LINEAR ALGEBRA II — Solutions to Practice 4.1

Conics — (Academic Year 2022–2023)

NOTE: This is a mostly unrevised, automatic translation from Spanish.

NOTE: All problems are assumed to be posed in the Euclidean affine plane endowed with a rectangular Cartesian system.

1.– *In the Euclidean affine plane and with respect to a rectangular reference, the conic of the equation is considered:*

$$x^2 - 4xy + y^2 - 6x + 2y = 0$$

Calculate the equation of the tangent lines to the conic that pass through the point $(-1,-3)$.

The matrix associated with the conic and the quadratic terms are respectively:

$$A = \begin{pmatrix} 1 & -2 & -3 \\ -2 & 1 & 1 \\ -3 & 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}.$$

We calculate the polar line of the point $(-1,-3)$:

$$(-1,-3,1)A(x,y,1)^t = 0 \iff x = 0$$

The intersection of this line with the conic are the points of tangency of the lines sought:

$$y^2 + 2y = 0 \qquad y = -2 \text{ or } y = 0.$$

The points of tangency are $(0,0)$ and $(0,-2)$. The tangent lines are the polar lines of these points:

$$(0,0,1)A(x,y,1)^t = 0 \iff -3x + y = 0,$$

and

$$(0,-2,1)A(x,y,1)^t = 0 \iff x - y - 2 = 0.$$

2.– *In the affine plane given the conic of equation:*

$$x^2 - 2xy + y^2 + 4x + 1 = 0$$

(i) Classify the conic.

The matrix associated with the conic and quadratic terms are respectively:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

We have $det(A) = -4$ and $det(T) = 0$. It is therefore a parable.

(ii) *Find its center, axes, vertices, and asymptotes.*

Since it is a parabola, it has neither a center nor asymptotes.

The axis is the polar line of the eigenvector associated with the nonzero eigenvalue of T.

The characteristic polynomial of T is:

$$|T - \lambda Id| = (1-\lambda)^2 - 1^2 = \lambda^2 - 2\lambda = \lambda(\lambda - 2).$$

Therefore the non-zero eigenvalue is $\lambda_1 = 2$. Its associated eigenvectors verify

$$(T - 2Id)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff -x - y = 0$$

Therefore $S_2 = \mathcal{L}\{(1,-1)\}$.

The axis will be the polar line of the vector $(1,-1)$:

$$\begin{pmatrix} 1 & -1 & 0 \end{pmatrix} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \iff 2x - 2y + 2 = 0 \iff x - y + 1 = 0.$$

The vertex is the intersection of the y-axis of the conic. We solve the system formed by their respective equations:

$$\begin{aligned} x - y + 1 &= 0 \\ x^2 - 2xy + y^2 + 4x + 1 &= 0 \end{aligned}$$

From the first equation $y = x + 1$. Substituting in the second:

$$\begin{aligned} x^2 - 2x(x+1) + (x+1)^2 + 4x + 1 &= 0 \\ x^2 - 2x^2 - 2x + x^2 + 2x + 1 + 4x + 1 &= 0 \\ 4x + 2 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$$

and $y = x + 1 = \frac{1}{2}$.

The vertex becomes $V = (-1/2, 1/2)$.

(iii) *Calculate its reduced equation, eccentricity and distance from the vertex to the focus.*

Because it is a parabola, the eccentricity is $e = 1$.

reduced equation is of the form:

$$\lambda_1 x''^2 - 2cy'' = 0 \iff 2x''^2 - 2cy'' = 0,$$

where

$$c = \sqrt{\frac{-|A|}{\lambda_1}} = \sqrt{2}.$$

The reduced equation is:

$$x''^2 - \sqrt{2}y'' = 0. \qquad (*)$$

Now we know that when it is expressed in the form $x^2 = 2py$ the focus is at the point $(0, p/2)$ and the vertex is at the origin. Therefore the focal length is $p/2$. In our case if we rewrite:

$$x''^2 - 2\frac{\sqrt{2}}{2}y'' = 0,$$

we see that $p = \sqrt{2}/2$ and thus the distance from the vertex to the focus is $p/2 = \sqrt{2}/4$.

