LINEAR ALGEBRA II

Solution to practice 3.1-3.2

Affine Geometry.

(Course 2022–2023)

Note: This is an unrevised, automatic translation from Spanish.

1.– *In the space affín* $\mathbb{R}^3$ *is considered the canonical reference* R *and the reference* $R' = \{(1,0,1);(1,1,0),(1,1,1),(1,0,0)\}$*. We denote respectively by* (x,y,z) *and* (x',y',z') *the coordinates of a point with respect to the references* R *and* R'*. Find the equations of the plane* $x+y-2z+1=0$ *at reference* R'*.* The reference change equations are:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + M_{CB} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, \qquad M_{CB} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The equation of the plane can be written as:

$$\begin{pmatrix} 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + 1 = 0.$$

We substitute using the reference change equation:

$$\begin{pmatrix} 1 & 1 & -2 \end{pmatrix} \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + M_{CB} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \right) + 1 = 0.$$

Operand remains:

$$-1 + \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + 1 = 0$$

and simplifying:

$$2x' + z' = 0.$$

2.– *In the ordinary affíny space and referred to an orthonormal system, the points* $A(2,-1,1)$*,* $B(-1,0,3)$*, the lines*

$$r : \frac{x+1}{3} = \frac{y-2}{1} = \frac{z}{1}, \qquad s : \frac{x}{1} = \frac{y}{2} = \frac{z}{-3}$$

and the planes $P : 3x - 2y + 4z + 8 = 0$, $Q : x + 5y - 6z - 4 = 0$.

The lines will be given in continuous form and the planes by their Cartesian equations. It is requested:

(a) *Line parallel to* r *that passes through* A*.* The directing vector is that of r, that is, $(3,1,1)$ and passes through $A(2,-1,1)$. Therefore its continuous equation is:

$$\frac{x-2}{3} = y + 1 = z - 1$$

(b) *Line that passes through* B *and is parallel to* P *and* Q*.* We must find the intersection of the directions of the planes P and Q. To do this we solve the system:

$$\begin{aligned} 3x - 2y + 4z &= 0 \\ x + 5y - 6z &= 0 \end{aligned}$$

We obtain $x = -8z/17$, $y = 22z/17$ and the direction ón that we are looking for is therefore the one determined by the vector $(-8, 22, 17)$. We can calculate it in another way. We know that $(3, -2, 4)$ and $(1, 5, -6)$ are the normal vectors (orthogonal) to the planes P and Q respectively. Therefore, the direction we are looking for is perpendicular to both and thus corresponds to the vector product:

$$\begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ 3 & -2 & 4 \\ 1 & 5 & -6 \end{vmatrix} = -8\bar{e}_1 + 22\bar{e}_2 + 17\bar{e}_3$$

Now the The line we are looking for is:

$$\frac{x+1}{-8} = \frac{y}{22} = \frac{z-3}{17}$$

(c) *Plane parallel to P that passes through A.* It has the same direction as P and therefore has the form

$$3x - 2y + 4z + \lambda = 0$$

We impose that it pass through the point $A(2, -1, 1)$, and we get $\lambda = -12$. The plane sought is:

$$3x - 2y + 4z - 12 = 0$$

(d) *Plane that passes through B and is parallel to r and s.* The direction ón of the plane is generated by the direction vectors of r and s, that is, $(3, 1, 1)$ and $(1, 2, -3)$. Therefore the normal (perpendicular) vector of the plane would be the vector product of both:

$$\begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ 3 & 1 & 1 \\ 1 & 2 & -3 \end{vmatrix} = -5\bar{e}_1 + 10\bar{e}_2 + 5\bar{e}_3$$

The plane would be of the form

$$-5x + 10y + 5z + \lambda = 0$$

If it passes through $B(-1, 0, 3)$, the result is $\lambda = -20$, and the plane is

$$-x + 2y + z - 4 = 0$$

Another way to calculate it directly is to take the points (x, y, z) in such a way that the vector that joins them with B is linearly dependent on the direction vectors:

$$\begin{vmatrix} x+1 & y & z-3 \\ 3 & 1 & 1 \\ 1 & 2 & -3 \end{vmatrix} = 0$$

(e) *Line perpendicular to Q and passing through A.* Since it is perpendicular to Q, the directing vector of the line is the normal to the plane Q, that is, $(1, 5, -6)$. Therefore, the requested straight line has the equation:

$$\frac{x-2}{1} = \frac{y+1}{5} = \frac{z-1}{-6}$$

(f) *Plane perpendicular to s and passing through B.* The normal vector to the plane we are looking for will be the direction vector of s, that is, $(1, 2, -3)$. The equation of the plane is of the form:

$$x + 2y - 3z + \lambda = 0$$

and assuming that $B(-1, 0, 3)$ is in the plane, $\lambda = 10 remains$:

$$x + 2y - 3z + 10 = 0$$

(g) *Line that passes through A and is perpendicular to r and to s.* If the line is perpendicular to r and s their direction vector would be the vector product of the direction vectors of r and s. We have calculated it in (d): $(-5, 10, 5)$. The equation of the line sought is:

$$\frac{x-2}{-5} = \frac{y+1}{10} = \frac{z-1}{5}$$

(h) *Plane perpendicular to P and Q and passing through B.* If the plane is perpendicular to P and Q its direction is generated by the normal vectors to both. The plane sought is then given by the equation:

$$\begin{vmatrix} x+1 & y & z-3 \\ 3 & -2 & 4 \\ 1 & 5 & -6 \end{vmatrix} = 0 \iff -8x + 22y + 17z - 59 = 0$$

