LINEAR ALGEBRA II — Solutions to Practice 1.1

Bilinear maps and quadratic forms — (Course 2022–2023)

NOTE. This is an unrevised automatic translation.

1.– *Check if the following applications are bilinear or not and in those that turn out to be, give the matrix that represents them in the corresponding canonical bases. Also decide whether the bilinear forms are symmetric or antisymmetric.*

(a)

(b)

(c) $h : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad h((x_1, x_2, x_3), (y_1, y_2, y_3)) = 5x_1y_1 + 4x_1y_2 + 1 - x_2y_1 + 5x_3y_1$

It is NOT BILINEAR because zero does not go to zero, that is:

$$h((0, 0, 0), (0, 0, 0)) = 1$$

(d)

(e) $m : \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}, m(p, q) = p(1)q(-1) - p(-1)q(1)$

Let's see if it is bilinear using the definition. Let $p, q, p_1, p_2, q_1, q_2 \in \mathcal{P}_2(\mathbb{R})$ and $\lambda, \mu \in \mathbb{R}$. Have:

$$\begin{aligned} m(\lambda p_1 + \mu p_2, q) &= (\lambda p_1(1) + \mu p_2(1))q(-1) - (\lambda p_1(-1) + \mu p_2(-1))q(1) \\ &= \lambda p_1(1)q(-1) - \lambda p_1(-1)q(1) + \mu p_2(1)q(-1) - \mu p_2(-1)q(1) = \lambda m(p_1, q) + \mu m(p_2, q) \end{aligned}$$

Analogously we see that:

$$m(p, \lambda q_1 + \mu q_2) = \lambda m(p, q_1) + \mu(p, q_2)$$

and therefore it is bilinear.

We calculate the matrix of the application in the canonical base $\{1, x, x^2\}$: The matrix is:

$$A = \begin{pmatrix} m(1,1) & m(1,x) & m(1,x^2) \\ m(x,1) & m(x,x) & m(x,x^2) \\ m(x^2,1) & m(x^2,x) & m(x^2,x^2) \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}$$

We see that the matrix is antisymmetric and therefore the bilinear mapping IS ANTISYMMETRIC.

2.– *Given the quadratic form $w : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $w(x, y) = x^2 + 4xy + 3y^2$:*

(i) *Classify it indicating its rank and signature.*

To classify it, we diagonalize the associated matrix with respect to the canonical basis by congruence:

$$F_C = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \overset{H_{21}(-2)\mu_{21}(-2)}{\longrightarrow\longrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The signature is $(1, 1)$ and the range is 2. It is therefore a non-degenerate and indefinite quadratic form.

(ii) *Find a basis of conjugate vectors.*

It is a basis B' in which the associated matrix is diagonal. In the previous section we have already diagonalized, therefore it is enough to apply the same column operations on the identity as in the previous process, thus obtaining the step matrix $M_{CB'}$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overset{\mu_{21}(-2)}{\longrightarrow} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = M_{CB'}$$

Therefore $B' = \{(1,0), (-2,1)\}$.

(iii) *Find the self-conjugate vectors, expressing them in the simplest way possible (give the result with respect to the canonical basis).*

Since w is indefinite with range 2, we know that the self-conjugates can be expressed as the union of two straight lines. If we work on the base B' in which it diagonalizes:

$$\begin{aligned} self-conj(w) &= \{(x',y')_{B'} | w((x',y')_B) = 0\} = \left\{ (x',y')_{B'} \middle| \begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 0 \right\} \\ &= \{(x',y')_{B'} | x'^2 - y'^2 = 0\} = \{(x',y')_B | x' - y' = 0\} \cup \{(x',y')_{B'} | x' + y' = 0\} = \\ &= \mathcal{L}\{(1,1)_{B'}\} \cup \mathcal{L}\{(1,-1)_{B'}\} \end{aligned}$$

Finally we pass the generators to the canonical base:

$$\underbrace{\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}}_{M_{CB'}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{B'} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}_C, \qquad \underbrace{\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}}_{M_{CB'}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}_{B'} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}_C,$$

and so:

$$self-conj(w) = \mathcal{L}\{(-1,1)\} \cup \mathcal{L}\{(3,-1)\}.$$

(iv) *Calculate the matrix associated to w in the base:*

$$B = \{(1,1), (1,-1)\}$$

We simply apply the base change formula:

$$F_B = M_{CB}^t F_C M_{CB} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^t \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 8 & -2 \\ -2 & 0 \end{pmatrix}.$$