3.– *Consider the conic given by the equation:*

$$3y^2 - 4xy + 12x - 14y + 19 = 0$$

a) *Find the asymptotes.*

The associated matrix and quadratic terms are respectively:

$$A = \begin{pmatrix} 0 & -2 & 6 \\ -2 & 3 & -7 \\ 6 & -7 & 19 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & -2 \\ -2 & 3 \end{pmatrix}.$$

We calculate the asymptotic directions. They are the vectors (x, y) fulfilling:

$$(x, y)T(x, y)^t = 0 \iff 3y^2 - 4xy = 0 \iff y(3y - 4x) = 0.$$

We obtain the directions given by the vectors $(1, 0)$ and $(3, 4)$. The asymptotes are their polar lines:

$$(1, 0, 0)A(x, y, 1)^t = 0 \quad \Rightarrow \quad -2y + 6 = 0 \quad \Rightarrow \quad y = 3$$

$$(3, 4, 0)A(x, y, 1)^t = 0 \quad \Rightarrow \quad -8x + 6y - 10 = 0 \quad \Rightarrow \quad 4x - 3y + 5 = 0$$

c) *Compute the exterior tangents to the conic passing through the point* $(0, 3)$.

We calculate the polar line of the given point. It intersects the conic at the points of tangency of the requested tangents. Then we will join those cut points with the initial point.

The polar line is:

$$(0, 3, 1)A(x, y, 1)^t = 0 \quad \Rightarrow \quad 2y - 2 = 0 \quad \Rightarrow \quad y = 1.$$

We cut with the equation of the hyperbola:

$$3 - 4x + 12x - 14 + 19 = 0 \quad \Rightarrow \quad 8x + 8 = 0 \quad \Rightarrow \quad x = -1$$

We get a single cut point $(-1, 1)$. The requested line is the one that joins said point with $(0, 3)$:

$$\frac{x - 0}{-1 - 0} = \frac{y - 3}{1 - 3} \iff 2x - y + 3 = 0.$$

Note: Only one exterior tangent appears, because the given point lies on the asymptote $y = 3$. The other outer "tangent" would be the asymptote itself.

4-5.– *For the following conics*

(1) $3x^2 + 3y^2 + 2xy - 4x - 4y = 0$

(2) $2x^2 + 3y^2 - 4x + 6y + 6 = 0$

(3) $6y^2 + 8xy - 8x + 4y - 8 = 0$

(4) $x^2 + 4y^2 - 4xy + 4 = 0$

(5) $x^2 - 2y^2 + xy + xy = 0$

(6) $x^2 + y^2 + 4x + 4 = 0$

(7) $x^2 + y^2 + 2xy - xy - 2 = 0$

(8) $x^2 + 4y^2 - 4xy + 6y = 0$

(9) $4x^2 + 4y^2 + 8xy + 4x + 4y + 1 = 0,$

is requested:

(a) *Classify them without finding the reduced equations.*

(1) We have that $a_{11} > 0, \quad det(T) = 8, \quad det(A) = -16$ and therefore it is a real ellipse.

(2) $det(T) = 6 > 0$ and $det(A) = 6 > 0$. So since the term in x^2 is positive, it is an imaginary ellipse.

(3) $det(T) = -16 < 0$ and $det(A) = -32 < 0$. It is a hyperbola.

(4) $det(T) = 0$ and $det(A) = 0$. So it's about two straight lines. Diagonalizing the associated matrix is:

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

and we see that they are two parallel imaginary lines ($x'^2 + 4 = 0$).

(5) $det(T) = -9/4 < 0$ and $det(A) = 0$. These are two real lines that intersect.

(6) $det(T) = 1 > 0$ and $det(A) = 0$. These are two imaginary lines that intersect.

(7) $det(T) = 0$ and $det(A) = 0$. It is two lines. Diagonalizing the associated matrix is:

$$\begin{pmatrix} 1 & 1 & -1/2 \\ 1 & 1 & -1/2 \\ -1/2 & -1/2 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -7/3 \end{pmatrix}$$

so we are dealing with two parallel real lines ($x'^2 - 7/3 = 0$).

(8) $det(T) = 0$ and $det(A) = -9 < 0$. It is a parable.

(9) $det(T) = 0$, $det(A) = 0$. It is two lines. diagonalizing remains :

$$\begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

a double line ($4x'^2 = 0$).