(i) *Plane that contains r and is perpendicular to P.* Because it contains r its direction ón contains the director vector of r, $(3, 1, 1)$. Besides being perpendicular to P, its direction also contains the normal vector to P, $(3, -2, 4)$. Finally we take any point of r that will also be in the plane we are looking for. For example $(-1, 2, 0)$. Now the requested equation is:

$$\begin{vmatrix} x+1 & y-2 & z \\ 3 & 1 & 1 \\ 3 & -2 & 4 \end{vmatrix} = 0 \iff 6x - 9y - 9z + 24 = 0 \iff 2x - 3y - 3z + 8 = 0$$

(k) *Line that contains B, is parallel to Q and intersects r.* We will take a plane π parallel to Q and passing through B. The straight line we are looking for will pass through B and through the intersection point of r and π. First we calculate π:

$$x + 5y - 6z + \lambda = 0$$

Assuming that it passes through B, $\lambda = 19$. We now intersect such a plane with r. A point of r is of the form $(-1 + 3\mu, 2 + \mu, \mu)$. You have to verify the equation of the plane π:

$$(-1 + 3\mu) + 5(2 + \mu) - 6\mu + 19 = 0$$

We obtain $\mu = -14$ and the intersection point of r and π is $(-43, -12, -14)$. Now the requested line is the one that joins this point with B:

$$\frac{x+1}{-43+1} = \frac{y}{-12} = \frac{z-3}{-14-3} \iff \frac{x+1}{-42} = \frac{y}{-12} = \frac{z-3}{-17}$$

(m) *Line parallel to the direction given by $\bar{v}(1, 1, 2)$ and that intersects the lines r and s.* A point of r has the form $(-1 + 3a, 2 + a, a)$ and one of s of the form $(b, 2b, -3b)$. We must fix two points so that the vector that joins them is parallel to $\bar{v}(1, 1, 2)$:

$$\frac{-1 + 3a - b}{1} = \frac{2 + a - 2b}{1} = \frac{a + 3b}{2}$$

from where $a = 17/15$ and $b = 11/15$. Therefore, the line we are looking for passes through the point $(11/15, 22/15, -33/15)$ and its equation is:

$$\frac{x - 11/15}{1} = \frac{y - 22/15}{1} = \frac{z - 33/15}{2}$$

Another way to find this line is to consider the plane π defined by the direction $\bar{v}$ and the line r. Then we intersect said planes with s and we will have a point on the line we are looking for. So the equation of the plane π is:

$$\begin{vmatrix} x+1 & y-2 & z \\ 1 & 1 & 2 \\ 3 & 1 & 1 \end{vmatrix} = 0 \iff -x+5y-2z-11=0$$

Compute the intersection with s; we take a point of said straight line $(b, 2b, -3b)$. We intersect with the plane:

$$-b+10b+6b-11=0 \quad \Rightarrow \quad b = 11/15$$

we see that we obtain the previously calculated point again. Now that a point and the direction are known, the equation of the line we are looking for is immediate.

(n) *Line that passes through A, intersects s and is perpendicular to r.* The lines that pass through A and are perpendicular to r, are 'an in the plane that passes through A and is perpendicular to r. Said plane has as normal vector the director of r, $(3, 1, 1)$. So your equation would be:

$$3x+y+z+\lambda=0$$

Assuming that $A(2, -1, 1)$ verify the equation, we obtain $\lambda = -6$. Now, since the straight line we are looking for has to intersect s, we calculate the intersection of s with this plane. A point in s is of the form $(a, 2a, -3a)$. Substituting in the equation of the plane:

$$3a+2a-3a-6=0$$

we see that $a = 3$. Therefore, the line we are looking for is the one that joins the points $A(2, -1, 1)$ and $(3, 6, -9)$:

$$\frac{x-2}{3-2} = \frac{y+1}{6+1} = \frac{z-1}{-9-1} \iff x-2 = \frac{y+1}{7} = \frac{z-1}{-10}$$

(o) *Plane perpendicular to P, parallel to r and passing through A.* The direction vectors of the plane we are looking for are the normal to P, $(3, -2, 4)$ and the principal of r, $(3, 1, 1)$. Furthermore, it must pass through $A(2, -1, 1)$. Therefore, the requested equation is given by the cancellation of the determinant:

$$\begin{vmatrix} x-2 & y+1 & z-1 \\ 3 & -2 & 4 \\ 3 & 1 & 1 \end{vmatrix} = 0 \iff -6x+9y+9z+12=0 \iff 2x-3y-3z-4=0$$

(q) *Distances from A to B, from A to r, from B to P and from r to s .* The distance from A to B is obtained directly:

$$d(A, B) = \sqrt{(-1-2)^2+(0+1)^2+(3-1)^2} = \sqrt{14}$$

The distance from A to r is obtained by the formula:

$$d(A, r) = \frac{||\overline{AC} \wedge \bar{v}||}{\|\bar{v}\|}$$

where C is a point of r and v its direction vector. In this case, $C(-1, 2, 0)$, $\bar{v}(3, 1, 1)$.

$$d(A, r) = \frac{\|(-3, 3, -1) \wedge (3, 1, 1)||}{\|(3, 1, 1)|} = \frac{4\sqrt{13}}{\sqrt{11}} = \frac{4\sqrt{143}}{11}$$