(v) *If f is the symmetric bilinear form associated with w calculate $f((2,1),(1,3))$.*

The matrix associated with f is the same as the matrix associated with w. Therefore:

$$f((2,1),(1,3)) = \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 25.$$

3.–

4.–

5.–

6.–

7.– *For each value of $k \in \mathbb{R}$ the quadratic form is defined:*

$$w : \mathbb{R}^3 \to \mathbb{R}, \qquad w(x,y,z) = x^2 + 2kxy - z^2 + 2yz$$

(i) *Classify w based on the values of k, indicating its rank and signature.*

To classify the quadratic form we diagonalize by congruence its associated matrix with respect to the canonical basis. This is found by appropriately translating coefficients.

$$F_C = \begin{pmatrix} 1 & k & 0 \\ k & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \overset{H_{21}(-k)\mu_{21}(-k)}{\longrightarrow\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -k^2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \overset{H_{23}\ \mu_{23}}{\longrightarrow\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -k^2 \end{pmatrix} \overset{H_{32}(1)\mu_{32}(1)}{\longrightarrow\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1-k^2 \end{pmatrix}$$

We analyze when the values of the diagonal can be annulled as a function of k, since they are the limit points of change of sign:

$$1-k^2 = 0 \iff k^2 = 1 \iff k = \pm 1.$$

We summarize in the following table the signature and range of the matrix and its classification:

k	signature	range	classification
$k < -1$	$(1,2)$	3	non-degenerate and indefinite
$k = -1$	$(1,1)$	2	degenerate and indefinite
$-1 < k < 1$	$(2,1)$	3	non-degenerate and indefinite
$k = 1$	$(1,1)$	2	degenerate and indefinite
$k > 1$	$(1,2)$	3	nondegenerate and indefinite

(ii) *For $k = 1$ find the self-conjugate vectors. Express the result in the simplest way possible and with respect to the canonical basis.*

For $k = 1$ the quadratic form is undefined and has range 2. In this case we know that the self-conjugate vectors can be expressed as the union of two planes.

If we work with the associated diagonalized matrix that we calculated before:

$$F_B \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We have to:

$$self-conj(w) = \{(x',y',z')_B | w((x',y',z')_B) =\} = \left\{ (x',y',z')_B \left| \begin{pmatrix} x' & y' & z' \end{pmatrix} F_B \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = 0 \right. \right\} =$$
$$= \{(x',y',z')_B | x'^2 - y'^2 = 0\} = \{(x',y',z')_B | (x'+y')(x'-y') = 0\} =$$
$$= \{(x',y',z')_B | x'+y' = 0\} \cup \{(x',y',z')_B | x'-y' = 0\} =$$
$$= \mathcal{L}\{(1,-1,0)_B, (0,0,1)_B\} \cup \mathcal{L}\{(1,1,0)_B, (0,0,1)_B\}$$

To pass the result to the canonical base we need to have the base change matrix M_{CB}. To find it, we perform the same column operations on the identity that we did in the diagonalization process:

$$Id \overset{\mu_{21}(-1)}{\longrightarrow} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{\mu_{23}}{\longrightarrow} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \overset{\mu_{32}(1)}{\longrightarrow} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = M_{CB}$$

We make the base change of the generators of the two planes:

$$M_{CB} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad M_{CB} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad M_{CB} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

We conclude that:

$$self-conj(w) = \mathcal{L}\{(1,0,-1), (-1,1,1)_B\} \cup \mathcal{L}\{(1,0,1)_B, (-1,1,1)_B\}$$

(iii) *Give a vector that is self-conjugate for any value of k.*

Of course the null vector $(0,0,0)$ is always self-conjugate for any quadratic form and therefore for any value of k.

Also, given that the associated matrix F_C takes the value 0 in position $2,2$ regardless of the vector of k, it follows that:

$$w(\vec{e}_2) = f(\vec{e}_2, \vec{e}_2) = (F_C)_{22} = 0$$

and therefore $\vec{e}_2 = (0,1,0)$ is self-conjugate for any value of k.

(iv) *Calculate k so that the vectors $(1,0,0)$ and $(0,1,0)$ are conjugate.*

In order for them to be conjugated, it must be true that $f((1,0,0),(0,1,0)) = 0$, being f the symmetric bilinear form associated with w. Since it has the same associated matrix as this has to be checked:

$$0 = f((1,0,0),(0,1,0)) = (1 \quad 0 \quad 0)\, F_C \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = k.$$

(v) *¿Is there any value of k for which there are no nonzero self-conjugate vectors?.*

No. We saw in (iii) that for any value of k, $w(0,1,0) = 0$ that is, $(0,1,0)$ is a self-conjugate non-zero vector.