(b) *Give the reduced equations of the nondegenerate and the lines that form the degenerate.*

1

The matrices associated with the conic and quadratic terms are respectively:

$$A = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 3 & -2 \\ -2 & -2 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

You have to:

$$det(A) = -16, \qquad det(T) = 8$$

and therefore it is a real ellipse.

To find the reduced equation we calculate the eigenvalues of T:

$$|T - \lambda Id| = (3 - \lambda)^2 - 1^2 = (2 - \lambda)(4 - \lambda)$$

so that the eigenvalues (the roots of the characteristic polynomial remain):

$$\lambda_1 = 2, \qquad \lambda_2 = 4.$$

The reduced form is:

$$2x'^2 + 4y'^2 + d = 0$$

with

$$d = \frac{det(A)}{\lambda_1 \lambda_2}) = \frac{-16}{8} = -2.$$

Is left over:

$$2x'^2 + 4y'^2 - 2 = 0 \iff x'^2 + 2y'^2 - 1 = 0$$

or in canonical form:

$$\frac{x'^2}{1^2} + \frac{y'^2}{(1/\sqrt{2})^2} = 1.$$

(2) This is an imaginary ellipse. As before we calculate the eigenvalues of the matrix T of quadratic terms:

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0 \iff \lambda = 2 \text{ or } \lambda = 3$$

To find the independent term we proceed as in the previous section:

$$\lambda_3 = \frac{det(A)}{det(T)} = \frac{6}{6} = 1$$

The reduced equation is:

$$2x'^2 + 3y'^2 + 1 = 0 \qquad \iff \qquad \frac{x'^2}{1/2} + \frac{y'^2}{1/3} = -1$$

(3) is now a hyperbola. But the method is the same. We can first simplify the equation:

$$6y^2 + 8xy - 8x + 4y - 8 = 0 \iff 3y^2 + 4xy - 4x + 2y - 4 = 0$$

We calculate the eigenvalues of T:

$$\begin{vmatrix} 0-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0 \iff \lambda = 4 \text{ or } \lambda = -1$$

Now we calculate the independent term:

$$\lambda_3 = \frac{det(A)}{det(T)} = \frac{-4}{-4} = 1$$

The reduced equation is:

$$-x'^2 + 4y'^2 + 1 = 0 \qquad \iff \qquad x'^2 - \frac{y'^2}{1/4} = 1$$

Observation : In the case of the hyperbola, we make the eigenvalue whose sign is the same as that of the determinant of the associated matrix, accompany the term x'^2. In this way the foci will be located on the $O'X'$ axis.

(4) These are two parallel imaginary lines. In the previous section we have diagonalized the associated matrix so that we obtain the equation:

$$x'^2 + 4 = 0 \iff (x' - 2\imath)(x' + 2\imath) = 0 \iff x' - 2\imath = 0 \text{ or } x' + 2\imath = 0$$

If we look at the elementary operations that we have performed, we see that the reference change matrix is:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

therefore , undoing the change, it remains that the lines have equations:

$$x - 2y - 2\imath = 0 \quad \text{or} \quad x - 2y + 2\imath = 0$$

(5) These are two intersecting lines. As before we can diagonalize the associated matrix, find the lines in the new reference, and then undo the change. But we will use another method. We calculate the center, which corresponds to the intersection of both lines. Then we calculate the intersection of an arbitrary line with the conic and we obtain one more point from each of the lines that form it.

Center first:

$$(x, y, 1)\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & -2 & -1/2 \\ 1/2 & -1/2 & 0 \end{pmatrix} = (0, 0, t) \quad \Rightarrow \quad \begin{cases} x + y/2 + 1/2 = 0 \\ x/2 - 2y - 1/2 = 0 \end{cases} \quad \Rightarrow \quad x = y = -1/3$$

We see that it is the point $(-1/3, -1/3)$.

Now we take the line $x = 0$ and intersect it with the conic. Is left over

$$-2y^2 - y = 0 \quad \Rightarrow \quad y = 0 \text{ or } y = -1/2,$$

then the intersection is the points $(0, 0)$ and $(0, -1/2)$. The lines sought are those that join the center with each of these points:

$$\frac{x-0}{-1/3-0} = \frac{y-0}{-1/3-0} \iff x - y = 0$$
$$\frac{x-0}{-1/3-0} = \frac{y+1/2}{-1/3+1/2} \iff x + 2y + 1 = 0$$

We can check that the equation of the conic is precisely $(x - y)(x + 2y + 1) = 0$.

