI-DERIVADAS)

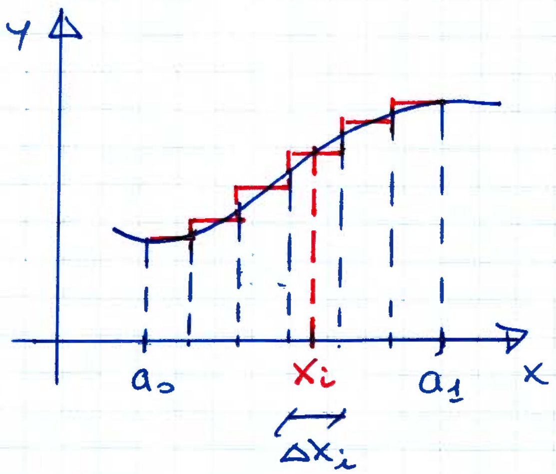
Dada $f(x)$, hallar $F(x)$ tal que $\boxed{\dfrac{dF(x)}{dx} = f(x)}$

Se escribe $$\boxed{\underbrace{\int f(x)\,dx}_{\text{Integral INDEFINIDA}} = \underbrace{F(x)+C}_{\text{Primitiva de } f(x)}}$$

INTEGRACIÓN (CÁLCULO DE "ÁREAS")



$\simeq$



Riemann: $I \simeq \sum_i f(x_i)\,\Delta x_i$; x_i en el subintervalo

$h = \max_i \{\Delta x_i\}$

Si $\lim_{h\to 0} \sum_i f(x_i)\,\Delta x_i = I$, entonces

se escribe $$\underbrace{\boxed{I = \int_{a_0}^{a_1} f(x)\,dx}}_{\text{Integral DEFINIDA}}$$

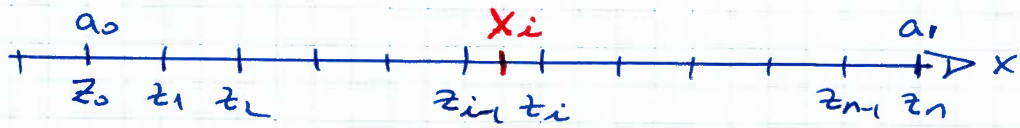
TEOREMA FUNDAMENTAL DEL CÁLCULO (Regla de Barrow)

$$\int f(x)\,dx = F(x) + C \quad \Longrightarrow \quad \boxed{\int_{a_0}^{a_1} f(x)\,dx = F(a_1) - F(a_0)}$$

Demostración: $F'(x) = f(x) \quad \forall x \in [a_0, a_1]$

Sean $\{z_i\}$ / $a_0 = z_0 < z_1 < z_2 < \dots < z_{n-1} < z_n = a_1$

los puntos que definen la partición del intervalo $[a_0, a_1]$



$\Delta x_i = (z_i - z_{i-1})$; $h = \max_i \{\Delta x_i\}$

$$F(a_1) - F(a_0) = F(z_n) - F(z_0) = F(z_n) \underbrace{- F(z_{n-1}) + F(z_{n-1})} \dots \underbrace{- F(z_1) + F(z_1)} - F(z_0)$$

$$= \sum_{i=1}^{n} \left[F(z_i) - F(z_{i-1}) \right]$$

Teorema del Valor Medio Diferencial:

$$\exists\, x_i \in (z_{i-1}, z_i) \ / \ F(z_i) - F(z_{i-1}) = \overbrace{F'(x_i)}^{f(x_i)} \Delta x_i$$

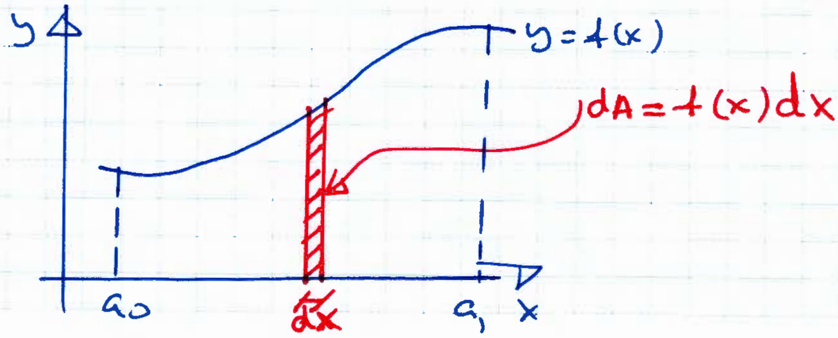
Elegimos estos puntos $\{x_i\}$ para realizar la suma de Riemann.

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} \left[F(z_i) - F(z_{i-1}) \right] = F(a_1) - F(a_0)$$

$$\Longrightarrow \lim_{h \to 0} \sum_i f(x_i)\Delta x_i = F(a_1) - F(a_0), \text{ luego:}$$

$$\int_{a_0}^{a_1} f(x)\,dx = F(a_1) - F(a_0)$$

Planteamiento Práctico:



INTEGRALES ITERADAS

SIMPLES

$$I = \int_{a_0}^{a_1} f(x)\, dx$$

DOBLES

$$I = \int_{a_0}^{a_1} \int_{b_0(x)}^{b_1(x)} f(x,y)\, dy\, dx$$

TRIPLES

$$I = \int_{a_0}^{a_1} \int_{b_0(x)}^{b_1(x)} \int_{c_0(x,y)}^{c_1(x,y)} f(x,y,z)\, dz\, dy\, dx$$

...

de orden n

$I = \ldots$ (generalización del concepto anterior)

Cálculo de Integrales Dobles

$$I = \int_{a_0}^{a_1} \int_{b_0(x)}^{b_1(x)} f(x,y)\, dy\, dx$$

¡ojo al orden!

$$= \int_{x=a_0}^{x=a_1} \underbrace{\left[\int_{y=b_0(x)}^{y=b_1(x)} f(x,y)\, dy \right]}_{g(x)} dx$$

$$= \int_{x=a_0}^{x=a_1} g(x)\, dx$$

$$g(x) = \int_{y=b_0(x)}^{y=b_1(x)} f(x,y)\, dy$$

Nota: Al calcular la integral en y, x se considera constante.

Cálculo de Integrales Triples

$$I = \int_{a_0}^{a_1} \int_{b_0(x)}^{b_1(x)} \int_{c_0(x,y)}^{c_1(x,y)} f(x,y,z)\, dz\, dy\, dx$$

¡OJO al orden!

$$= \int_{x=a_0}^{x=a_1} \underbrace{\left[\int_{y=b_0(x)}^{y=b_1(x)} \underbrace{\left[\int_{z=c_0(x,y)}^{z=c_1(x,y)} f(x,y,z)\, dz \right]}_{g(x,y)} dy \right]}_{h(x)} dx$$

$$= \int_{x=a_0}^{x=a_1} h(x)\, dx$$

$$h(x) = \int_{y=b_0(x)}^{y=b_1(x)} g(x,y)\, dy$$

$$g(x,y) = \int_{z=c_0(x,y)}^{z=c_1(x,y)} f(x,y,z)\, dz$$

Notas: 1) Al calcular la integral en z, x e y se consideran constantes

2) Al calcular la integral en y, x se considera constante

Recolocación de Términos

Los factores comunes que sean independiente de una variable de integración se pueden "sacar" fuera de la correspondiente integral.