8.–

9.– *Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of polynomials of degree less than or equal to two with real coefficients. We consider the bilinear form:*

$$\phi : \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}, \quad \phi(p(x), q(x)) = \int_0^1 p(x)q'(x)dx + \int_0^1 p'(x)q(x)dx$$

(i) *Prove that ϕ is symmetric.*

It is necessary to verify that for any two polynomials $p(x), q(x) \in \mathcal{P}_2(\mathbb{R})$ it is true that:

$$\phi(p(x), q(x)) = \phi(q(x), p(x)).$$

But:

$$\phi(p(x), q(x)) = \int_0^1 p(x)q'(x)dx + \int_0^1 p'(x)q(x)dx = \int_0^1 q'(x)p(x)dx + \int_0^1 q(x)p'(x)dx =$$
$$= \int_0^1 q(x)p'(x)dx + \int_0^1 q'(x)p(x)dx = \phi(q(x), p(x)$$

(ii) *Find the matrix associated to ϕ with respect to the canonical base.*

The canonical basis of $\mathcal{P}_2(\mathbb{R})$ is $C = \{1, x, x^2\}$. The associated matrix in such a base is:

$$F_C = \begin{pmatrix} \phi(1,1) & \phi(1,x) & \phi(1,x^2) \\ \phi(x,1) & \phi(x,x) & \phi(x,x^2) \\ \phi(x^2,1) & \phi(x^2,x) & \phi(x^2,x^2) \end{pmatrix}.$$

To make the calculations more quickly, we can make two observations:

- Because ϕ is symmetric:

$$\phi(x,1) = \phi(1,x), \quad \phi(x^2,1) = \phi(1,x^2), \quad \phi(x^2,x) = \phi(x,x^2).$$

- By the properties of differential and integral calculus:

$$\int_0^1 p(x)q'(x)dx + \int_0^1 p'(x)q(x)dx = \int_0^1 (p(x)q(x))'dx = p(1)q(1) - p(0)q(0).$$

So:

$$\begin{aligned}\phi(1,1) &= 1\cdot 1 - 1\cdot 1 = 0.\\ \phi(1,x) &= 1\cdot 1 - 1\cdot 0 = 1.\\ \phi(1,x^2) &= 1\cdot 1^2 - 1\cdot 0^2 = 1.\\ \phi(x,x) &= 1\cdot 1 - 0\cdot 0 = 1.\\ \phi(x,x^2) &= 1\cdot 1^2 - 0\cdot 0^2 = 1.\\ \phi(x^2,x^2) &= 1^2\cdot 1^2 - 0^2\cdot 0^2 = 1.\end{aligned}$$

and so:

$$F_C = \begin{pmatrix} 0 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}.$$

(iii) *Calculate the range and signature of the quadratic form associated with ϕ.*

To calculate the range and signature we diagonalize the associated matrix by congruence:

$$\begin{pmatrix} 0 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} \xrightarrow{H_{13}} \xrightarrow{\nu_{13}} \begin{pmatrix} 1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 0\end{pmatrix} \xrightarrow{H_{21}(-1)} \xrightarrow{\nu_{21}(-1)} \xrightarrow{H_{31}(-1)} \xrightarrow{\nu_{31}(-1)} \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Therefore $range(\phi) = 2$ and $sign(\phi) = (1, 1)$.

(iv) *Find a base of polynomials of $\mathcal{P}_2(\mathbb{R})$, with respect to which the matrix associated to ϕ is diagonal.*

It is enough to carry out on the identity matrix the same row operations that we have done to diagonalize the associated matrix; the rows of the matrix that we obtain are the coordinates of the vectors of the searched basis with respect to the canonical basis:

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \xrightarrow{H_{13}} \begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix} \xrightarrow{H_{21}(-1)} \xrightarrow{H_{31}(-1)} \begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & -1\\ 1 & 0 & -1\end{pmatrix}.$$

The base sought is therefore formed by the polynomials:

$$\{0\cdot 1 + 0\cdot x + 1\cdot x^2, 0\cdot 1 + 1\cdot x - 1\cdot x^2, 1\cdot 1 + 0\cdot x - 1\cdot x^2\} = \{x^2, x - x^2, 1 - x^2\}.$$

Notation: $p'(x), q'(x)$ respectively denote the derivatives of $p(x), q(x)$.

