Note: This is an unrevised automatic translation.

4.– *In the vector space* $\mathbb{R}^3$ *we consider the basis* $B = \{(1,0,0),(1,1,0),(1,1,1)\}$ *and a scalar product whose Gram matrix with respect to the base* B *is:*

$$G_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Given the vectors $\vec{v} = (1,0,1)$, $\vec{u} = (0,1,1)$ *calculate* $\vec{u}\cdot\vec{v}, \|u\|, \|v\|$ *and the angle that they form.*

To be able to do the calculations, first we obtain the Gram matrix of the scalar product with respect to the canonical base:

$$G_C = M_{BC}^t G_B M_{BC}$$

where

$$M_{BC} = M_{CB}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Operating:

$$G_C = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

Now:

$$\vec{u}\cdot\vec{v} = (1 \quad 0 \quad 1)\, G_C \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1$$

$$\|\vec{u}\| = +\sqrt{\vec{u}\cdot\vec{u}} = +\sqrt{(1 \quad 0 \quad 1)\, G_C \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} = 1$$

$$\|\vec{v}\| = +\sqrt{\vec{v}\cdot\vec{v}} = +\sqrt{(0 \quad 1 \quad 1)\, G_C \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} = \sqrt{3}$$

Finally:

$$ang(\vec{u},\vec{v}) = arccos\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) = arccos\left(\frac{1}{\sqrt{3}}\right) = arccos\left(\frac{\sqrt{3}}{3}\right).$$

5.– *In* $\mathbb{R}^3$ $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ *is considered a scalar product fulfilling:*

- *The vector subspaces* $\mathcal{L}\{(1,0,1)\}$ *and* $\mathcal{L}\{(1,1,0),(0,0,1)\}$ *are orthogonal.*
- *The vectors* $(1,1,0)$ *and* $(0,0,1)$ *form an angle of* $\pi/3$.
- *The three previous vectors are unitary.*

(i) Calculate the Gram matrix of the scalar product with respect to the canonical base.

We consider the base formed by the three vectors about which we have information:

$$B = \{u_1 = (1,0,1), u_2 = (1,1,0), u_3 = (0,0,1)\}$$

They form a base because we have as many vectors as the dimension of the space $\mathbb{R}^3$ and are also independent because the matrix formed by their coordinates has rank 3. By definition of a Gram matrix:

$$G_B = \begin{pmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 & u_1 \cdot u_3 \\ u_2 \cdot u_1 & u_2 \cdot u_2 & u_2 \cdot u_3 \\ u_3 \cdot u_1 & u_3 \cdot u_2 & u_3 \cdot u_3 \end{pmatrix}$$

From the orthogonality indicated in the first section we know that $u_1 \cdot u_2 = u_1 \cdot u_3 = 0$. From the fact that they are unitary we know that $u_1 \cdot u_1 = u_2 \cdot u_2 = u_3 \cdot u_3 = 1$. Finally we use the angle data:

$$u_2 \cdot u_3 = \|u_2\| \|u_3\| cos(\pi/3) = \frac{1}{2}.$$

From all this we deduce that:

$$G_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{pmatrix}$$

Finally we pass it to the base canonic:

$$G_C = (M_{BC})^t G_B M_{BC}$$

where

$$M_{BC} = (M_{CB})^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

Operating we obtain:

$$G_C = \begin{pmatrix} 2 & -5/2 & -1 \\ -5/2 & 4 & 3/2 \\ -1 & 3/2 & 1 \end{pmatrix}$$

(ii) *Given $U = \mathcal{L}\{(1,1,1),(1,1,2),(2,2,3)\}$ compute a basis of its orthogonal subspace $U^\perp$ with respect to the given dot product.*

First we calculate a basis of U by eliminating the possible dependent vectors between its generators:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \xrightarrow{H_{21}(-1)} \xrightarrow{H_{31}(-2)} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{H_{32}(-1)} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

By therefore $U = \mathcal{L}\{(1,1,1),(0,0,1)\}$. Now:

$$\begin{aligned} U^\perp &= \{(x,y,z) \in \mathbb{R}^3 | (x,y,z) \cdot (1,1,1) = 0,\ (x,y,z) \cdot (0,0,1) = 0\} = \\ &= \{(x,y,z) \in \mathbb{R}^3 | (x,y,z) G_c (1,1,1)^t = 0,\ (x,y,z) \cdot (0,0,1)^t = 0\} = \\ &= \{(x,y,z) \in \mathbb{R}^3 | -\frac{3}{2}x + 3y + \frac{3}{2}z = 0,\ -x + \frac{3}{2}y + z = 0\} = \\ &= \{(x,y,z) \in \mathbb{R}^3 | y = 0,\ x = z\} = \mathcal{L}\{(1,0,1)\}. \end{aligned}$$

