LINEAR ALGEBRA II — Solutions to Problem Set 2.3

Orthogonal transformations — (Academic Year 2022–2023)

Note: This is an unrevised, automatic translation.

1.– *The Euclidean vector space $\mathbb{R}^3$ referred to an orthonormal basis is considered. Get the matrix expression in this base of:*

(a) *the orthogonal symmetry with respect to the subspace $\mathcal{L}\{(1,1,1)\}$*

We compute a basis of the subspace orthogonal to $\mathcal{L}\{(1,1,1)\}$:

$$(x,y,z)\cdot(1,1,1)=0 \quad\Leftrightarrow\quad x+y+z=0 \quad\Rightarrow\quad \mathcal{L}\{(1,1,1)\}^{\perp}=\mathcal{L}\{(1,-1,0),(1,0,-1)\}$$

The symmetry matrix in the base $B=\{(1,1,1),(1,-1,0),(1,0,-1)\}$ is:

$$S_{BB}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$$

(**Observation**: For *symmetries* it is not necessary that the base used be *orthonormal*, it is enough to take any base formed by arbitrary bases of the direction with respect to which the symmetry and its orthogonal subspace. However, for the *rotations*, choose a base *orthonormal*.)

To calculate the matrix in the initial base we make the base change:

$$\begin{aligned}S_{CC}&=M_{CB}S_{BB}M_{CB^{-1}}=\\&=\begin{pmatrix}1&1&1\\1&-1&0\\1&0&-1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}1&1&1\\1&-1&0\\1&0&-1\end{pmatrix}^{-1}=\begin{pmatrix}-1/3&2/3&2/3\\2/3&-1/3&2/3\\2/3&2/3&-1/3\end{pmatrix}\end{aligned}$$

2.– *In the Euclidean vector space $\mathbb{R}^3$ find the equations of a rotation of angle 90^o and semiaxis generated by the vector $(3,0,4)$.*

First we build a well oriented orthogonal basis with the first vector the semi-axis of rotation $\vec{u}_1=(3,0,4)$.

We look for a second vector orthogonal to the first:

$$(x,y,z)\cdot(3,0,4)=0 \iff 3x+4z=0.$$

Take for example $\vec{u}_2=(0,1,0)$. We choose a vector orthogonal to the previous two:

$$\begin{aligned}(x,y,z)\cdot(3,0,4)=0 &\iff 3x+4z=0\\(x,y,z)\cdot(0,1,0)=0 &\iff y=0\end{aligned}$$

We take $\vec{u}_3=(4,0,-3)$.

We check if the base $\{\vec{u}_1,\vec{u}_2,\vec{u}_3\}$ is well oriented, studying the sign of the determinant of the change of base matrix with respect to the canonical base:

$$\begin{vmatrix}3&0&4\\0&1&0\\4&0&-3\end{vmatrix}=-25<0$$

Therefore it is not well oriented. We change the sign of the third vector to correct it and normalize:

$$\frac{(3,0,4)}{\|(3,0,4)\|}=(3/5,0,4/5),\quad \frac{(0,1,0)}{\|(0,1,0)\|}=(0,1,0),\quad \frac{(-4,0,3)}{\|(-4,0,3)\|}=(-4/5,0,3/5)$$

Then the basis $B=\{(3/5,0,4/5),(0,1,0),(-4/5,0,3/5)\}$ is a well-oriented orthonormal basis with the first vector in the same direction and sense as the semi-axis.

On that basis the rotation matrix is:

$$T_B=\begin{pmatrix}1 & 0 & 0\\ 0 & cos(90^o) & -sin(90^o)\\ 0 & sin(90^o) & cos(90^o)\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}.$$

Finally we change the matrix to the canonical basis:

$$T_C=M_{CB}T_BM_{BC}=M_{CB}T_BM_{CB}^{-1}=M_{CB}T_BM_{CB}^t$$

where

$$M_{CB}=\frac{1}{5}\begin{pmatrix}3 & 0 & -4\\ 0 & 5 & 0\\ 4 & 0 & 3\end{pmatrix}$$

and $M_{CB}^{-1}=M_{CB}^t$ for being a base change matrix between orthonormal bases.