10.– *Let $w : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a quadratic form. It's known that:*

- *The vectors $\{(1,1,0),(0,1,0),(1,1,1)\}$ are a basis of conjugate vectors.*

- *w has range 1.*

- *$(0,1,0)$ is a self-conjugate vector.*

-*$w(1,2,0) = 1$.*

i) *Find the matrix associated to w with respect to the canonical base.*

Since $B = \{(1,1,0),(0,1,0),(1,1,1)\}$ is a basis of conjugate vectors, the matrix associated to w with respect to said basis is diagonal:

$$F_B = \begin{pmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{pmatrix}.$$

Since $(0,1,0)$ is self-conjugate, $w(0,1,0) = 0$. We observe that $(0,1,0)_C = (0,1,0)_B$ and thus in matrix form:

$$(0,1,0)F_B(0,1,0)^t = 0$$

where $b = 0$.

Also $w(1,2,0) = 1$. We pass the vector $(1,2,0)$ to the base B to express the given condition in matrix form:

$$M_{BC}\begin{pmatrix}1\\2\\0\end{pmatrix} = M_{CB}^{-1}\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}.$$

Therefore $(1,2,0)_C = (1,1,0)_B$ and $w(1,2,0) = 1$ is expressed as:

$$(1,1,0)F_B(1,1,0)^t = 0$$

where $a + b = 1$. Since $b = 0$, we deduce that $a = 1$.

Finally, since w has range 1 and:

$$F_B = \begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & c\end{pmatrix}.$$

we deduce that $c = 0$.

To finish we make a base change of the associated matrix:

$$F_C = M_{BC}^t F_B M_{BC} = M_{CB}^{-t} F_B M_{CB}^{-1}$$

where

$$M_{CB} = \begin{pmatrix}1 & 0 & 1\\1 & 1 & 1\\0 & 0 & 1\end{pmatrix}.$$

Operating remains:

$$F_C = \begin{pmatrix}1 & 0 & -1\\0 & 0 & 0\\-1 & 0 & 1\end{pmatrix}.$$

ii) *Classify* w.

The diagonal form associated with w is:

$$F_B = \begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}.$$

Therefore the range is 1, the signature $(1,0)$. It is a positive semidefinite degenerate quadratic form.

iii) *Find all the self-conjugate vectors of* w.

These are the vectors whose image times w is zero.

Method I: Since the quadratic form is positive semidefinite we know that the self-conjugate vectors coincide with the kernel:

$$F_C\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \iff xz = 0, \quad -x + z = 0 \iff x - z = 0$$

Going from implicit to parametric remains:

$$Aut(w) = \mathcal{L}\{(0,1,0),(1,0,1)\}.$$

Method II: Working on base B:

$$\begin{aligned} Aut(w) &= \{(x',y',z')_B | (x',y',z')F_B(x',y',z')^t = 0\} = \\ &= \{(x',y',z')_B | x'^2 = 0\} = \{(x',y',z')_B | x' = 0\} = \\ &= \mathcal{L}\{(0,1,0)_B, (0,0,1)_B\}. \end{aligned}$$

We express the result in the canonical basis:

$$(0,1,0)_B = (0,1,0)_C, \qquad (0,0,1)_B = (1,1,1)_C.$$

Therefore:

$$Aut(w) = \mathcal{L}\{(0,1,0),(1,1,1)\}.$$

11.–

12.– *Let $w : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a quadratic form and $u, v \in \mathbb{R}^n$ verifying $w(u) = 1, w(v) = -1$. Prove that $\{u, v\}$ are linearly independent. ¿Is w necessarily an indefinite quadratic form?.*

If they were dependent, they would be a multiple of another, that is, we would have, for some $\lambda \in R$:

$$u = \lambda v \quad \Rightarrow \quad 1 = w(u) = w(\lambda v) = \lambda^2 w(v) = -1 \quad \Rightarrow \quad \lambda^2 = -1$$

Impossible. Therefore they are necessarily dependent.

On the other hand, it is always indefinite because there are vectors for which the quadratic form takes positive values (thus it is neither definite nor negative semidefinite) and negative values (thus it is neither definite nor positive semidefinite).