8.– *Given the symmetric matrix* $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ *find an orthogonal matrix* P *(*$P^{-1} = P^t$*) such that* $P^{-1}AP$ *is a diagonal matrix.*

Since A is symmetric we know that we can diagonalize it orthogonally, that is, with respect to a basis orthonormal; equivalently find an orthogonal matrix P such that $P^{-1}AP = P^tAP$ is diagonal. The steps are: - Calculate the eigenvalues. - Calculate the eigenvectors. - The eigenvectors associated to different eigenvalues are orthogonal; those associated with the same eigenvalue must be orthogonalized by Gram-Schmidt. - Finally we normalize the ortgonal basis of eigenvectors by dividing each one of them by its norm. Let's start by computing the characteristic polynomial:

$$p_A(\lambda) = det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} = det \begin{pmatrix} 3-\lambda & 3-\lambda & 3-\lambda \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} =$$

$$= (3-\lambda) det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} = (3-\lambda) det \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = \lambda^2(3-\lambda).$$

The eigenvalues are their roots. $\lambda_1 = 0$ with algebraic multiplicity 2. $\lambda_2 = 3$ with algebraic multiplicity 1. Because it is symmetric we know that algebraic and geometric coincide. We calculate the eigenvectors. Associated with $\lambda_1 = 0$:

$$(A - 0 \cdot Id) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff x + y + z = 0.$$

Going from implicits to parametric we get:

$$x = \alpha, \quad y = \beta, \quad z = -\alpha - \beta$$

and from there

$$S_0 = \mathcal{L}\{(1, 0, -1), (0, 1, -1)\}.$$

Associated with $\lambda_2 = 3$:

$$(A - 3 \cdot Id) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff -2x + y + z = 0, \quad x - 2y + z = 0, \quad x + y - 2x = 0.$$

Eliminating dependent equations and solving the system we obtain the parametrics:

$$x = \alpha, \quad y = \alpha, \quad z = \alpha$$

and from this

$$S_3 = \mathcal{L}\{(1, 1, 1)\}.$$

We orthogonalize the vectors of $S_0 = \mathcal{L}\{(1, 0, -1), (0, 1, -1)\}$, by the Gram-Schmidt method:

$$\vec{u}_1 = (1, 0, -1)$$
$$\vec{u}_2 = (0, 1, -1) + a(1, 0, -1)$$

Imposing that $\vec{u}_1 \cdot \vec{u}_2 = 0$ and isolating a:

$$a = \frac{-(1, 0, -1) \cdot (0, 1, -1)}{(1, 0, -1) \cdot (1, 0, -1)} = \frac{-1}{2}.$$

from
where $\vec{u}_2 = (0, 1, -1) - (1/2)(1, 0, -1) = (-1/2, 1, -1/2)$. Then $S_0 = \mathcal{L}\{(1, 0, -1), (-1/2, 1, -1/2)\}$ and:

$$B = \{\underbrace{(1, 0, -1), (-1/2, 1, -1/2)}_{S_0}, \underbrace{(1, 1, 1)}_{S_3}\}$$

is an orthogonal basis. We normalize it by dividing each vector by its norm:

$$\frac{(1,0,-1)}{\|(1,0,-1)\|} = (1/\sqrt{2}, 0, -1/\sqrt{2})$$
$$\frac{(-1/2,1,-1/2)}{\|(1,0,-1)\|} = (-2/\sqrt{5}, 1/\sqrt{5}, -2\sqrt{5})$$
$$\frac{(1,1,1)}{\|(1,1,1)\|} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$$

Placing the three vectors as columns of a matrix we have:

$$P = \begin{pmatrix} 1/\sqrt{2} & -2/\sqrt{5} & 1/\sqrt{3} \\ 0 & 1/\sqrt{5} & 1/\sqrt{3} \\ -1/\sqrt{2} & -2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}, \quad with P^{-1}AP = D$$

being D the diagonal matrix formed by the eigenvalues:

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

9.– *In* $\mathbb{R}^3$ *a bilinear form* f *is considered whose associated matrix in the canonical base is:*

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}$$

(i) *Prove that it is a dot product.*

For it to be a dot product it has to be a bilinear form, symmetric and positive definite:

- Bilinearity is a given of the statement.

- The symmetry is a consequence of the fact that the associated matrix is symmetric.

- To see that it is positive definite we can use Sylvester's criterion: we must verify that the successive determinants of the first k rows and columns are positive for $k = 1, 2, 3$:

$$|\,1\,| = 1 > 0, \qquad \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1 > 0, \qquad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 4 \end{vmatrix} = 4 - 1 = 3 > 0.$$

(ii) *With respect to the dot product defined by* f*:*

(ii.a) *Find the associated matrix with respect to the canonical basis of the orthogonal projection over* $\mathcal{L}\{(1,0,0)\}$.