Operating results:

$$T_C=\frac{1}{25}\begin{pmatrix}9 & -20 & 12\\ 20 & 0 & -15\\ 12 & 15 & 16\end{pmatrix}.$$

And the equations of rotation are:

$$t\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\frac{1}{25}\begin{pmatrix}9 & -20 & 12\\ 20 & 0 & -15\\ 12 & 15 & 16\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}.$$

6.– *In the vector space* $\mathbb{R}^3$ *the usual scalar product and the positive orientation given by the canonical basis are considered:*

(ii) *Calculate the matrix associated with a symmetry with respect to the equation plane* $x+y-z=0$. (1 point)

We begin by calculating a basis formed by the generators of the plane of symmetry and a vector orthogonal to it. The latter can be the normal vector to the plane, formed by the coefficients of the implicit equation $(1,1,-1)$.

The other two are obtained by parametrically solving the given equation:

$$x+y-z=0 \quad\Rightarrow\quad y=-x+z$$

where from:

$$x=a,\quad y=-a+b,\quad z=b$$

and the generators are $(1,-1,0)$ and $(0,1,1)$. So we take:

$$B=\{\underbrace{(1,-1,0),(0,1,1)}_{plane},\underbrace{(1,1,-1)}_{plane^{\perp}}\}$$

On such basis:

$$T_B=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$$

Finally we change it to the canonical:

$$T_C = M_{CB}T_B(M_{CB})^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

(iii) *¿Is it possible to achieve the above symmetry by properly composing two rotations? Reason your answer* (0.5 points).

It is impossible. The composition of two rotations is a rotation, because both are direct transformations with associated matrix of positive determinant. However, a symmetry with respect to a plane is an inverse transformation, with a negative determinant.

7.– *Reasonably indicate the falsity or truth of the following questions:*

(i) *An orthogonal transformation on $\mathbb{R}^3$ with two distinct real eigenvalues is a composition rotation with symmetry.*

FALSE. For example $T_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ has two distinct real eigenvalues but is the matrix associated with a rotation of 180 degrees.

(ii) *The sum of two orthogonal transformations is an orthogonal transformation.*

FALSE. For example, in $\mathbb{R}^2$, Id and $-Id$ are matrices associated to two orthogonal transformations but their sum is the zero matrix that does not correspond to an orthogonal transformation (the determinant of an orthogonal matrix is 1 or -1).

(iii) *In $\mathbb{R}^2$ a dot product can be defined with respect to which the vectors $(1,1)$ and $(1,0)$ are orthogonal.*

TRUE. It is enough to consider the base $B = \{(1,1),(1,0)\}$ and take as the Gram matrix of the scalar product in that base the identity: $G_B = Id$. By definition of a Gram matrix with respect to the base B, $(1,1)\cdot(1,0) = (G_B)_{12} = 0$ and thus $(1,1)\perp(1,0)$.

(iv) *In $\mathbb{R}^2$ a scalar product can be defined with respect to which the vector $(1,1)$ has zero norm.*

FALSE. The only zero norm vector is the null vector.

8.– *Reason the falsity or truth of the following questions:*

(i) *If T is the matrix of an orthogonal transformation in $\mathbb{R}^2$ and $trace(T) = 1$ then T is a rotation.* (0.5 points)

TRUE. In the plane, an orthogonal transformation is either a rotation or a symmetry about a line. If it is a symmetry, the associated matrix in a suitable basis is:

$$T_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $trace(T_B) = 1 + (-1) = 0$. The trace is preserved by changing the base, so in that case it would have to happen that $trace(T) = trace(T_B) = 0$. But we are told that $trace(T) = 1$. Therefore it is not a symmetry and it is necessarily a rotation .

(ii) *If T is the matrix of a rotation in $\mathbb{R}^2$ then it has no real eigenvalues.* (0.5 points)

FALSE. For example , if it is a rotation of zero degrees, the associated matrix is the identity $T = Id$ and 1 is an eigenvalue.

(iii) *If we consider the Euclidean space* $\mathbb{R}^3$ *with the usual conditions,* $T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *is the matrix of a symmetry with respect to the plane* $x - y = 0$. (0.5 points)

TRUE. In $\mathbb{R}^3$ an orthogonal transformation is:

- One rotation, if the determinant of its associated matrix is 1.