Ej: $f(x,y,z) = 5x\,(x^2+y^2)(x^3+y^3+z^3)$

$$I = \int_{a_0}^{a_1}\int_{b_0(x)}^{b_1(x)}\int_{c_0(x,y)}^{c_1(x,y)} 5x\,(x^2+y^2)(x^3+y^3+z^3)\,dz\,dy\,dx$$

$$= 5\int_{x=a_0}^{x=a_1} x\left[\int_{y=b_0(x)}^{y=b_1(x)} (x^2+y^2)\left[\int_{z=c_0(x,y)}^{z=c_1(x,y)} (x^3+y^3+z^3)\,dz\right]dy\right]dx$$

Notas:

1) Se puede proceder de igual forma con los términos que aparecen al calcular las integrales internas.

2) ¡OJO! El resultado debe tener sentido.
En el ejemplo anterior:

$$I \neq \int_{x=a_0}^{x=a_1} (x^2+y^2)\left[\int_{y=b_0(x)}^{y=b_1(x)} 5x\left[\int_{z=c_0(x,y)}^{z=c_1(x,y)} (x^3+y^3+z^3)\,dz\right]dy\right]dx$$

Correcto, aunque se puede sacar fuera de la integral en y

INCORRECTO: ¿qué hace aquí la variable y?
⟹ ESTÁ MAL

Permutación del orden de integración

1) En general, el orden de integración no se puede cambiar directamente. Además, si se hace esto el resultado no tendrá sentido.

Ej:

$$I = \int_{a_0}^{a_1} \int_{b_0(x)}^{b_1(x)} \int_{c_0(x,y)}^{c_1(x,y)} f(x,y,z)\, dz\, dy\, dx$$

$$\neq \underbrace{\int_{c_0(x,y)}^{c_1(x,y)} \underbrace{\int_{b_0(x)}^{b_1(x)} \underbrace{\int_{a_0}^{a_1} f(x,y,z)\, dx}_{g(y,z)\text{, puede estar bien}}\, dy}_{h(z)\text{, está mal: ¿qué significan } b_0(x),\, b_1(x)?}\, dz}_{\text{está mal: ¿qué significan } c_0(x,y),\, c_1(x,y)}$$

2) Para permutar el orden de integración, hay que redefinir adecuadamente el <u>recinto de integración</u> (ver apartados siguientes.

3) Sí se puede permutar el orden cuando los límites de integración son constantes.

Ej:
$$I = \int_{a_0}^{a_1} \int_{b_0}^{b_1} f(x,y)\, dy\, dx = \int_{b_0}^{b_1} \int_{a_0}^{a_1} f(x,y)\, dx\, dy$$

INTEGRALES MULTIPLES

INTEGRAL DE LONGITUD (SIMPLE)

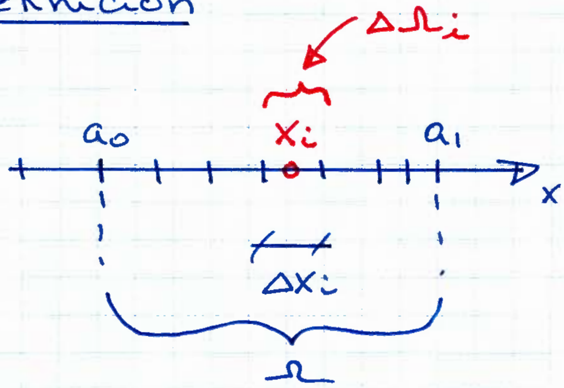
Sea $\Omega = \{x \,/\, x \in [a_0, a_1]\} \subset \mathbb{R}$

$x \in \Omega \longrightarrow f(x) \equiv$ "masa" por unidad de longitud

$\boxed{I = \int_\Omega f(x)\, dL}$ = "masa" total de Ω

En particular, $\int_\Omega dL = L \equiv$ longitud de Ω

Definición



$$\left.\begin{cases} \Omega = \bigcup_i (\Delta\Omega_i) \\ (\mathring{\Delta\Omega_i}) \cap (\mathring{\Delta\Omega_j}) = \emptyset \quad \forall i \neq j \end{cases}\right\} (**)$$

$$\begin{cases} \Delta x_i \equiv \text{longitud de } \Delta\Omega_i \\ x_i \in \Delta\Omega_i \end{cases}$$

$h = \max_i \{\Delta x_i\}$

$$I \simeq \sum_i f(x_i)\, \Delta x_i$$

Si $\lim_{h \to 0} \sum_i f(x_i)\, \Delta x_i = I$, entonces

$$I = \int_\Omega f(x)\, dL$$

Cálculo (~ Teorema fundamental del cálculo)

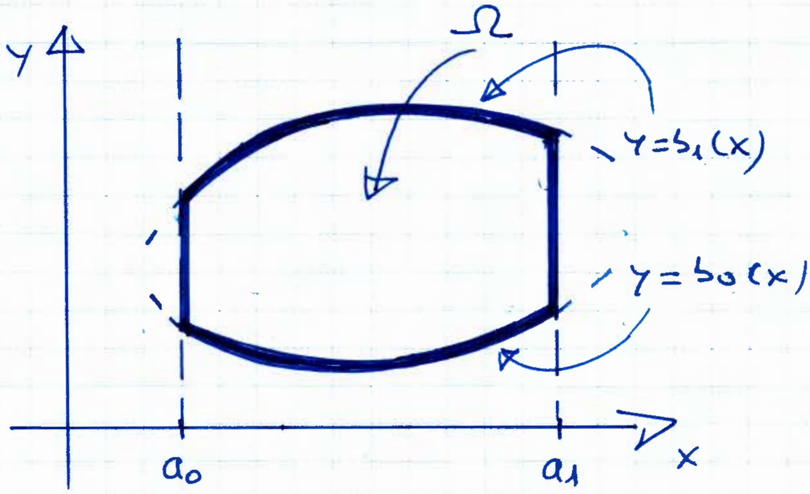
Obviamente, mediante una integral definida (simple):

$$\boxed{\int_\Omega f(x)\, dL = \int_{a_0}^{a_1} f(x)\, dx}$$

(**) Partición del dominio Ω

INTEGRAL DE ÁREA (DOBLE)

Sea $\Omega = \{(x,y) \,/\, x \in [a_0, a_1]; y \in [b_0(x), b_1(x)]\} \subset \mathbb{R}^2$ (*)

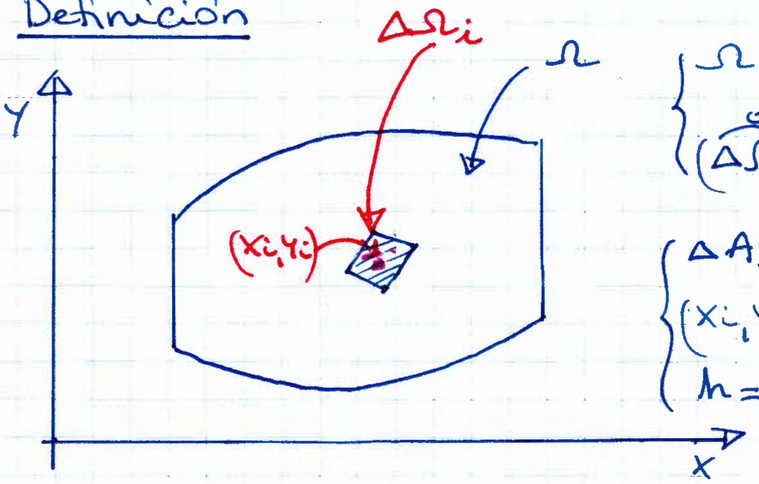


$(x,y) \in \Omega \longrightarrow f(x,y) \equiv$ "masa" por unidad de área

$\boxed{I = \iint_\Omega f(x,y)\, dA}$ $\equiv$ "masa" total de Ω

En particular, $\iint_\Omega dA = A \equiv$ área de Ω

Definición



$$\left.\begin{cases} \Omega = \bigcup_i (\Delta\Omega_i) \\ (\mathring{\Delta\Omega_i}) \cap (\mathring{\Delta\Omega_j}) = \phi \quad \forall i \neq j \end{cases}\right\} (**)$$