(6) These are two imaginary lines that intersect. We can proceed as in the previous case. Just keep in mind that now we will work with complex numbers.

We calculate the center:

$$(x, y, 1)\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix} = (0, 0, t) \quad \Rightarrow \quad \begin{cases} x + 2 = 0 \\ y = 0 \end{cases} \quad \Rightarrow \quad (x, y) = (-2, 0)$$

We intersect the line $x = 0$ with the conic:

$$y^2 + 4 = 0 \quad \Rightarrow \quad (x, y) = (0, 2\imath) \text{ or } (x, y) = (0, -2\imath)$$

Now we calculate the lines that join these points with the conic:

$$\frac{x+2}{0+2} = \frac{y-0}{2\imath-0} \iff x + \imath y + 2 = 0$$
$$\frac{x+2}{0+2} = \frac{y-0}{-2\imath-0} \iff x - \imath y + 2 = 0$$

Again we can see that the equation of the conic is $(x + \imath y + 2)(x - \imath y + 2) = 0$

(7) These are two parallel real lines. We can act as in the case (4) . But let's use another method. Here we know that there is a line of centers whose direction is that of the lines we are looking for:

$$(x, y, 1)\begin{pmatrix} 1 & 1 & -1/2 \\ 1 & 1 & -1/2 \\ -1/2 & -1/2 & -2 \end{pmatrix} = (0, 0, t) \quad \Rightarrow \quad x + y - 1/2 = 0$$

Therefore these straight lines are of the form $x + y + a = 0$.

We compute the intersection of the conic with an arbitrary line , eg, $x = 0$. We obtain:

$$y^2 - y - 2 = 0 \quad \Rightarrow \quad y = 2 \text{ or } y = -1$$

Therefore the lines contain respectively the points $(0,2)$ and $(0,1)$. They are therefore:

$$\begin{aligned} x+y-2=0 \\ x+y+1=0 \end{aligned}$$

Once again we can verify that the equation of the conic is $(x+y-2)(x+y+1)=0$.

(8) This is a parabola. Remember that, in its reduced form $ax^2+2by=0$, the associated matrix is:

$$A' = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & b & 0 \end{pmatrix}$$

where a is the non-zero eigenvalue of matrix T and $det(A') = det(A)$. Therefore we calculate the eigenvalues of T:

$$det(T-\lambda I)=0 \iff \begin{vmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda = 0 \text{ or } \lambda = 5$$

and also:

$$-9 = det(A) = det(A') = -ab^2 = -5b^2 \quad \Rightarrow \quad b = \pm 3\sqrt{5}/5$$

Therefore the reduced equation is :

$$5x^2 - \frac{6\sqrt{5}}{5}y = 0$$

(9) This is a double line. The equation is necessarily of the form

$$(ax+by+c)^2 = 0 \iff a^2x^2+b^2y^2+2abxy+2acx+2acy+c^2=0.$$

Comparing with the equation we have, we deduce that the line is:

$$2x+2y+1=0$$

We can also calculate the double line taking into account that it corresponds to a center line, that is, it is given by the equation:

$$(x,y,1)A = 0$$

We are left with three dependent equations. Taking only one of them is:

$$4x+4y+2=0 \iff 2x+2y+1=0$$

(c) *In cases where they exist, determine: centers, singular points, asymptotic directions, asymptotes, axes, vertices, foci, directives, and eccentricity.*

Singular points only appear when the conic is formed by two intersecting lines (the singular point is the intersection) or by two coincident lines (the singular place is the line itself). In those cases we have already found these points and the centers.

(1) The center is the point (a,b) verifying:

$$(a,b,1)A = (0,0,h) \iff 3a+b-2=0, \quad a+3b-2=0.$$

solving we get the point $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$.

The axes are the polar lines of the eigenvectors of t associated with nonzero eigenvalues.