13.– *In each of the following sections, give a matrix non-diagonal associated with a quadratic form w of $\mathbb{R}^3$ that also satisfies the indicated condition (justify the answers).*

(i) *w is positive definite.*

A quadratic form in $\mathbb{R}^3$ is positive definite if its associated matrix in some basis is diagonal with all terms in it positive: for example the identity. To obtain an associated matrix that is not diagonal, we make a congruence that we know is equivalent to a change of base and therefore retains the property of being positive definite.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{H_{21}(1)\mu_{21}(1)}{\longrightarrow\longrightarrow} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(ii) *w is negative semidefinite.*

A negative semidefinite diagonal matrix has negative signs on the diagonal and also zeros. To achieve that it is NOT diagonal while maintaining the property of being negative semidefinite, we use the same technique as in the previous section:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{H_{21}(1)\mu_{21}(1)}{\longrightarrow\longrightarrow} \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(iii) *w is indefinite and not degenerate.*

In its diagonal form, for it to be indefinite, positive and negative signs must appear; In order for it to be non-degenerate, it must have a maximum range:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \overset{H_{21}(1)\mu_{21}(1)}{\longrightarrow\longrightarrow} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(iv) *w is indefinite and degenerate.*

As before, for it to be indefinite, positive and negative signs must appear in the diagonal form; to be degenerate it has to have rank less than 3:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{H_{21}(1)\mu_{21}(1)}{\longrightarrow\longrightarrow} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

14.– *Let $w : \mathbb{R}^3 \to \mathbb{R}$ be a quadratic form* **non-degenerate** *in $\mathbb{R}^3$ and F_C its associated matrix in the canonical basis. Reason the truth or falsity of the following questions:*

(i) *If all diagonal elements of F_C are positive then w is positive definite.*

FAKE. For example, if $F_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ it is true that all its elements on the diagonal are positive. However, if we diagonalize it by congruence to classify it, we arrive at:

$$F_C \overset{H_{32}(-1)\mu_{32}(-1)}{\longrightarrow\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

That is, we see that its signature is $(2, 1)$ and therefore it is non-degenerate but indefinite. It is NOT positive definite.

(ii) *If any element on the diagonal of F_C is null then w is undefined.*

TRUE. Because it is not degenerate, it can only be positive definite, negative definite, or indefinite.

But remember that the elements of the diagonal are $(F_C)_{i,i} = f(e_i, e_i) = w(e_i)$ where $\{e_1, e_2, e_3\}$ are the canonical basis vectors . If it is positive definite, $w(e_i) > 0$ would hold, that is, the elements of the diagonal will beípositive. Similarly, if it were negatively defined, the elements of the diagonal would be negative. Therefore if some is null the only possibility is that it is indefinite.

(iii) *F_C^2 is the matrix associated with a positive definite quadratic form.*

TRUE. Since F_C is associated with a quadratic form, it is symmetric. Therefore F_C^2 is also symmetric.

In order for it to be positive definite, it is necessary to verify that if $u \neq 0$ then $u^t F_C^2 u > 0$. But:

$$u^t F_C^2 u = u^t F_C F_C u = u^t F_C^t F_C u = (F_C u)^t (F_C u) = \|F_C u\|^2$$

where the norm is with the usual dot product. Since F_C is non-degenerate it has maximum rank and if $u \neq 0$ then $F_C u \neq 0$. Therefore, $\|F_C u\|^2 > 0$ and we tried what we wanted.

(iv) *If $F_C = Id$ then $f((x_1, x_2, x_3)_B, (y_1, y_2, y_3)_B) = x_1y_1 + x_1y_3 + 2x_2y_2 + x_3y_1 + 3x_3y_3$ can be the expression of a symmetric bilinear form associated with w, in a base B.*

TRUE. The matrix associated with the bilinear form given in base B is $F_B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$. For two symmetric matrices to correspond to the same bilinear form, they must be congruent, that is, have the same signature. The signature of $F_C = Id$ is obviously $(3, 0)$. For F_B we find it diagonalizing by congruence:

$$F_B \overset{H_{31}(-1)\mu_{31}(-1)}{\longrightarrow\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

We see that $sign(f) = (3, 0)$ and therefore F_B is congruent with F_C.

(v) *$f((x, y, z), (x', y', z')) = xx' + xy' + yx' + yy' + zz'$ can be the expression of a symmetric bilinear form associated with w.* (0.5 points)

FAKE. The matrix associated with f is

$$F_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We see that it has range < 3 (since the first two rows are equal) and thus it is degenerate; therefore it is impossible for it to be associated to w on any basis since w is NOT degenerate.

15.–