$$\begin{cases} \Delta A_i \equiv \text{área de } \Delta\Omega_i \\ (x_i, y_i) \in \Delta\Omega_i \\ h = \max_i \{\phi(\Delta\Omega_i)\} \end{cases}$$

$$I \simeq \sum_i f(x_i, y_i)\, \Delta A_i$$

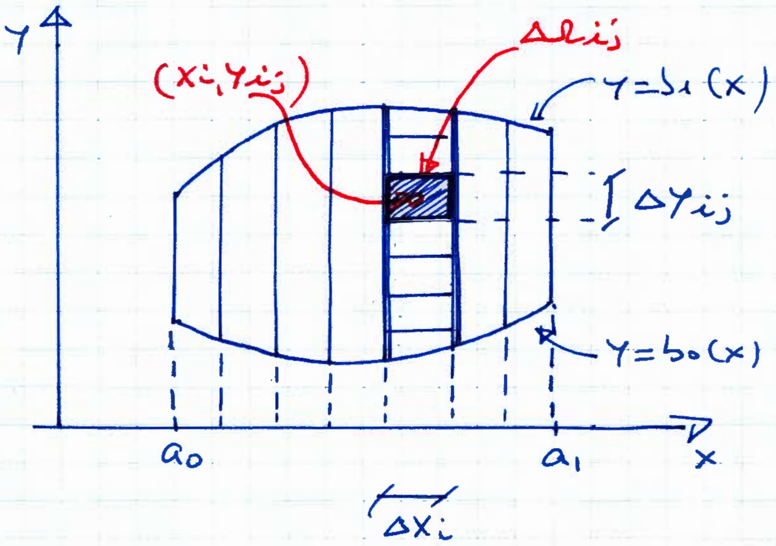
Si $\lim_{h \to 0} \sum_i f(x_i, y_i)\, \Delta A_i = I$, entonces

$$I = \iint_\Omega f(x,y)\, dA$$

(*) Cualquier dominio $\Omega \subset \mathbb{R}^2$ suficientemente regular puede expresarse así.
(**) Partición del dominio Ω

Cálculo (extensión del T.F.C. a 2D)

Tomamos particiones del tipo:



$$\begin{cases} \Delta A_{ij} = \Delta y_{ij}\, \Delta x_i \\ h = \max\limits_{i,j} \{\Delta x_i, \Delta y_{ij}\} \end{cases}$$

$$I \simeq \sum_i \sum_j f(x_i, y_{ij})\, \Delta y_{ij}\, \Delta x_i$$

$$= \sum_i \Big[\underbrace{\sum_j f(x_i, y_{ij})\, \Delta y_{ij}}_{\simeq g(x_i)} \Big] \Delta x_i$$

Si $\lim\limits_{h \to 0} \sum\limits_i \sum\limits_j f(x_i, y_{ij})\, \Delta y_{ij}\, \Delta x_i = I$, entonces

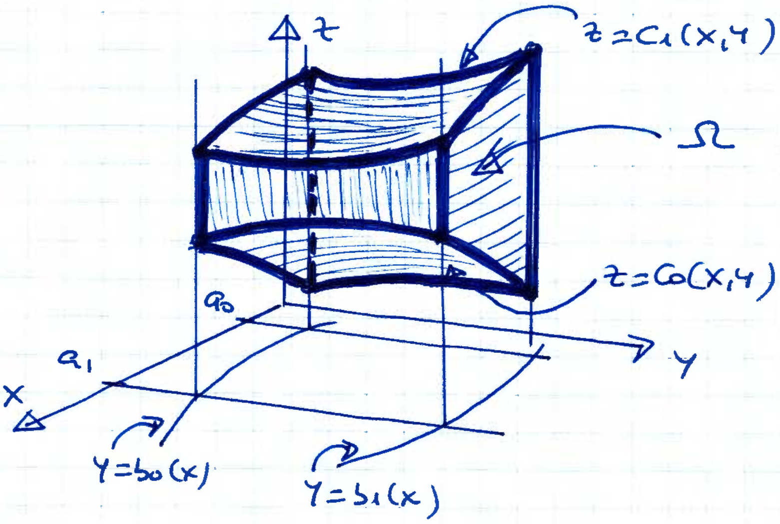
$$I = \iint_\Omega f(x,y)\, dA = \int_{a_0}^{a_1} \Big[\underbrace{\int_{b_0(x)}^{b_1(x)} f(x,y)\, dy}_{g(x)} \Big] dx$$

Luego:

$$\boxed{\iint_\Omega f(x,y)\, dA = \int_{a_0}^{a_1} \int_{b_0(x)}^{b_1(x)} f(x,y)\, dy\, dx}$$

INTEGRAL DE VOLUMEN (TRIPLE)

Sea $\Omega = \{(x,y,z) \,/\, x \in [a_0, a_1];\ y \in [b_0(x), b_1(x)];\ z \in [c_0(x,y), c_1(x,y)]\} \subset \mathbb{R}^3$ (*)

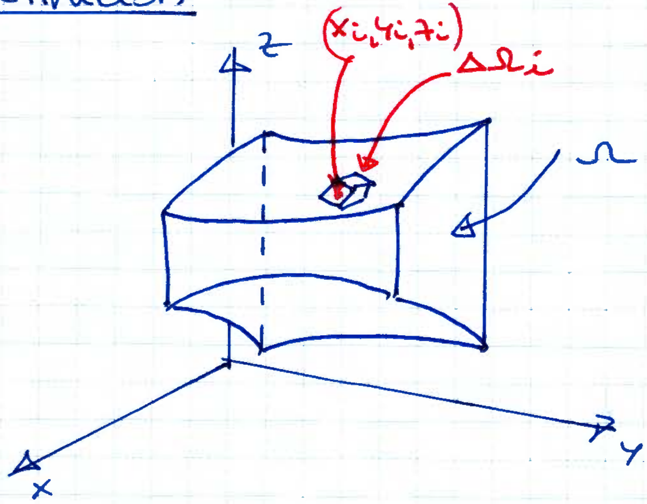


$(x,y,z) \in \Omega \rightarrow f(x,y,z) \equiv$ "masa" por unidad de volumen

$$\boxed{I = \iiint_\Omega f(x,y,z)\, dV} \equiv \text{"masa" total de } \Omega$$