We find the eigenvectors :

- Associated with $\lambda_1 = 2$:

$$(T - 2Id)(x,y)^t = 0 \iff (x,y) \in \mathcal{L}\{(1,-1)\}.$$

- Associated with $\lambda_4 = 4$:

$$(T - 4Id)(x,y)^t = 0 \iff (x,y) \in \mathcal{L}\{(1,1)\}.$$

The axes remain:

$$\begin{aligned}(1,-1,0)A(x,y,1)^t = 0 &\iff 2x - 2y = 0 \iff x - y = 0\\ (1,1,0)A(x,y,1)^t = 0 &\iff 4x + 4y - 4 = 0 \iff x + y - 1 = 0\end{aligned}$$

The vertices are the intersection of the axes with the conic.

We intersect the first axis:

$$\begin{cases} x - y = 0 \\ 3x^2 + 3y^2 + 2xy - 4x - 4y = 0 \end{cases}$$

solving we get $V_1 = (0,0)$, $V_2 = (1,1)$.

We intersect the second axis:

$$\begin{cases} x + y = 1 \\ 3x^2 + 3y^2 + 2xy - 4x - 4y = 0 \end{cases}$$

solving we get $V_3 = (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$, $V_4 = (\frac{1+\sqrt{2}}{2}, \frac{1-\sqrt{2}}{2})$.

The foci in the new reference are $(c,0)$ and $(-c,0)$ with:

$$c = \sqrt{a^2 - b^2} = \sqrt{1 - 1/2} = \sqrt{2}/2.$$

The reference change equation is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}}_{center} + \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{normalized eigenvectors\quad of\quad T} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

We change the reference foci using this equation:

$$\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + \frac{1}{-\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -\sqrt{2}/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

that is, the foci are the points $(1,0)$ and $(0,1)$.

Finally the eccentricity is:

$$\frac{c}{a} = \frac{\sqrt{2}/2}{1} = \frac{\sqrt{2}}{2}.$$

(2) We saw that it is an imaginary ellipse.

The center is calculated as in any other conic :

$$(a,b,1)\begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 3 \\ -2 & 3 & 6 \end{pmatrix} = (0,0,t) \iff \begin{cases} 2a - 2 = 0 \\ 3b + 3 = 0 \end{cases} \iff (a,b) = (1,-1)$$

It has no singular points, nor asymptotes.

The axes correspond again to the conjugates of the directions of the eigenvectors of T:

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda = 2 \text{ or } \lambda = 3$$

The eigenvectors are respectively $(1,0)$ and $(0,1)$. Therefore the axes remain :

$$\begin{aligned} (1,0,0)A(x,y,1)^t = 0 &\iff 2x-2=0 \\ (0,1,0)A(x,y,1)^t = 0 &\iff 3y+3=0 \end{aligned}$$

The rest does not make sense to calculate it, because it is an imaginary ellipse.

(3) This is now a hyperbola. We proceed as before. The reduced equation we saw was:

$$-x'^2 + 4y'^2 + 1 = 0 \qquad \iff \qquad x'^2 - \frac{y'^2}{1/4} = 1$$

We calculate the eigenvectors of T and the center to write the reference change.

eigenvalue associated with -1:

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x + 2y = 0$$

The normalized eigenvalue is $(2/\sqrt{5}, -1/\sqrt{5})$.

eigenvalue associated with 4:

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 2x - y = 0$$

The normalized eigenvalue is $(1/\sqrt{5}, 2/\sqrt{5})$.

The center is:

$$(x,y,1) \begin{pmatrix} 0 & 2 & -2 \\ 2 & 3 & 1 \\ -2 & 1 & -4 \end{pmatrix} = (0,0,t) \quad \Rightarrow \quad (x,y) = (-2,1)$$

The asymptotic directions correspond to the intersection of the conic with the points of infinity:

$$(x,y,0)A(x,y,0)^t = 0 \iff 6y^2 + 8xy = 0 \iff y = 0 \text{ or } 4x + 3y = 0 \iff (1,0) \text{ or } (-3,4)$$

The asymptotes are the polar lines of such directions:

$$(1,0,0)A(x,y,0)^t = 0 \iff 2y - 2 = 0 \iff y - 1 = 0$$

and

$$(-3,4,0)A(x,y,0)^t = 0 \iff 8x + 6y + 10 = 0 \iff 4x + 3y + 5 = 0$$

- The axes correspond to the polar lines of the eigenvectors :

$$\begin{aligned} (2,-1,0) \begin{pmatrix} 0 & 2 & -2 \\ 2 & 3 & 1 \\ -2 & 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 &\iff -2x + y - 5 = 0 \\ (1,2,0) \begin{pmatrix} 0 & 2 & -2 \\ 2 & 3 & 1 \\ -2 & 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 &\iff 4x + 8y = 0 \end{aligned}$$

- The vertices are found by intersecting the axes with the conic. Specifically, only one of the axes intersects the hyperbola and if we have ordered the eigenvalues putting first the one with the same sign as $|A|$, the axis that intersects the vertices is the polar line of the second eigenvector . It remains: $(2/\sqrt{5}-2,-1/\sqrt{5}+1)$ and $(-2/\sqrt{5}-2,1/\sqrt{5}+1)$.