- A composition rotation with a symmetry with respect to the plane perpendicular to the axis of rotation, if the determinant of its associated matrix is -1.

In this case $det(T) = -1$, so we are in the second case. The angle α of rotation satisfies:

$$-1 + 2cos(\alpha) = trace(T) \qquad \Longleftrightarrow \qquad \alpha = \pm arcs\left(\frac{1 + trace(T)}{2}\right) = \pm arcs\left(\frac{1+1}{2}\right) = 0$$

Therefore the rotation is zero degrees and in reality it is simply symmetry with respect to a plane . The plane of symmetry are the eigenvectors associated with 1:

$$(T - Id)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \iff xy = 0.$$

(iv) *In the Euclidean space* $\mathbb{R}^3$ *(usual conditions) there is an orthogonal transformation* t. *Gauss holds that it is a rotation of* 90 *degrees and Euler that it is a rotation of* -90 *degrees. ¿Can they both be right?* (0.5 points)

YES. It is enough to take into account that if we change the semi-axis of rotation of direction, the sign of the angle changes. For example, we consider the matrix of a transformation:

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(90) & -sin(90) \\ 0 & sin(90) & cos(90) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

the angle of rotation is:

$$\alpha = \pm arcs\left(\frac{trace(T) - 1}{2}\right) = \pm 90^o$$

The semiaxis of rotation is any eigenvector associated with 1. We can take $(1,0,0)$ but also $(-1,0,0)$.

- If the semi-axis is $\vec{u}_1 = (1,0,0)$ to see the sign of the angle we take a base:

$$B = \{\vec{u}_1, \vec{v}, T\vec{v}\} = \{(1,0,0), (0,1,0), (0,0,1)\}$$

where $\vec{v}$ is any vector independent of the semi-axis. The sign of the angle is $sign(det(M_{CB})) = sign(1)$. It is therefore a rotation of semi-axis $(1,0,0)$ and angle $+90^o$.

- If the semi-axis is $\vec{u}_1 = (-1,0,0)$ we do the same thing:

$$B = \{\vec{u}_1, \vec{v}, T\vec{v}\} = \{(-1,0,0), (0,1,0), (0,0,1)\}$$

where $\vec{v}$ is any vector independent of the semi-axis. The sign of the angle is $sign(det(M_{CB})) = sign(-1)$. It is therefore a rotation of semi-axis $(-1,0,0)$ and angle -90^o.

11.– *In the Euclidean space* $\mathbb{R}^3$ *with the usual dot product we consider the planes* π_1 *and* π_2 *of equations:*

$$\begin{array}{rcc} \pi_1 & : & x + yz = 0 \\ \pi_2 & : & 2x - y + z = 0 \end{array}$$

Find the equations of a rotation that takes the plane π_1 *in the plane* π_2*.*

The axis of rotation corresponds to the straight intersection of both planes:

$$\left\{ \begin{array}{l} x + yz = 0 \\ 2x - y + z = 0 \end{array} \right. \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} x = 0 \\ y = z \end{array} \right.$$

A direction vector of it is $(0, 1, 1)$.

The rotation has to carry the normal vector of one plane over the other, that is, the $(1, 1, -1)$ on the direction given by $(2, -1, 1)$. Let's see the angle α that they form:

$$cos(\alpha) = \frac{(1, 1, -1) \cdot (2, -1, 1)}{\|(1, 1, -1)\| \|(2, -1, 1)\|} = 0.$$

They are perpendicular. Therefore we will choose a semi-axis rotation $(0, 1, 1)$, of 90 degrees and in such a way that the positive orientation is given by the base:

$$B = \{(0, 1, 1), (1, 1, -1), (2, -1, 1)\}.$$

Let's build that rotation. We obtain an orthogonal base whose first vector is the semi-axis of rotation:

$$B' = \{(0, 1, 1), (0, 1, -1), (1, 0, 0)\}$$

We check that B and B' have the same orientation. For this, it is enough that the determinant of the matrix of base change with respect to the canonical one has the same sign:

$$det(M_{CB}) = det \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} = -6 < 0, \qquad det(M_{CB'}) = det \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} = -2 < 0.$$