En particular, $\iiint_\Omega dV = V \equiv$ volumen de Ω

Definición



$$\left\{ \begin{array}{l} \Omega = \bigcup_i (\Delta\Omega_i) \\ (\mathring{\Delta\Omega_i}) \cap (\mathring{\Delta\Omega_j}) = \emptyset \quad \forall i \neq j \end{array} \right\} (**)$$

$$\left\{ \begin{array}{l} \Delta V_i \equiv \text{volumen de } \Delta\Omega_i \\ (x_i, y_i, z_i) \in \Delta\Omega_i \\ h = \max_i \{\phi(\Delta\Omega_i)\} \end{array} \right.$$

$$I \simeq \sum_i f(x_i, y_i, z_i)\, \Delta V_i$$

Si $\lim_{h \to 0} \sum_i f(x_i, y_i, z_i)\, \Delta V_i = I$, entonces

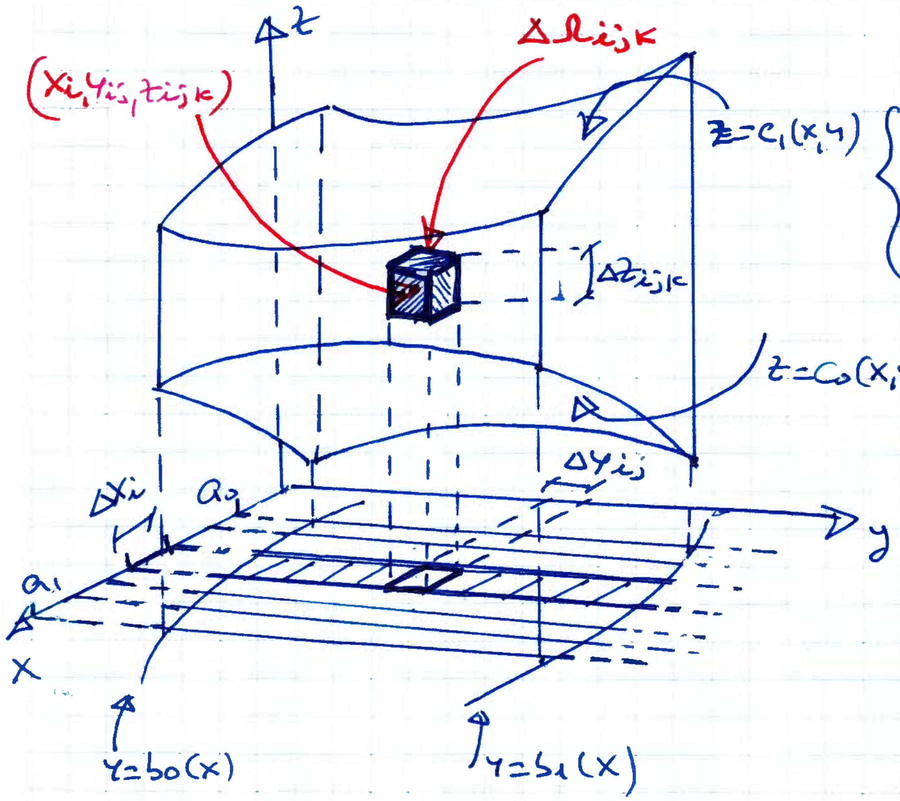
$$I = \iiint_\Omega f(x,y,z)\, dV$$

(*) Cualquier dominio $\Omega \subset \mathbb{R}^3$ suficientemente regular puede expresarse así.

(**) Partición del dominio Ω

Cálculo (extensión del T.F.C. a 3D)

Tomamos particiones del tipo:



$$\begin{cases} \Delta V_{ijk} = \Delta z_{ijk} \Delta y_{ij} \Delta x_i \\ h = \max\limits_{i,j,k} \{\Delta x_i, \Delta y_{ij}, \Delta z_{ijk}\} \end{cases}$$

$$I \simeq \sum_i \sum_j \sum_k f(x_i, y_{ij}, z_{ijk}) \Delta z_{ijk} \Delta y_{ij} \Delta x_i$$

$$= \sum_i \underbrace{\left[\sum_j \underbrace{\left[\sum_k f(x_i, y_{ij}, z_{ijk}) \Delta z_{ijk} \right]}_{\simeq g(x_i, y_{ij})} \Delta y_{ij} \right]}_{\simeq h(x_i)} \Delta x_i$$

Si $\lim\limits_{h \to 0} \sum_i \sum_j \sum_k f(x_i, y_{ij}, z_{ijk}) \Delta z_{ijk} \Delta y_{ij} \Delta x_i = I$, entonces

$$I = \iiint_\Omega f(x,y,z)\, dV = \int_{a_0}^{a_1} \underbrace{\left[\int_{b_0(x)}^{b_1(x)} \underbrace{\left[\int_{c_0(x,y)}^{c_1(x,y)} f(x,y,z)\, dz \right]}_{g(x,y)} dy \right]}_{h(x)} dx$$

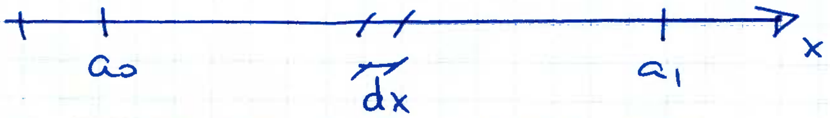
Luego:

$$\boxed{\iiint_\Omega f(x,y,z)\, dV = \int_{a_0}^{a_1} \int_{b_0(x)}^{b_1(x)} \int_{c_0(x,y)}^{c_1(x,y)} f(x,y,z)\, dz\, dy\, dx}$$

PLANTEAMIENTO PRÁCTICO DE INTEGRALES MÚLTIPLES

INTEGRAL DE LONGITUD (SIMPLE)

$$\Omega = \{ x \,/\, x \in [a_0, a_1] \} \subset \mathbb{R}$$

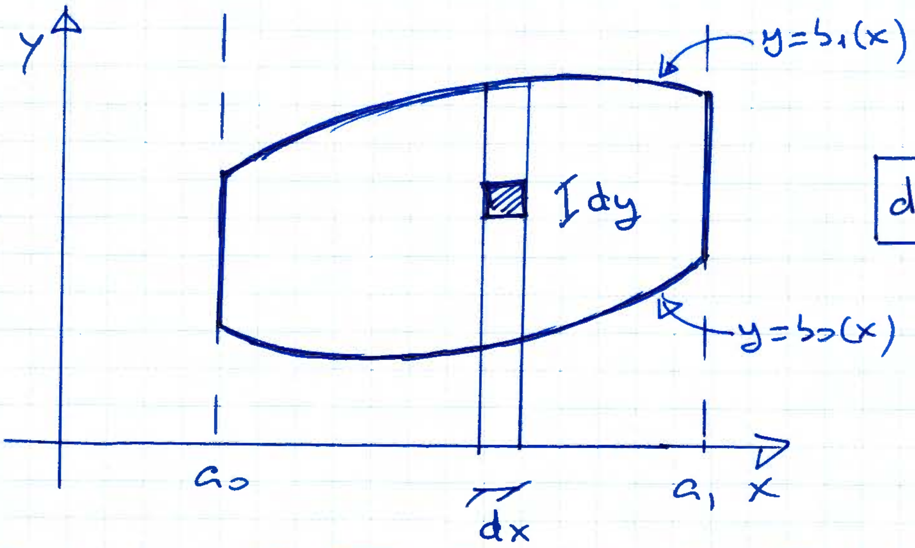


$$\boxed{dL = dx}$$

$$\boxed{\int_{\Omega} f(x)\, dL = \int_{a_0}^{a_1} f(x)\, dx}$$

INTEGRAL DE ÁREA (DOBLE)

$$\Omega = \{ (x,y) \,/\, x \in [a_0, a_1];\ y \in [b_0(x), b_1(x)] \} \subset \mathbb{R}^2$$