We first compute a basis of the subspace orthogonal to the given $V = \mathcal{L}\{(1,0,0)\}$:

$$V^{\perp} = \{(x,y,z) \in \mathbb{R}^3 | (x,y,z) \cdot (1,0,0) = 0\}$$

where

$$(x,y,z) \cdot (1,0,0) = 0 \iff \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \iff x + y + z = 0.$$

We go from implicits to parametrics and then to generators:

$$z = -x - y$$

from where the parametrics are:

$$x = a, \quad y = b, \quad z = -a - b$$

y

$$V^{\perp} = \mathcal{L}\{(1,0,-1),(0,1,-1)\}.$$

We form a base with the generators of V and $V^{\perp}$:

$$B = \{\underbrace{(1,0,0)}_{V}, \underbrace{(1,0,-1),(0,1,-1)}_{V^{\perp}}\}$$

on this basis we know that the projection matrix is:

$$P_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally we pass it to the canonical:

$$P_B = M_{CB} P_B M_{CB}^{-1}, \qquad \text{where} \qquad M_{CB} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

Operating we get:

$$P_C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(ii.b) *Find an orthogonal basis of the subspace* $U = \{(x,y,z) \in \mathbb{R}^3 | x + y + z = 0\}$.

First we find a basis of the subspace. We have done it in the previous section since it coincides with $V^{\perp}$:

$$U = \mathcal{L}\{(1,0,-1),(0,1,-1)\}.$$

Now we orthogonalize the base by the Gram-Schmidt method. The first vector remains equal to $v_1 = (1,0,-1)$. we seek a second vector of the form:

$$v_2 = (0,1,-1) + a(1,0,-1)$$

We choose a by requiring that it be orthogonal to the first:

$$v_2 \cdot v_1 = 0 \iff (0,1,-1)\cdot(1,0,-1) + a(1,0,-1)\cdot(1,0,-1) = 0 \iff a = \frac{-(0,1,-1)\cdot(1,0,-1)}{(1,0,-1)\cdot(1,0,-1)}$$

We do the math:

$$(0,1,-1)\cdot(1,0,-1) = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 4$$

$$(1,0,-1)\cdot(1,0,-1) = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 4$$

Result:

$$a = \frac{-(0,1,-1)\cdot(1,0,-1)}{(1,0,-1)\cdot(1,0,-1)} = \frac{-4}{4} = -1, \qquad \Rightarrow \quad v_2 = (0,1,-1) - (1,0,-1) = (-1,1,0).$$

The requested base is:

$$\{(1,0,-1),(-1,1,0)\}$$

10.– *Let $\mathcal{P}_1(\mathbb{R})$ the vector space of polynomials of degree less than or equal to 1. A bilinear form $f : \mathcal{P}_1(\mathbb{R}) \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ is considered, whose associated matrix with respect to the canonical basis is:*

$$F_C = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}.$$

(i) *Prove that f is a dot product.*

A dot product is a bilinear, symmetric, positive definite form. That it is bilinear says the statement. It is symmetric if its associated matrix is. And in this case it is evident that $F_C = F_C^t$. To see that it is positive definite, we diagonalize the associated matrix by congruence and check that it has signature $(2, 0)$:

$$F_C = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \overset{H_{21}(-1)\mu_{21}(-1)}{\longrightarrow \longrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \quad \Rightarrow \quad sign(f) = (2, 0).$$

(ii) *With respect to the dot product defined by f:*

(ii.a) *Give two polynomials that form an orthonormal basis of $\mathcal{P}_1(\mathbb{R})$.*

A basis B is orthonormal if and only if the associated matrix of the dot product with respect to it is the identity. So we start by completing the previous diagonalization until we reach the identity:

$$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \overset{H_{21}(1/2)\mu_{21}(1/2)}{\longrightarrow \longrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = F_B$$

Now we perform on the identity the same column operations performed in the diagonalization process:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overset{\mu_{21}(-1)}{\longrightarrow} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \overset{\mu_{21}(1/2)}{\longrightarrow} \begin{pmatrix} 1 & -1/2 \\ 0 & 1/2 \end{pmatrix} = M_{CB}$$

The columns of the matrix M_{CB} are the coordinates of the vectors of the orthonormal basis B with respect to the canonical basis $C = \{1, x\}$:

$$B = \{(1, 0)_C, (-1/2, 1/2)_C\} = \left\{1, -\frac{1}{2} + \frac{1}{2}x\right\}$$

(ii.b) *Find the angle formed by the polynomials $p(x) = 1 + x$ and $q(x) = 1 - x$.*