Now the reference change is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

- To find the foci we know that in the new reference they are $(c,0)$ and $(-c,0)$, where $c=\sqrt{a^2+b^2}$, $a=1$ and $b=1/2$. $(\sqrt{5}/2,0)$ and $(-\sqrt{5}/2,0)$ remain. Using the reference change formulas, we get: $(-1,1/2)$ and $(-3,3/2)$.

- The guidelines are the polar lines of the foci:

$$(-1,1/2,1)\begin{pmatrix} 0 & 2 & -2 \\ 2 & 3 & 1 \\ -2 & 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \iff -x+y/2-3/2=0$$

$$(-3,3/2,1)\begin{pmatrix} 0 & 2 & -2 \\ 2 & 3 & 1 \\ -2 & 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \iff x-y/2+7/2=0$$

- The eccentricity is $e=\dfrac{c}{a}=\dfrac{\sqrt{5}}{2}$.

(4) We saw that these were two imaginary parallel lines.

We know that there is a line of centers:

$$(x,y,1)A=(0,0,h) \iff x-2y=0$$

There are no (real) singular points because in fact the lines are imaginary. If there would be if we worked with the imaginary affine plane.

The axis corresponds to the conjugate of the direction of the eigenvector associated with the nonzero eigenvalue of T:

$$\begin{vmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = 0 \iff \lambda=0 \text{ or } \lambda=5$$

The eigenvalue associated with 5 is $(-1,2)$. So the axis is :

$$(-1,2,0)A(x,y,1)^t=0 \iff -5x+10y=0$$

Everything else does not make sense to calculate in this case.

(5) We saw that these are two intersecting real lines.

The center has already been calculated by $(-1/3,-1/3)$.

The only singular point is precisely the intersection of the two lines, that is, the center.

The asymptotes are the two lines.

The axes correspond to the bisectors of these lines. In any case , we calculate them as usual: taking the conjugates of the directions marked by the eigenvectors of T.

We obtain that the eigenvalues are:

$$\frac{-1+\sqrt{10}}{2} \text{and} \frac{-1-\sqrt{10}}{2}$$

and the corresponding eigenvectors :

$$(-1, 3+\sqrt{10})\text{and}(3+\sqrt{10}, 1)$$

Therefore the axes remain :

$$(-1, 3+\sqrt{10}, 0)A(x, y, 1)^t = 0 \iff -x + (3+\sqrt{10})y + \frac{2+\sqrt{10}}{3} = 0$$
$$(3+\sqrt{10}, 1, 0)A(x, y, 1)^t = 0 \iff (3+\sqrt{10})x + y + \frac{4+\sqrt{10}}{3} = 0$$

(6) We saw that there were two intersecting imaginary lines:

The center has already been calculated:

$$(-2.0)$$

The only singular point is the center.

The axes are again the conjugates of the eigenvectors of T.

These eigenvectors are $(1, 0)$ and $(0, 1)$ and therefore the axes:

$$(1, 0, 0)A(x, y, 1)^t = 0 \iff x + 2 = 0$$
$$(0, 1, 0)A(x, y, 1)^t = 0 \iff y = 0$$

(7) These are two parallel real lines.

There is a line of centers that we have already calculated:

$$x + y - 1/2 = 0$$

There are no singular (proper) points.

The asymptotic directions are those of the lines.

The axis coincides in this case with the line of centers.

(8) This is a parabola. We saw that the reduced form was

$$5x^2 - \frac{6\sqrt{5}}{5}y = 0 \iff x^2 = 2\frac{3\sqrt{5}}{25}y$$

In this case (because it is a parabola) it does not have a center. Now, the translation vector of the reference change corresponds to the vertex of the parabola. For the rest we proceed as in the previous cases:

Eigenvalue associated with 5:

$$(x, y)\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} = 0 \quad \Rightarrow \quad 2x + y = 0$$

The normalized eigenvalue is $(1/\sqrt{5}, -2/\sqrt{5})$.