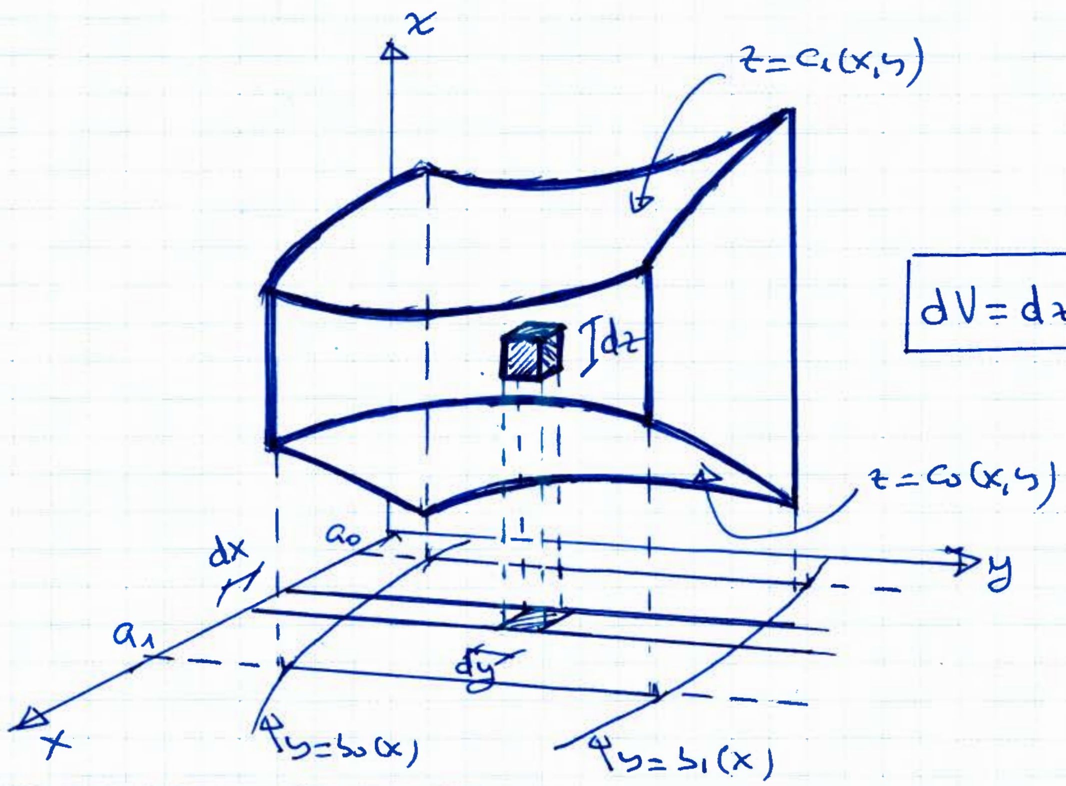


$$\boxed{dA = dy\, dx}$$

$$\boxed{\iint_{\Omega} f(x,y)\, dA = \int_{a_0}^{a_1} \int_{b_0(x)}^{b_1(x)} f(x,y)\, dy\, dx}$$

INTEGRAL DE VOLUMEN (TRIPLE)

$$\Omega = \{ (x,y,z) \,/\, x \in [a_0,a_1];\ y \in [b_0(x), b_1(x)];\ z \in [c_0(x,y), c_1(x,y)] \} \subset \mathbb{R}^3$$



$$\boxed{dV = dz\,dy\,dx}$$

$$\boxed{\iiint_\Omega f(x,y,z)\,dV = \int_{a_0}^{a_1}\int_{b_0(x)}^{b_1(x)}\int_{c_0(x,y)}^{c_1(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

NOTAS SOBRE EL PLANTEAMIENTO DE INTEGRALES MÚLTIPLES

1) Cualquier integral múltiple en un espacio de dimensión "n" puede reducirse a "n" integrales iteradas.

2) La principal dificultad consiste en escribir adecuadamente el recinto de integración Ω

$$\begin{cases} 1D: \ \Omega = \{x \,/\, x \in [a_0, a_1]\} \subset \mathbb{R} \ \rightsquigarrow \text{Trivial} \\ 2D: \ \Omega = \{(x,y) \,/\, x \in [a_0, a_1];\ y \in [b_0(x), b_1(x)]\} \subset \mathbb{R}^2 \rightsquigarrow \text{Fácil} \\ 3D: \ \Omega = \{(x,y,z) \,/\, x \in [a_0, a_1];\ y \in [b_0(x), b_1(x)];\ z \in [c_0(x,y), c_1(x,y)]\} \subset \mathbb{R}^3 \\ \qquad \rightsquigarrow \text{Difícil (en general)} \end{cases}$$

El concepto puede generalizarse a espacios de dimensión $= n$

3) A medida que aumenta n hay más alternativas posibles:

$$\begin{cases} n=2 \implies \begin{cases} dA = dy\,dx \\ dA = dx\,dy \end{cases} \\ n=3 \implies \begin{cases} dV = dz\,dy\,dx \\ dV = dy\,dz\,dx \\ dV = dz\,dx\,dy \\ dV = dx\,dz\,dy \\ dV = dy\,dx\,dz \\ dV = dx\,dy\,dz \end{cases} \end{cases}$$

Dilema: ¿Cuál es la mejor?

EJEMPLOS: APLICACIONES DE INTEGRALES MÚLTIPLES

en 3D:

1) Volumen

$$f(x,y,z)=1 \longrightarrow V=\iiint_\Omega dv$$

2) Masa

$$f(x,y,z)=\rho(x,y,z) \equiv \text{densidad} \rightarrow M=\iiint_\Omega \rho\, dv$$

3) Posición del Centro de Gravedad

$$\bar{f}(x,y,z)=\bar{r}\,\rho(x,y,z) \rightarrow \bar{r}_G=\frac{1}{M}\iiint_\Omega \bar{r}\,\rho\, dv$$

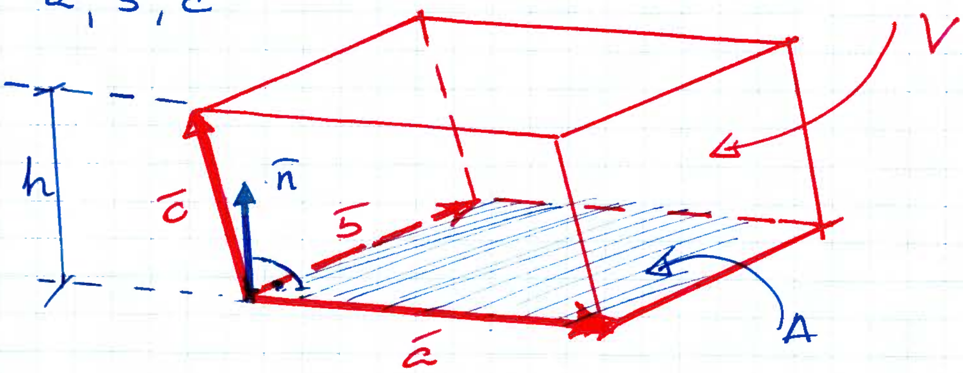
4) Tensor de Inercia

$$\underset{\sim}{f}(x,y,z)=\left(|\bar{r}|^2\,\underset{\sim}{I}-\bar{r}\,\bar{r}^T\right)\rho(x,y,z) \rightarrow \underset{\sim}{I}=\iiint_\Omega \left(|\bar{r}|^2\underset{\sim}{I}-\bar{r}\bar{r}^T\right)\rho\, dv$$

CAMBIO DE VARIABLE

PRODUCTO TRIPLE

Sea V el volumen del prisma generado por los vectores $\bar{a}, \bar{b}, \bar{c}$