Now we normalize the base B':

$$B'' = \{\frac{(0, 1, 1)}{\|(0, 1, 1)\|}, \frac{(0, 1, -1)}{\|(0, 1, -1)\|}, \frac{(1, 0, 0)}{\|(1, 0, 0)\|}\} = \{(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (1, 0, 0)\}.$$

The rotation matrix in base B'' is:

$$F_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(90) & -sin(90) \\ 0 & sin(90) & cos(90) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We pass it to the canonical base:

$$F_C = M_{CB''} F_{B''} M_{BC''} = M_{CB''} F_{B''} M_{CB''}^{-1} = M_{CB''} F_{B''} M_{CB''}^{t}$$

where we use that the bases B'' and C are both orthonormal and therefore the inverse of the change of base matrix coincides with its transpose. Operating remains:

$$F_C = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \\ 1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix}.$$

The equations of rotation are:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = F_C \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

12.– *In $\mathbb{R}^2$ with respect to the usual scalar product, consider a linear transformation $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ whose associated matrix with respect to the canonical basis is:*

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

(i) *Find a and b so that t is a symmetry about a line.*

For it to be a symmetry it must be an orthogonal transformation with associated determinant matrix -1. For it to be orthogonal, $T_C T_C^t = Id$ must be fulfilled:

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix}^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \iff a^2 + b^2 = 1, \quad 2ab = 0.$$

For the determinant to be 1:

$$det \begin{pmatrix} a & b \\ b & a \end{pmatrix} = 1 \iff a^2 - b^2 = -1.$$

From the equations $a^2 + b^2 = 1$ and $a^2 - b^2 = -1$ we obtain $a = 0$ and $b = \pm 1$. In this case, it also holds that $2ab = 0$.

Therefore there are two cases:

i) $a = 0$ and $b = 1$.

ii) $a = 0$ $b = -1$.

(ii) *For each of the values of a and b obtained in the previous section, calculate the axis of symmetry.*

The axis of symmetry corresponds to the eigenvectors of T_C associated with 1:

i) $a = 0$ and $b = 1$.

$$(T_C - 1Id) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff xy = 0.$$

The axis of symmetry has the equation $x - y = 0$, that is, it is the subspace $\mathcal{L}\{(1,1)\}$.

ii) $a = 0$ and $b = -1$.

$$(T_C - 1Id) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x + y = 0.$$

The axis of symmetry has the equation $x + y = 0$, that is, it is the subspace $\mathcal{L}\{(1,-1)\}$.

14.– *Reason the truth or falsity of the following statements.*

(i) *The trace of the matrix associated with a symmetry with respect to a line in the plane is* 0.

TRUE. The trace of an application matrix does not depend on the basis from which it works, because the trace is preserved by similarity. So if it is a symmetry in the plane with respect to a line, in a suitable basis (first vector the axis of symmetry and second orthogonal to it) the associated matrix is:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and therefore its trace (with respect to any base) is $1 + (-1) = 0$.

(ii) *If the matrix associated with an orthogonal transformation in the plane has trace* 0 *then it is a symmetry with respect to a line.*

FALSE. For example , if we consider a rotation matrix of angle $\pi/2$:

$$\begin{pmatrix} cos(\pi/2) & -sin(\pi/2) \\ sin(\pi/2) & cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has trace 0 and is obviously NOT a symmetry with respect to a line.

(iii) *A symmetry in space with respect to a line is an inverse transformation.*

FALSE. A symmetry with respect to a straight line in space corresponds to a rotation of angle π. A rotation matrix has a positive determinant and therefore it is a direct transformation.

(iv) *In space, the composition of a symmetry with respect to a point and a symmetry with respect to a plane is always a rotation.*

TRUE. The matrix associated with a symmetry with respect to a point with respect to any base is $-Id$, it has a determinant -1. The matrix associated with symmetry with respect to a plane has a determinant -1 as well. Therefore its composition is represented by a matrix product of these two and then with determinant $(-1)(-1) = 1$: it is a rotation.