Observation: If we do not remember the formula, we you can calculate the plane perpendicular to r passing through A. The intersection of said plane with r gives us a point M, which is the projection of A onto r. The requested distance is the distance between A and M. The distance from B to P is given by the expression:

$$d(B, P) = \frac{3 \cdot (-1) - 2 \cdot 0 + 4 \cdot 3 + 8}{\sqrt{3^2+(-2)^2+4^2}} = \frac{17}{\sqrt{29}} = \frac{17\sqrt{29}}{29}$$

Observation: Again if we want to reduce the calculation to the distance between two points, we find the intersection point of the line perpendicular to the plane and that passes through B with the plane P . Finally, to calculate the distance between r and s, we find the intersection point of the perpendicular to both with one of the lines. We saw in section (p) that the cut point with s is $(7/30, 7/15, -7/10)$. Now the requested distance is from this point to the line r:

$$d(r,s) = \frac{\|(7/30+1, 7/15-2, -7/10) \wedge (3,1,1)\|}{\|(3,1,1)|} = \frac{5\sqrt{66}/6}{\sqrt{11}} = \frac{5\sqrt{6}}{6}$$

Another way is to directly use the formula:

$$d(r,s) = \frac{|[(-1,2,0)-(0,0,0),(3,1,1),(1,2,-3)]|}{\|(3,1,1)\wedge(1,2,-3)\|} = \frac{5\sqrt{6}}{6}$$

(r) *Ángles formed by r and s, by s and Q and by P and Q.* The ángle formed by r and s is the ángle formed by their directions:

$$cos(\alpha(r,s)) = \frac{|((3,1,1)(1,2,-3))|}{\|(3,1,1)\|\|(1,2,-3)\|} = \frac{2}{\sqrt{11}\sqrt{14}} \quad \Rightarrow \quad \alpha(r,s) = 80,72^o$$

The ángle formed by s and Q is the complement of the one formed by the normal direction to Q and the address of s. Therefore:

$$sin(\alpha(s,Q)) = \frac{|((1,2,-3)(1,5,-6))|}{\|(1,2,-3)\|\|(1,5,-6)\|} = \frac{29}{\sqrt{14}\sqrt{62}} \quad \Rightarrow \quad \alpha(s,Q) = 79.84^o$$

The ángle formed by P and Q is the one formed by their normal directions:

$$cos(\alpha(P,Q)) = \frac{|((3,-2,4)(1,5,-6))|}{\|(3,-2,4)\|\|(1,5,-6)\|} = \frac{31}{\sqrt{29}\sqrt{62}} \quad \Rightarrow \quad \alpha(P,Q) = 43,02^o$$

3.– *In the space affín give the equationón of a line passing through the point $P = (1,0,1)$ and cutting perpendicular to the line:*

$$s \equiv \begin{cases} y+z=4 \\ x+3y=11 \end{cases}$$

Also find the distance between P and s. We will calculate the plane π perpendicular to s and passing through the point P, which contains the line sought. Then we find $Q = \pi \cap S$ and finally the ideal line is the line PQ. The parametric equations of s, solving the system formed by its implicit equations are:

$$x = -1+3t, \quad y = 4-t, \quad z = t.$$

Therefore the direction vector of s is $u_s = (3,-1,1)$ and a plane perpendicular to it is of the form $3x-y+z+d=0$. We impose that it passes through $P=(1,0,1)$, $3\cdot 1 - 0 + 1 + d = 0$, from where , $d=-4$ and $\pi \equiv 3x-y+z-4$ =0. Now to find $Q = \pi \cap S$ we substitute the parametric equations of S in the plane π:

$$3(-1+3t)-(4-t)+t-4=0 \iff 11t-11=0 \iff t=1.$$

Then $Q = (-1+3\cdot 1, 4-1, 1) = (2,3,1)$. The requested line is the line PQ:

$$\frac{x-1}{2-1} = \frac{y-0}{3-0} = \frac{z-1}{1-1} \iff \begin{cases} 3x-y-3=0 \\ z-1=0 \end{cases}$$

The distance from point P to line s is precisely the distance between points P and Q:

$$d(P,s) = d(P,Q) = \sqrt{(2-1)^2+(3-0)^2+(1-1)^2} = \sqrt{10}.$$

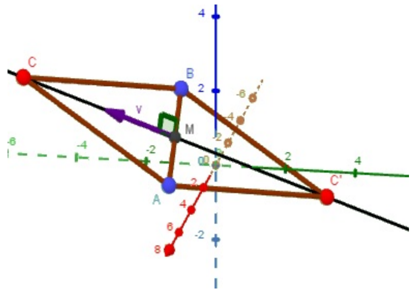
3.– *In the affín space* $\mathbb{R}^3$ *an isósceles triangle* ABC *with únequal angle in the vertices* C*. It is known that* $x+y+z=1$*,* $A=(2,-1,0)$*,* $B=(0,-1,2) is contained in the plane of equation$ *and has an area of* $4\sqrt{3}$*.*

(i) *Calculate the coordinates of vertex* C*.* (1 point) Since the triángle is isósceles with únequal angle in C, t́his belongs to bisector r of side AB. T́his is the line that passes through the midpoint M of A and B, is perpendicular to $\vec{AB}$ and is contained in the given plane. Also $h = \|\vec{MC}\|$ satisfies:

$$4\sqrt{3} = \text{área} = \frac{\|\vec{AB}\|\cdot h}{2} = \frac{\|(2,0,2)\|\cdot h}{2} \quad\Rightarrow\quad h = 2\sqrt{6}.$$