$V = A \cdot h$

módulo

$$\begin{cases} A = |\bar{a} \wedge \bar{b}| \\ h = |\bar{c} \cdot \bar{n}| \end{cases}$$

valor absoluto

con $\bar{n} = \dfrac{\bar{a} \wedge \bar{b}}{|\bar{a} \wedge \bar{b}|}$

Luego $V = |\bar{a} \wedge \bar{b}| \left| \left(\dfrac{\bar{a} \wedge \bar{b}}{|\bar{a} \wedge \bar{b}|} \right) \cdot \bar{c} \right| = |(\bar{a} \wedge \bar{b}) \cdot \bar{c}|$

valor absoluto

$$\boxed{V = |(\bar{a} \wedge \bar{b}) \cdot \bar{c}|}$$ (producto triple)

valor absoluto

Notas: $|(\bar{a} \wedge \bar{b}) \cdot \bar{c}| = |(\bar{a} \wedge \bar{c}) \cdot \bar{b}| = |(\bar{b} \wedge \bar{c}) \cdot \bar{a}|$

EXPRESIÓN EN FORMA DE DETERMINANTE

Si $\bar{a} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}$, $\bar{b} = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}$, $\bar{c} = \begin{Bmatrix} c_x \\ c_y \\ c_z \end{Bmatrix}$

$$(\bar{a} \wedge \bar{b}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \bar{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \bar{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \bar{k}$$

$$(\bar{a} \wedge \bar{b}) \cdot \bar{c} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} c_x - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} c_y + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} c_z$$

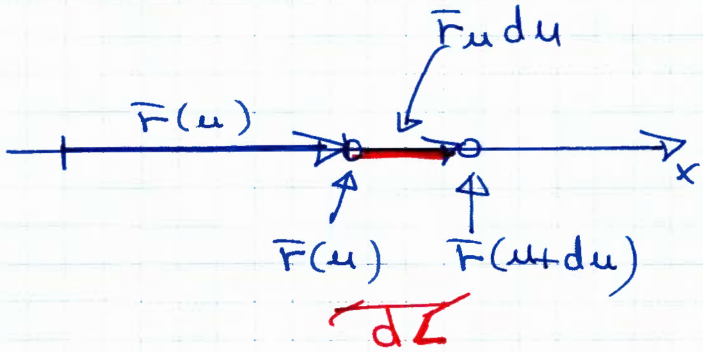
$$= \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{vmatrix}$$

luego:

$$\boxed{V = \left| (\bar{a} \wedge \bar{b}) \cdot \bar{c} \right| = \left| \det \left(\bar{a} \mid \bar{b} \mid \bar{c} \right) \right|}$$

1D (u)

Sea $x = x(u)$



$$\bar{F} = \{x\}, \quad \bar{u} = \{u\} \longrightarrow \bar{F} = \bar{F}(\bar{u})$$

$$\bar{F}(u+du) - \bar{F}(u) = \bar{F}_u \, du, \quad \text{con} \quad \bar{F}_u = \left\{\frac{\partial x}{\partial u}\right\}$$

$$dL = \left| \bar{F}_u \, du \right| = \left| \bar{F}_u \right| du = \left| \frac{\partial x}{\partial u} \right| du$$

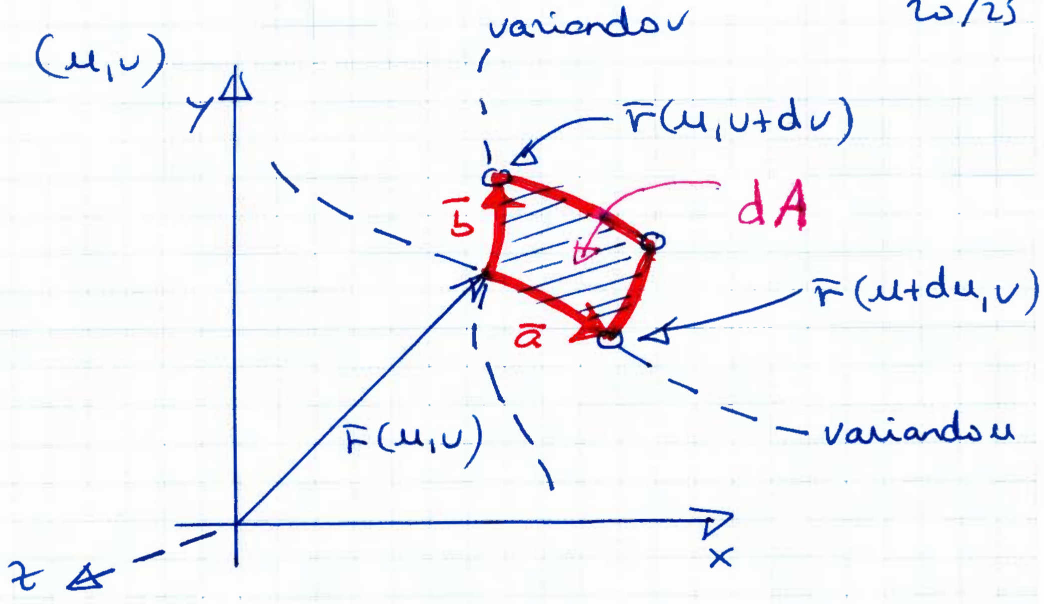
$$= \left| \det\left(\frac{\partial x}{\partial u}\right) \right| du$$

Luego:

$$\boxed{dL = |J| \, du, \quad J = \det(\underset{\sim}{J}), \quad \underset{\sim}{J} = \left[\frac{\partial x}{\partial u}\right] = \frac{d\bar{F}}{d\bar{u}}}$$

CURVILÍNEAS 2D

(u,v)



Sea:

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

$$\bar{F} = \begin{Bmatrix} x \\ y \end{Bmatrix}, \quad \bar{u} = \begin{Bmatrix} u \\ v \end{Bmatrix} \longrightarrow \bar{r} = \bar{F}(\bar{u})$$

$$\left\{\begin{aligned} \bar{a} &= \bar{F}(u+du,v) - \bar{F}(u,v) = \bar{F}_u\, du \\ \bar{b} &= \bar{F}(u,v+dv) - \bar{F}(u,v) = \bar{F}_v\, dv \end{aligned}\right\} \text{ con } \bar{F}_u = \begin{Bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{Bmatrix}, \ \bar{F}_v = \begin{Bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \end{Bmatrix}$$

$$dA = |\bar{a} \wedge \bar{b}| = |(\bar{F}_u\, du) \wedge (\bar{F}_v\, dv)| = |\bar{F}_u \wedge \bar{F}_v|\, du\, dv$$

$$= \left| \det \begin{pmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{pmatrix} \right| du\, dv$$

$$= \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} \right| du\, dv = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| du\, dv$$