We know that:

$$ang(p(x), q(x)) = arcs\left(\frac{< p(x), q(x) >}{\|p(x)\|\|q(x)\|}\right)$$

To be able to do the calculations using the matrix F_C of the scalar product, we express the given polynomials in coordinates with respect to the canonical base C:

$$p(x) = 1 + x = (1, 1)_C, \qquad q(x) = 1 - x = (1, -1)_C.$$

Then:

$$< p(x), q(x) >= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -4$$

Now the norms:

$$\|p(x)\| = \sqrt{\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}} = 2\sqrt{2}$$

and

$$\|q(x)\| = \sqrt{\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}} = 2$$

Finally:

$$ang(p(x), q(x)) = arcos\left(\frac{-4}{2\sqrt{2} \cdot 2}\right) = arcos\left(\frac{-1}{\sqrt{2}}\right) = 135^o.$$

11.– *Consider the real vector space $\mathbb{R}^3$ endowed with the ordinary scalar product. Find the matrix F, in the canonical basis, of a symmetric endomorphism f of $\mathbb{R}^3$, knowing that the kernel of f is the subspace $\mathcal{L}\{(1,1,1)\}$ and 3 is a double eigenvalue of f.*

We know that the eigenvalues of f are 0 and 3 with multiplicities 1 and 2 respectively. Since the characteristic subspaces are orthogonal to each other, the characteristic subspace relative to the eigenvalue 3 is precisely the one orthogonal to $S_0 = \mathcal{L}\{(1,1,1)\}$:

$$S_3 = S_0^{\perp} = \{(x,y,z)/x+y+z=0\} = \mathcal{L}\{(1,-1,0),(1,0,-1)\}$$

We look for an orthogonal basis of this subspace. We take $\bar{u}_1 = (1,-1,0)$ and

$$\bar{u}_2 = a\bar{u}_1 + (1,0,-1);$$

such that $\bar{u}_1 \cdot \bar{u}_2 = 0$. We get $a = -1/2$ and:

$$S_3 = \mathcal{L}\{(1,-1,0),(1/2,1/2,-1)\}$$

Therefore, the matrix of f in the base $\{(1,1,1),(1,-1,0),(1/2,1/2,-1)\}$ is:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Changing the base gives us the matrix we are looking for:

$$\begin{pmatrix} 1 & 1 & 1/2 \\ 1 & -1 & 1/2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1/2 \\ 1 & -1 & 1/2 \\ 1 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

(Final exam, September 2002)

14.– *Find the Gram matrix with respect to the canonical basis of a scalar product, knowing that:*

- *The vectors* $(1,0)$ *and* $(0,1)$ *form a angle of* 60 *degrees.*

- $\|(1,1)\| = \sqrt{3}$. - $B = \{(1,0),(1,-2)\}$ *is an orthogonal basis.*

We know that the Gram matrix of a scalar product is symmetric:

$$G_C = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Since b is an orthogonal basis:

$$(1,0)\cdot(1,-2) = 0 \iff (1 \quad 0)\, G_C \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 0 \iff a - 2b = 0.$$

Given that $\|(1,1)\| = \sqrt{3}$:

$$3 = \|(1,1)\|^2 = (1,1)\cdot(1,1) \iff 3 = (1 \quad 1)\, G_C \begin{pmatrix} 1 \\ 1 \end{pmatrix} = a + 2b + c = 3$$

From these two equations we already have that :

$$a = 2b, \quad c = 3 - 4b \qquad \Rightarrow \qquad G_C = \begin{pmatrix} 2b & b \\ b & 3-4b \end{pmatrix}$$

Finally if the vectors $(1,0)$ and $(0,1)$ form a angle of 60 degrees:

$$(1,0)\cdot(1,0) = \|(1,0)\|\|(0,1)\| cos(60)$$

where:

$$\|(1,0)\|^2 = (1,0)\cdot(1,0) = (\,1 \quad 0\,)\, G_C \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2b$$

$$\|(0,1)\|^2 = (0,1)\cdot(0,1) = (\,0 \quad 1\,)\, G_C \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3-4b$$

$$(1,0)\cdot(0,1) = (\,1 \quad 0\,)\, G_C \begin{pmatrix} 0 \\ 1 \end{pmatrix} = b$$

We are left with:

$$b = \sqrt{2b(3-4b)}\cdot\frac{1}{2}$$

Removing denominators and squaring:

$$4b^2 = 6b - 8b^2 \iff 2b^2 = b$$

where $b = 0$ ór $b = 1/2$. If $b = 0$ then $G_C = 0$ and that is not possible because it is the matrix of a positive definite scalar product. So $b = 1/2$ and $G_C = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$.