Therefore, if $\vec{v}$ is the director vector of perpendicular bisector r, the point C can be obtained by adding to M a vector in the direction of $\vec{v}$ and the same sense or the opposite and module h:

$$C = M + h\frac{\vec{v}}{\|\vec{v}\|} \text{ ó } C = Mh\frac{\vec{v}}{\|\vec{v}\|}.$$



Let's go with the calculations. The midpoint M is:

$$M = \frac{A+B}{2} = \frac{(2,-1,0)+(0,-1,2)}{2} = (1,-1,1)$$

The vector $\vec{v}$ director of the bisector is a vector (a,b,c) fulfilling: i) It is perpendicular to $\vec{AB} = BA = (-2,0,2)$:

$$(a,b,c)\cdot(-2,0,2) = 0 \quad\Longleftrightarrow\quad -2a+2c=0 \quad\Longleftrightarrow\quad ac=0$$

ii) is in the plane $x+y+z=1$. It is perpendicular to its normal vector $(1,1,1)$:

$$(a,b,c)\cdot(1,1,1) = 0 \quad\Longleftrightarrow\quad a+b+c=0$$

Solving the system formed by the two equations yields $a=c$, $b=-2a$. Taking $a=1$ we can choose $\vec{v}=(1,-2,1)$. So a possible solution for C is:

$$C = M + h\frac{\vec{v}}{\|\vec{v}\|} = (1,-1,1) + 2\sqrt{6}\frac{(1,-2,1)}{\|(1,-2,1)\|} = (3,-5,3)$$

Another seriesía:

$$C = Mh\frac{\vec{v}}{\|\vec{v}\|} = (1,-1,1) - 2\sqrt{6}\frac{(1,-2,1)}{\|(1,-2,1)\|} = (-1,3,-1)$$

(ii) *Calculate the volume of the triangular pyramid that has as its base the tri ángle* ABC *and vertex the origin.* (0.5 points) The volume is the área of the base (which is a given value) times the height H of the pyramid divided by three. The height is the distance from the vertex $(0,0,0)$ to the plane that contains the base, that is, to the plane $x+y+z-1=0$. We use the formula for the distance from a point to a plane:

$$H = \frac{|0+0+0-1|}{\|(1,1,1)\|} = \frac{1}{\sqrt{3}}$$

Then:

$$Vol = \frac{base\cdot H}{3} = \frac{4\sqrt{3}\cdot\frac{1}{\sqrt{3}}}{3} = \frac{4}{3}.$$

(iii) *Find the equations of a symmetry with respect to a line that transforms point A into point B and leaves C fixed.* (1 point) The line of symmetry has to be precisely the bisector of the segment A and B that passes through C. We saw in section (i) that it is the line that passes through $M = (1, -1, 1)$ and has a directing vector $\vec{v} = (1, -2, 1)$. Then the equations of symmetry are $f(P) = M + t(PM)$ where t is the symmetry with respect to the vector line generated by $\vec{v}$ y M is a fixed point of the transformation; matrix:

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + T_C \begin{pmatrix} x-1 \\ y+1 \\ z-1 \end{pmatrix}$$

where T_C is the matrix of symmetry with respect to the subspace $\mathcal{L}\{(1, -2, 1)\}$. We calculate this matrix. We build a base B with the vector that generates the line of symmetry and two other perpendiculars to it; that is, two independent vectors fulfilling:

$$(x, y, z) \cdot (1, -2, 1) = 0 \quad \iff \quad x - 2y + z = 0.$$

For example:

$$B = \{(1, -2, 1), (1, 0, -1), (1, 1, 1)\}$$

In this base the symmetry matrixía is:

$$T_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We make the base change:

$$T_C = M_{CB} T_B M_{CB}^{-1}, \qquad \text{where } M_{CB} = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

Operand remains:

$$T_C = \frac{1}{3}\begin{pmatrix} -2 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & -2 \end{pmatrix}$$

And the equation of symmetry is:

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -2 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & -2 \end{pmatrix} \begin{pmatrix} x-1 \\ y+1 \\ z-1 \end{pmatrix}$$

5.– *In space afín $\mathbb{R}^3$ is considered a scalar product whose Gram matrix with respect to the canonical base is:*

$$G_C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Calculate the distance between the point $(-4, 1, 0)$ and the line r of equations:

$$r \equiv \begin{cases} x + y - z = 1 \\ x - y + z = 3 \end{cases}$$

Although we know that there is a direct formula to find the distance from a point to a plane in space, it is not advisable to use it because it uses the vector product and the usual calculation method of the same it is not valid under scalar products other than the usual one. Then, to calculate the distance, we will directly find a point Q on the line r so that the vector that joins it with $P = (-4, 1, 0)$ is orthogonal to the line. In that case we know that $d(P, r) = d(P, Q)$. We begin by finding the parametric equations of the line by solving the system formed by its two implicit equations as a function of a parameter. Adding them together we get $x = 2$ and after the first $y = z - 1$. We are left with:

$$x = 2, \quad y = \lambda - 1, \quad z = \lambda.$$

We also see that the direction vector of the line is $(0, 1, 1)$. Now a point Q on the line would be:

$$Q = (2, \lambda - 1, \lambda)$$

We impose that PQ be orthogonal to $(0, 1, 1)$:

$$PQ = QP = (6, \lambda - 2, \lambda)$$

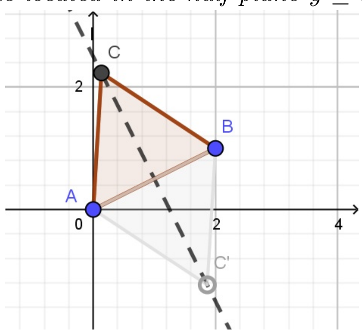
y

$$0 = PQ \cdot (0, 1, 1) = (6, \lambda - 2, \lambda)G_C(0, 1, 1)^t = 6\lambda$$

We deduce that $\lambda = 0$ and:

$$d(P, r) = d(P, Q) = \|PQ\| = \|(6, -2, -0)\| = \sqrt{(6, -2, 0)G_C(6, -2, 0)} = \sqrt{20} = 2\sqrt{5}.$$

7.– *In plane afín euclídeo* $\mathbb{R}^2$ *let* A, B, C *be the vertices of an equilateral triangle located in the half-plane* $y \geq 0$, *with* $A = (0, 0)$ *and* $B = (2, 1)$.