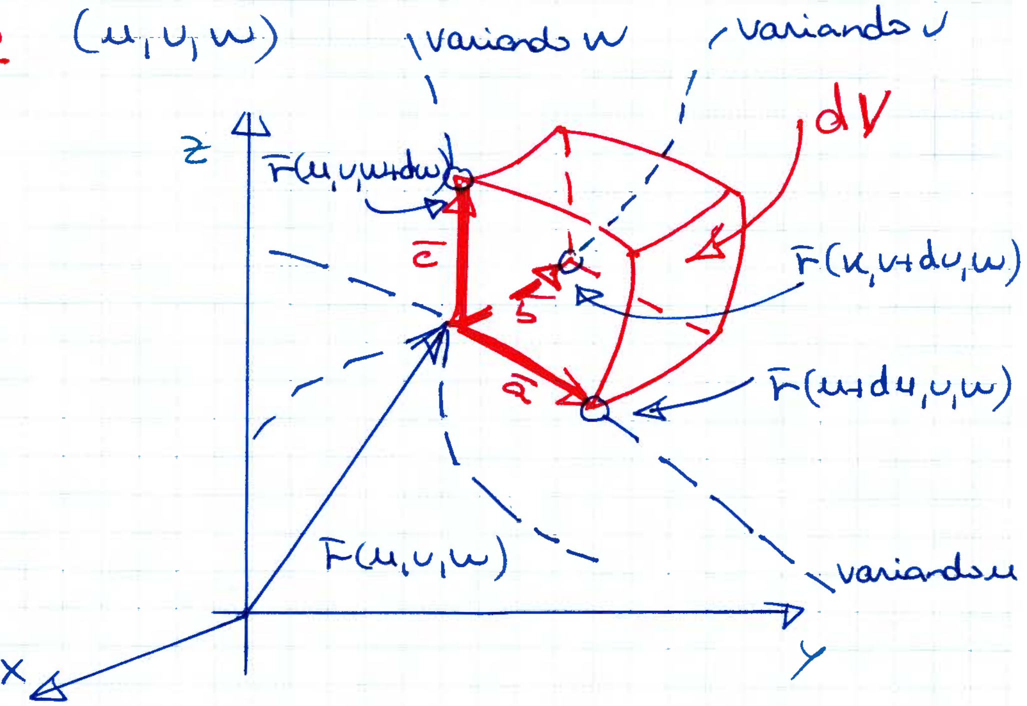
Luego:

$$\boxed{dA = |J|\, du\, dv, \quad J = \det(\underset{\sim}{J}), \quad \underset{\sim}{J} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{d\bar{r}}{d\bar{u}}}$$

CURVILÍNEAS 3D (u,v,w)

Sea:

$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$$



$$\bar r = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}, \quad \bar u = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \longrightarrow \bar r = \bar r(\bar u)$$

$$\left.\begin{aligned} \bar a &= \bar r(u+du,v,w) - \bar r(u,v,w) = \bar r_u\, du \\ \bar b &= \bar r(u,v+dv,w) - \bar r(u,v,w) = \bar r_v\, dv \\ \bar c &= \bar r(u,v,w+dw) - \bar r(u,v,w) = \bar r_w\, dw \end{aligned}\right\} \text{ con } \bar r_u = \begin{Bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{Bmatrix}, \bar r_v = \begin{Bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{Bmatrix}, \bar r_w = \begin{Bmatrix} \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial w} \end{Bmatrix}$$

$$dV = \left| \left((\bar r_u\, du) \wedge (\bar r_v\, dv) \right) \cdot \bar r_w\, dw \right| = \left| \underbrace{(\bar r_u \wedge \bar r_v) \cdot \bar r_w}_{\text{producto triple}} \right| du\, dv\, dw$$

$$= \left| \det \left(\bar r_u \,\vdots\, \bar r_v \,\vdots\, \bar r_w \right) \right| du\, dv\, dw$$

$$= \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} \right| du\, dv\, dw$$

Luego:

$$\boxed{dV = |J| \underbrace{du\, dv\, dw}_{\text{y sus permutaciones}}, \quad J = \det(\underset{\sim}{J}), \quad \underset{\sim}{J} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} = \frac{d\bar r}{d\bar u}}$$

EN GENERAL

Sea $\overline{r} = \overline{r}(\overline{u})$ un cambio de variable (transformación de coordenadas)

con
$$\begin{cases} \overline{r} = \begin{Bmatrix} x_1 \\ | \\ x_n \end{Bmatrix} \equiv \text{coordenadas cartesianas} \\ \overline{u} = \begin{Bmatrix} u_1 \\ | \\ u_n \end{Bmatrix} \equiv \text{coordenadas curvilíneas} \end{cases}$$

Sean:
$$\begin{cases} \underset{\sim}{J} = \dfrac{d\overline{r}}{d\overline{u}} = \begin{bmatrix} \dfrac{\partial x_1}{\partial u_1} & \text{—} & \dfrac{\partial x_1}{\partial u_n} \\ | & & | \\ \dfrac{\partial x_n}{\partial u_1} & \text{—} & \dfrac{\partial x_n}{\partial u_n} \end{bmatrix}, \text{ la matriz JACOBIANA de la transformación} \\ J = \det(\underset{\sim}{J}), \text{ el determinante JACOBIANO de la transformación} \end{cases}$$

En cartesianas:

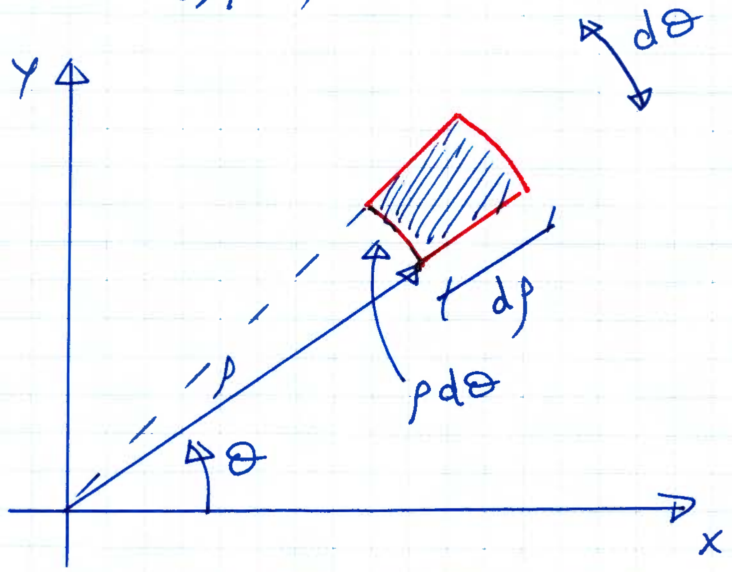
$$dV = \underbrace{dx_1 \, dx_2 \cdots dx_n}_{\text{y sus permutaciones}}$$

En curvilíneas:

$$dV = |J| \, \underbrace{du_1 \, du_2 \cdots du_n}_{\text{y sus permutaciones}}$$

Notas:
1) Esto es válido para $n=1$, $n=2$, $n=3$, $n \geq 4$
2) Para $n \geq 4$ V es un "hipervolumen"
3) J no debe cambiar de signo ni anularse en el dominio de integración

POLARES (ρ, θ)



Directamente:

$$\boxed{dA = (\rho\, d\theta)(d\rho) = \rho\, d\rho\, d\theta}$$

Analíticamente:

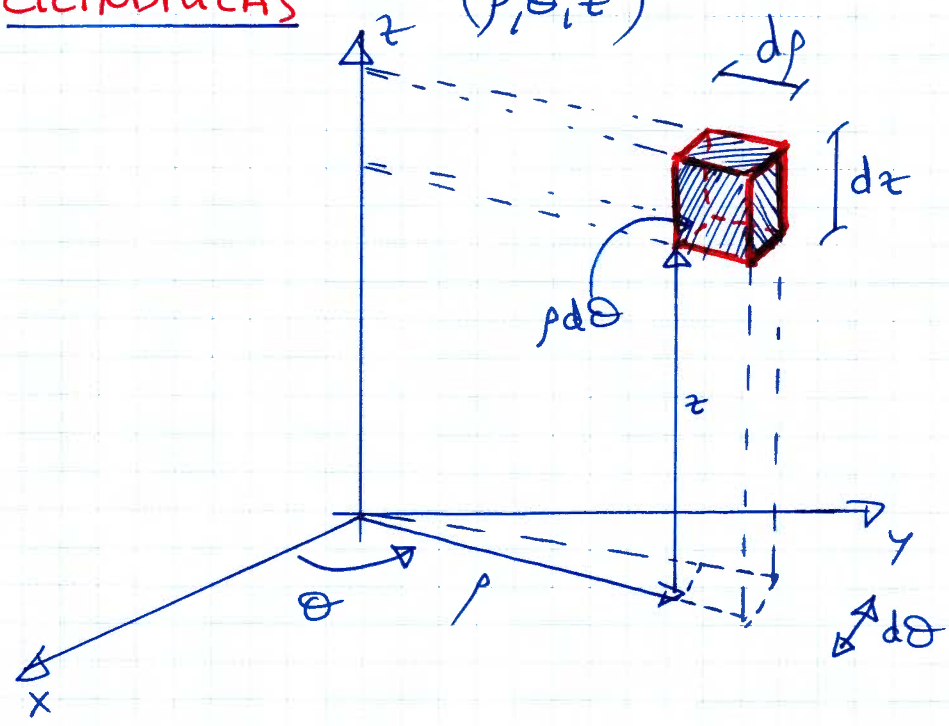
$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\operatorname{sen}\theta \end{cases} \qquad F = \begin{Bmatrix} x \\ y \end{Bmatrix} \qquad \bar{u} = \begin{Bmatrix} \rho \\ \theta \end{Bmatrix}$$

$$\underset{\sim}{J} = \frac{dF}{d\bar{u}} = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\rho\,\operatorname{sen}\theta \\ \operatorname{sen}\theta & \rho\cos\theta \end{bmatrix}$$

$$J = \det(\underset{\sim}{J}) = \rho\cos^2\theta + \rho\,\operatorname{sen}^2\theta = \rho$$

$$\boxed{dA = |J|\, d\rho\, d\theta = \rho\, d\rho\, d\theta}$$

CILÍNDRICAS (ρ, θ, z)



Directamente:

$$\boxed{dV = (\rho d\theta)(d\rho)(dz) = \underbrace{\rho\, d\rho\, d\theta\, dz}_{\text{y sus permutaciones}}}$$

Analíticamente

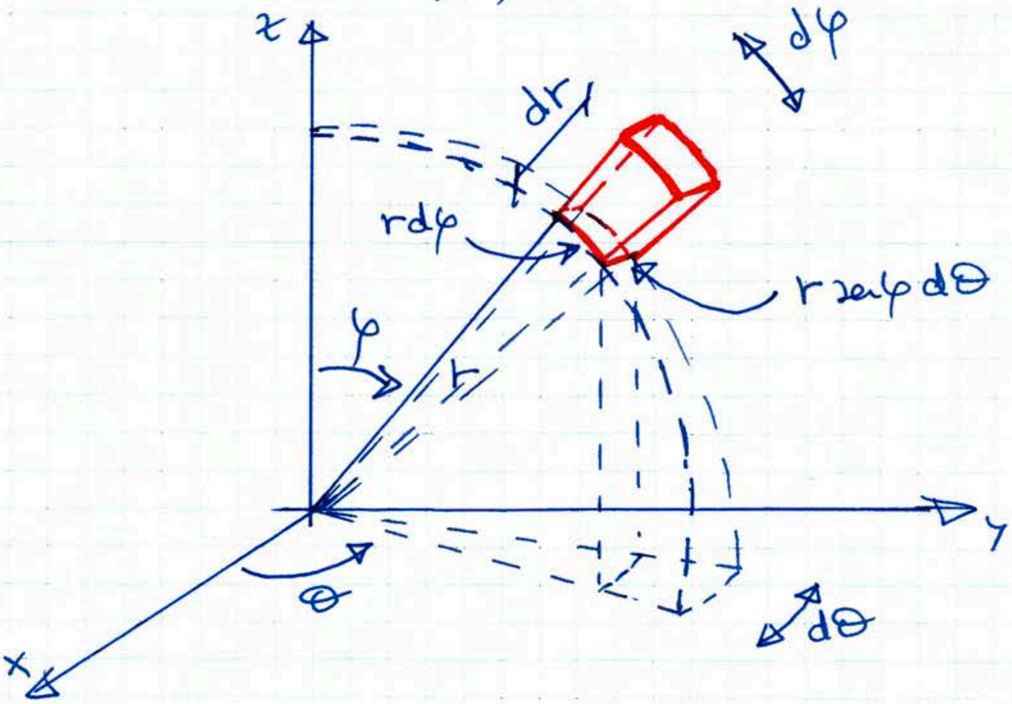
$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \\ z = z \end{cases} \qquad \bar{r} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \qquad \bar{u} = \begin{Bmatrix} \rho \\ \theta \\ z \end{Bmatrix}$$

$$\underset{\sim}{J} = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\rho\,\mathrm{sen}\,\theta & 0 \\ \mathrm{sen}\,\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$J = \det(\underset{\sim}{J}) = \rho\cos^2\theta + \rho\,\mathrm{sen}^2\theta = \rho$$

$$\boxed{dV = |J|\, d\rho\, d\theta\, dz = \underbrace{\rho\, d\rho\, d\theta\, dz}_{\text{y sus permutaciones}}}$$

ESFÉRICAS (r, θ, φ)



Directamente

$$\boxed{dV = (r\,d\varphi)(r \operatorname{sen}\varphi\, d\theta)\, dr = r^2 \operatorname{sen}\varphi\, \underbrace{dr\, d\theta\, d\varphi}_{\text{y sus permutaciones}}}$$

Analíticamente

$$\begin{cases} x = r \operatorname{sen}\varphi \cos\theta \\ y = r \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = r \cos\varphi \end{cases} \qquad \bar{r} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \qquad \bar{u} = \begin{Bmatrix} r \\ \theta \\ \varphi \end{Bmatrix}$$

$$\underset{\sim}{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \operatorname{sen}\varphi \cos\theta & -r \operatorname{sen}\varphi \operatorname{sen}\theta & r\cos\varphi\cos\theta \\ \operatorname{sen}\varphi \operatorname{sen}\theta & r \operatorname{sen}\varphi \cos\theta & r \cos\varphi \operatorname{sen}\theta \\ \cos\varphi & 0 & -r \operatorname{sen}\varphi \end{bmatrix}$$

$$J = \det(\underset{\sim}{J}) = (-r \operatorname{sen}\varphi)\left(r \operatorname{sen}^2\varphi \cos^2\theta + r \operatorname{sen}^2\varphi \operatorname{sen}^2\theta\right)$$
$$+ (\cos\varphi)\left(-r^2 \operatorname{sen}\varphi \cos\varphi \operatorname{sen}^2\theta - r^2 \operatorname{sen}\varphi \cos\varphi \cos^2\theta\right)$$
$$= -r^2 \operatorname{sen}^3\varphi - r^2 \operatorname{sen}\varphi \cos^2\varphi = -r^2 \operatorname{sen}\varphi\left(\operatorname{sen}^2\varphi + \cos^2\varphi\right) = -r^2 \operatorname{sen}\varphi$$

$$\boxed{dV = |J|\, dr\, d\theta\, d\varphi = r^2 \operatorname{sen}\varphi\, \underbrace{dr\, d\theta\, d\varphi}_{\text{y sus permutaciones}}}$$