(i) *Calculate the coordinates of* C. Let (a, b) the coordinates of C. Since it is an equilateral triangle, it must be true that:

$$d(A, B) = d(A, C) = d(B, C)$$

Where:

$$\begin{aligned} d(A, B) &= \sqrt{(2-0)^2 + (1-0)^2} = \sqrt{5} \\ d(A, C) &= \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2} \\ d(B, C) &= \sqrt{(a-2)^2 + (b-1)^2} = \sqrt{a^2 + b^2 - 4a - 2b + 5} \end{aligned}$$

Equating distances and squaring we obtain the equations:

$$\begin{aligned} a^2 + b^2 &= 5 \\ a^2 + b^2 - 4a - 2b + 5 &= 5 \end{aligned}$$

Subtracting $4a + 2b = 5$, that is, $b = (5 - 4a)/2$. Substituting into the first equation:

$$a^2 + \left(\frac{5 - 4a}{2}\right)^2 = 5$$

Operating and simplifying:

$$4a^2 - 8a + 1 = 0$$

where $a = 1 \pm \dfrac{\sqrt{3}}{2}$; Using that $b = (5 - 4a)/2$, we get two possible solutions:

$$C = (1 + \sqrt{3}/2, 1/2 - \sqrt{3}) \qquad \text{ór} \qquad C = (1 - \sqrt{3}/2, 1/2 + \sqrt{3}).$$

As the triangle must be in the half plane $y \geq 0$ the only possibility is:

$$C = (1 - \sqrt{3}/2, 1/2 + \sqrt{3}).$$

(ii) *Find the equations of a symmetry with respect to a line that has vertex A in B.* The axis of symmetry is the bisector of a point and its image, that is, the line that passes through the midpoint of both and is orthogonal to the segment that joins them. To find the equations of symmetry we first need any point on the axis. We can take the midpoint of A and B:

$$M = \frac{A + B}{2} = (2, 1/2).$$

The equations are then:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1/2 \end{pmatrix} + T_C \begin{pmatrix} x - 2 \\ y - 1/2 \end{pmatrix}$$

where T_C is the matrix of the orthogonal transformation corresponding to a symmetry with respect to the subspace generated by the director vector of the axis $\vec{u}$ To compute T_C we first work on an auxiliary basis $B = \{\vec{u}, \vec{v}\}$ where $\vec{v}$ is orthogonal to $\vec{u}$. As we have indicated at the beginning, a vector orthogonal to the axis is the vector $\vec{AB} = (2, 1) = \vec{v}$. We impose that $\vec{u} = (x, y)$ is perpendicular to $\vec{v}$:

$$(x, y) \cdot (2, 1) = 0 \iff 2x + y = 0 \iff y = -2x$$

We take $x = 1$ and $y = -2$, that is, $\vec{u} = (1, -2)$. In the base $B = \{(1, -2), (2, 1)\}$ the symmetry matrixía is:

$$T_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We change the base to the canonical one:

$$T_C = M_{CB} T_B M_{CB}^{-1}, \qquad \text{where } M_{CB} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

Operand results in:

$$T_C = \frac{1}{5}\begin{pmatrix} -3 & -4 \\ -4 & 3 \end{pmatrix}$$

and finally the equations of symmetry:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1/2 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} -3 & -4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} x - 2 \\ y - 1/2 \end{pmatrix}$$

8.– *In the Euclidean affín plane find the equations of a symmetryí What does the line $x = 0$ lead to in the line $3x + 4y - 4 = 0$. ¿Is the solution unique?.* The axis of symmetry is equidistant from the original line and its symmetry. Equivalently it is any of the two bisectors of the two ángles that t́hese determine. So it is clear that there are two possible solutions. To find the bisectors we equate the distances from an arbitrary point to both lines:

$$\frac{|x|}{\sqrt{1^2 + 0^2}} = \frac{|3x + 4y - 4|}{\sqrt{3^2 + 4^2}}$$

Simplifying:

$$5|x| = |3x + 4y - 4|$$

from where either:

$$5x = 3x + 4y - 4 \iff 2x - 4y + 4 = 0 \iff x - 2y + 2 = 0,$$

or:

$$-5x = 3x + 4y - 4 = 0 \iff 8x + 4y - 4 = 0 \iff 2x + y - 1 = 0.$$

Let us calculate, for example, the equations of symmetry with respect to the line $2x + y - 1 = 0$. We choose any point of the same (verifying the equation that defines it) for example $P = (0, 1)$. The equations of symmetryía are then:

$$t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + T_c \begin{pmatrix} x \\ y-1 \end{pmatrix}$$

where Tc is the symmetry matrix. To find the matrix we take a base B where the first vector is the director of the axis of symmetry and the second is orthogonal to it. The normal vector of the axis is $(2, 1)$ and therefore the director $(1, -2)$. We then consider the base B:

$$B = \{(1, -2), (2, 1)\}$$

In this base we know that:

$$T_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We pass it to the canonical base:

$$T_C = M_{CB} T_B (M_{CB})^{-1})$$

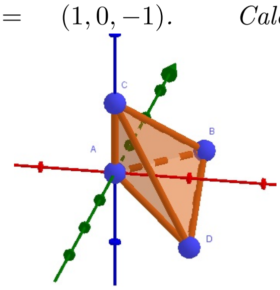
where $M_{CB} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Operand results in:

$$T_C = \begin{pmatrix} -3/5 & -4/5 & \\ -4/5 & 3 & 5 \end{pmatrix}.$$

Finally, the equation of symmetry is:

$$t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3/5 & -4/5 & \\ -4/5 & 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y-1 \end{pmatrix}$$

9.– *In the space affín euclídeo* $\mathbb{R}^3$ *let the tetrahedron of vertices* $A = (0, 0, 0)$, $B = (1, 1, 0)$, $C = (0, 0, 1)$ *and* $D = (1, 0, -1)$. *Calculate its área and its volume.*



By default we work with the usual scalar product. In this case we know that the volume of the tetradron defined by three vectors is given by:

$$Vol = \frac{1}{6}|[\vec{A}B, \vec{A}C, \vec{A}D]| = \frac{1}{6}\left\|\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix}\right\| = \frac{1}{6}.$$

The área is the sum of the áreas of each of the four faces. These are triangles. We will use that the área of a triangle given by two vectors $\vec{u}$ and $\vec{v}$ is $\frac{1}{2}\|\vec{u}\times\vec{v}\|$. Face ABC:

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{e}_1 - \vec{e}_2 = (1,-1,0).$$

y

$$area(ABC) = \frac{1}{2}\|(1,-1,0)\| = \frac{1}{2}\sqrt{1^2+(-1)^2+0^2} = \frac{1}{2}\sqrt{2}.$$

Face ABD:

$$\overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -\vec{e}_1 + \vec{e}_2 - \vec{e}_3 = (-1,1,-1).$$

y

$$area(ABD) = \frac{1}{2}\|(-1,1,-1)\| = \frac{1}{2}\sqrt{(-1)^2+(1)^2+(-1)^2} = \frac{1}{2}\sqrt{3}.$$

Face ACD:

$$\overrightarrow{AC}\times\overrightarrow{AD} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \vec{e}_2 = (0,1,0).$$

and

$$area(ACD) = \frac{1}{2}\|(0,1,0)\| = \frac{1}{2}.$$

Face BCD:

$$\overrightarrow{BC}\times\overrightarrow{BD} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ -1 & -1 & 1 \\ 0 & -1 & -1 \end{vmatrix} = 2\vec{e}_1 - \vec{e}_2 + \vec{e}_3 = (2,-1,1).$$

y

$$area(BCD) = \frac{1}{2}\|(2,-1,1)\| = \frac{1}{2}\sqrt{(2)^2+(-1)^2+(-1)^2} = \frac{1}{2}\sqrt{6}.$$

The t́otal area is:

$$\frac{1}{2}(\sqrt{2}+\sqrt{3}+1+\sqrt{6}).$$

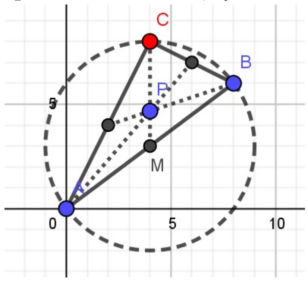
10.– *In the space affín* $\mathbb{R}^2$ *we consider the points* $A=(0,0)$*,* $B=(8,6)$*.*

(i) *Calculate the implicit equation of the locus of points in the plane* C *that make* AB *the hypotenuse of a right triangle* ABC*.* (0.9 points) Let $C=(x,y)$ be the coordinates of point C. For the triangle ABC to be a right angle with hypotenuse AB, the vectors $\vec{AC}$ and $\vec{BC}$ must be orthogonal:

$$\vec{AC}\cdot\vec{BC}=0 \iff (x-0,y-0)\cdot(x-8,y-6)=0 \iff x^2-8x+y^2-6y=0 \iff (x-4)^2+(y-3)^2=5^2.$$

It is a circle with center $(4,3)$ and radius 5.

(ii) *If* $P = (4,14/3)$ *is the centroid of one of the triangles indicated in the previous section, find its área and perímeter.* (0.9 points)



The centroid is the intersection of the medians, that is, the lines that join each vertex with the point middle of the opposite side. The midpoint of side AB is:

$$M = \frac{A+B}{2} = \frac{(0,0)+(8,6)}{2} = (4,3).$$

The vertex C belongs to the median MP and therefore we can find it as the intersection of such line and the locus calculated in 1. The straight line MP has the equation $x-4=0$. We substitute in the equation of the conic y:

$$(4-4)^2 + (y-3)^2 = 25 \quad \Rightarrow \quad y = 3 \pm 5$$

Given that the centroid is 'a in the upper half-plane $y>0$ and así $C=(4,3+5)=(4,8)$. So the sides of the triangle measure:

$$AB = \sqrt{(8-0)^2+(6-0)^2} = 10, \quad AC = \sqrt{(4-0)^2+(8-0)^2} = 4\sqrt{5}, \quad BC = \sqrt{(8-4)^2+(6-8)^2} = 2\sqrt{5}.$$

From where:

$$\text{perímeter} = AB + AC + BC = 10 + 6\sqrt{5}$$

and since the area is a rectangle:

$$\text{área} = \frac{AC \cdot BC}{2} = 20.$$

(iii) *Find the equations of a dilation that takes point A to point B and point B to point A.* (0.7 points) Given Since the dilation interchanges the points A and B, the length of the segment AB remains invariant. Therefore the reason has to satisfy $|k|=1$. Since the dilation is not the identity, $k=-1$. If Q is the center, the following must be true:

$$B = f(A) = Q - 1(AQ) \quad \Rightarrow \quad 2Q = A + B \quad \Rightarrow \quad Q = \frac{A+B}{2} = M = (4,3).$$

It is then a dilation of center $(4,3)$ and ratio -1. Its equations are:

$$f(x,y) = (4,3) - ((x,y)-(4,3)) = (8-x, 6-y).$$

11.– *In the affín space we consider a regular pyramid with a square base. We denote by A,B,C,D the four vertices of the base and by E the upper vertex. Knowing that the base is contained in the plane $z=0$, that $A=(0,0,0)$ and $C=(4,2,0)$ are opposite vertices of the base and that the height of the pyramid is 5 calculate:*

(i) *The coordinates of the three remaining vertices.* The length of the diagonal d of the square is the distance between A and C:

$$d = d(A,C) = \sqrt{(4-0)^2+(2-0)^2+(0-0)^2} = \sqrt{20}.$$

By the Pitágoras Theorem the side l of the square satisfies:

$$l^2 + l^2 = d^2 \quad \Rightarrow \quad l = \frac{d}{\sqrt{2}} = \sqrt{10}.$$

The vertices B and D are distant from C and A by the distance l. Furthermore, by estra in the plane $z=0$ they are of the form $(a,b,0)$. Therefore we set the equations:

$$d(B,A) = \sqrt{10} \iff \sqrt{(a-0)^2+(b-0)^2+(0-0)^2} = \sqrt{10} \iff a^2+b^2 = 10$$
$$d(B,C) = \sqrt{10} \iff \sqrt{(a-4)^2+(b-2)^2+(0-0)^2} = \sqrt{10} \iff a^2 - 8a + 16 + b^2 - 4b + 4 = 10$$

Subtracting the equations:

$$8a + 4b = 20 \quad \Rightarrow \quad b = 5 - 2a$$

Substituting in $a^2 + b^2 = 10$:

$$a^2 + (5-2a)^2 = 10 \iff 5a^2 - 20a + 15 = 0 \iff a^2 - 4a + 3 = 0 \iff$$
$$\iff a = \frac{4 \pm \sqrt{4^2 - 12}}{2} \iff a = 1 \text{ ó } a = 3.$$

If $a = 1$ then $b = 5 - 2 \cdot 1 = 3$ and if $a = 3$ then $b = 5 - 2 \cdot 3 = -1$. The vertices B and D are $B = (1,3,0)$ and $D = (3,-1,0)$. The vertex E is on the midpoint M of the base, which is the midpoint of A and D and at height 5:

$$M = \frac{A + C}{2} = (2,1,0).$$

Given that the vector $(0,0,1)$ is unitary and perpendicular to the base

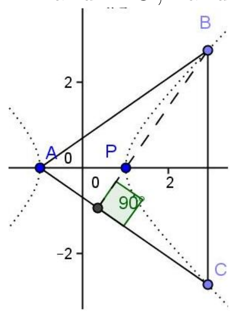
$$E = M \pm 5(0,0,1) = (2,1,\pm 5).$$

(there are two possible solutions for vertex E).

(ii) *The volume of the pyramid.* The volume of a pyramid is:

$$V = \frac{\text{báse area} \cdot \text{ height}}{3} = \frac{l^2 \cdot h}{3} = \frac{10 \cdot 5}{3} = \frac{50}{3}.$$

13.– *In the plane affín let be the points $A(-1,0)$ and $P(1,0)$. Calculate the implicit equation of the locus of points B in the plane such that AB is one of the equal sides of an isosceles triangle whose orthocenter is P. ¿What kind of curve is it?.+* There are two possible interpretations: - The equal sides are AB and AC, and therefore the únequal angle is A:



In an isosceles triangle, the line that joins the vertex of the unequal angle with the orthocenter is an axis of symmetry. In our case, such a straight line, the one that joins A and P, is the OX axis. If B has coordinates (x,y) its symmetric with respect to the axis OX is the point C, $(x,-y)$, so that the three vertices of the triangle are ABC. For P to be the orthocenter, the side AC must be perpendicular to the height of the vertex B, that is, to the vecotr BP. Therefore:

$$\vec{AC} \cdot \vec{BP} = 0$$

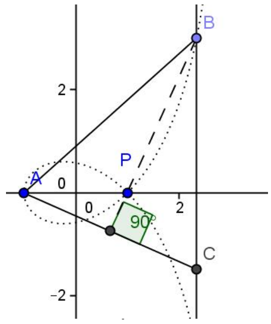
that is:

$$(x - (-1), -y - 0) \cdot (1 - x, 0 - y) = 0 \iff (1 + x)(1 - x) + y^2 = 0 \iff x^2 - y^2 = 1$$

We see that the equation of the requested locus corresponds to the hipérbola:

$$x^2 - y^2 = 1.$$

- The equal sides are AB and BC and therefore the únequal angle B:



The height on the side BC is the line that joins A with the orthocenter; therefore BC is perpendicular to the axis OX. Thusí if B has coordinates (x, y) then C has coordinates (x, y'). Since the distance AB is the same as BC:

$$(x+1)^2 + y^2 = (yy'^2) \quad \iff \quad (x+1)^2 = y'^2 - 2yy'$$

On the other hand, the line BP has to be perpendicular to the side AC, that is:

$$\vec{AC} \cdot \vec{BP} = 0 \iff (x-(-1), -y-0) \cdot (1-x, 0-y') = 0 \iff (1+x)(1-x) + yy' = 0 \iff x^2 - 1 + yy' = 0$$

Solving y' in the second equation and substituting in the first one we have:

$$(x+1)^2 = \frac{(1-x^2)^2}{y^2} - 2(1-x^2)$$

Dividing by $x+1$:

$$(x+1) = \frac{(x^2-1)(x-1)}{y^2} + 2(x-1)$$

Removing denominators and simplifying we obtain

$$x^3 - x^2 - x + 1 - xy^2 - 3y^2 = 0,$$

which is the equation of a c´locate.

14.– *In the plane affín consider the circumference $c : (x-3)^2 + y^2 = 3^2$ and the line $r : y - 3 = 0$. For each line h passing through the origin, let A be the point of intersection (other than the origin) of c and h and B be the point of intersection of r and h. Calculate the implicit equation of the locus of points of intersection of the parallel to the OX axis by A and the parallel to the OY axis by B.* The circle has center $(3, 0)$ and radius 3; the simplified equation is: $x^2 - 6x + y^2 = 0$. We take an arbitrary line h passing through the origin $x = ay$ (we note that the line $y = 0$ need not be considered because it does not intersect r). We find the cut point between c and h.

$$\begin{cases} 0 = x^2 - 6x + y^2 \\ x = ay \end{cases}$$

Remains:

$$a^2y^2 - 6ay + y^2 = 0 \iff y((1+a^2)y - 6a) = 0$$

from where $y = 0$ ó $y = \dfrac{6a}{1+a^2}$. We are left with the different cut from the origin. Since $x = ay$ is:

$$A = \left(\frac{6a^2}{1+a^2}, \frac{6a}{1+a^2} \right).$$

Now we cut r and h:

$$\begin{cases} 0 = y - 3 \\ x = ay \end{cases}$$

Remains:

$$B = (3a, 3).$$

Parallel to the OX axis through A is $y = \dfrac{6a}{1+a^2}$. Parallel to the OY axis through B is $x = 3a$. And the intersection of both:

$$x = 3a, \qquad y = \frac{6a}{1+a^2}$$

which are the parametric equations of the locus. To find the implicit one, we clear the parameter a in the first one and we substitute in the second one. It remains:

$$y = \frac{2x}{1+(x/3)^2} \iff y = \frac{18x}{x^2+9} \iff yx^2 + 9y - 18x = 0.$$

16.– In the Euclidean affinity space and with respect to the canonical reference, the points $A = (1, 0, 0)$ and $B = (1, 2, 2)$.

(i) *Find the locus of points that are equidistant from A and B.* **Method I:** A point $P = (x, y, z)$ equidistant from A and B if:

$$d(P, A) = d(P, B) \iff \sqrt{(x-1)^2 + y^2 + z^2} = \sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2}$$

Squaring and operating results in:

$$x^2 - 2x + 1 + y^2 + z^2 = x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 - 4z + 4$$

Simplifying we obtain the plane of equation:

$$y + z - 2 = 0.$$

Method II: The points that are equidistant from two dice in space are those that lie on the bisector plane; that is, the plane that passes through the midpoint and is perpendicular to the segment that joins them. We have:

$$M = \frac{A+B}{2} = (1, 1, 1).$$

The vector $\vec{AB} = (0, 2, 2)$ is perpendicular to the searched plane. Equivalently the vector $(0, 1, 1)$ is the normal vector of such a plane. Therefore ́this is of the form:

$$y + z + d = 0. We impose$$

to pass through M:

$$1 + 1 + d = 0 \quad \Rightarrow \quad d = -2.$$

Thusí the requested locus is:

$$y + z - 2 = 0.$$

(ii) *Find the coordinates of a point C in the plane $z = 2$, such that the triangle ABC is isosceles with AB the unequal side and have área $2\sqrt{3}$. ¿Is the solution unique?.* Being in the plane $z = 2$, the point has the form $C = (a, b, 2)$. Now to form an isosceles triangle with AB, equidistant from A and B and therefore it is in the geometric place found in the previous section:

$$b + 2 - 2 = 0 \Rightarrow b = 0.$$

That is, $C = (a, 0, 2)$. Finally $Area(ABC) = 2\sqrt{3}$. But:

$$Area(ABC) = \frac{base \cdot h}{2} = \frac{\|\vec{AB}\|\|\vec{MC}\|}{2} = \frac{\sqrt{0^2+2^2+2^2}\sqrt{(a-1)^2+(-1)^2+1^2}}{2} = \sqrt{2((a-1)^2+2)}$$

Equating $Area(ABC) = 2\sqrt{3}$ is:

$$2\sqrt{3} = \sqrt{2((a-1)^2+2)} \quad \Rightarrow \quad 6 = (a-1)^2 + 2 \quad \Rightarrow \quad (a-1)^2 = 4 \quad \Rightarrow \quad (a-1)^2 = \pm 2.$$

We see that there are two solutions (the solution 'on is therefore not únique):

$$a = 3 \text{ ó } a = -1.$$

That is:

$$C = (3, 0, 2) \text{ ó } C = (-1, 0, 2).$$