Eigenvalue associated with 0:

$$(x, y)\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} = 0 \quad \Rightarrow \quad x - 2y = 0$$

The normalized eigenvalue is $(2/\sqrt{5}, 1/\sqrt{5})$.

Observation: For the parabola to be oriented on the positive OY' axis, we verify that we have chosen the correct orientation of the eigenvalue associated with 0. The sign of the coefficient of y' in the expression $\lambda_1 x'^2 - 2cy' = 0$ is:

$$(2/\sqrt{5}, 1/\sqrt{5})(0,3)^t = 3/\sqrt{5} > 0$$

then instead of the vector $(2/\sqrt{5}, 1/\sqrt{5})$ we take the vector $(-2/\sqrt{5}, -1/\sqrt{5})$.

Now we calculate the axis. Corresponds to the conjugate of the eigenvector associated with the nonzero eigenvalue:

$$(1,-2,0)\begin{pmatrix} 1 & -2 & 0 \\ -2 & 4 & 3 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \iff 5x - 10y - 6 = 0$$

The vertex is the intersection of this axis with the conic:

$$\left(\frac{10y+6}{5}\right)^2 + 4y^2 - 4y\left(\frac{10y+6}{5}\right) + 6y = 0$$

It turns out the point $(18/25, -6/25)$

The reference change formulas are:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18/25 \\ -6/25 \end{pmatrix} + \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ -2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

Besides:

- Because it is a parabola, it has no center.

- The asymptotic directions correspond to the intersection of the conic with the points of infinity:

$$(x,y,0)A(x,y,0)^t = 0 \iff x^2 + 4y^2 - 4xy = 0 \iff (x-2y)^2 = 0 \iff (2,1)$$

(actually we know that it is the address corresponding to the eigenvector associated with 0).

- The focus in the new reference is $(0, p/2)$, where $p = (3\sqrt{5}/5)/5$. $(0.3\sqrt{5}/50)$ remains. If we change it to the original with the reference change formula, it results: $(3/5, -3/10)$.

- The directrix is the polar line of the focus:

$$(3/5, -3/10, 1)\begin{pmatrix} 1 & -2 & 0 \\ -2 & 4 & 3 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \iff 12x + 6y - 9 = 0$$

is always 1.

(9) This is a double real line.

The line of centers, the singular points, asymptotes and the axis coincide with the line itself, which we have already calculated before.

6.– *Given the curve of equation*

$$3y^2 + 4xy - 4x - 6y - 1 = 0.$$

(i) *Classify the conic and give its reduced equation.*

The associated matrices and matrices of quadratic terms of the conic are respectively:

$$A = \begin{pmatrix} 0 & 2 & -2 \\ 2 & 3 & -3 \\ -2 & -3 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}.$$

We have that $|A| = 16, \quad |T| = -4$. It is therefore a hyperbola.

To find the reduced equation we calculate the eigenvalues of T:

$$|T - \lambda Id| = 0 \iff \lambda^2 - 3\lambda - 4 = 0 \iff \lambda = -1 \text{ or } \lambda = 4.$$

Given that $|A| > 0$ we take as the first eigenvalues the positive one:

$$\lambda_1 = 4, \qquad \lambda_2 = -1.$$

The reduced equation is:

$$4x''^2 - y''^2 + d = 0, \quad d = \frac{|A|}{\lambda_1\lambda_2} = -4.$$

That is to say:

$$4x''^2 - y''^2 - 4 = 0 \iff \frac{x''^2}{1} - \frac{y''^2}{4} = 1.$$

(ii) *Find the point whose polar line is $x - 1 = 0$.*

Let (a, b) be the pole of the given polar line. We know that:

$$(a, b, 1)A(x, y, 1)^t \text{is equivalent to the line} x - 1 = 0.$$

Operating:

$$(2b - 2)x + (2a + 3b - 3)y + (-2a - 3b - 1) = 0 \text{is equivalent to the line} x - 1 = 0.$$

We deduce that:

$$2a + 3b - 3 = 0, \qquad \frac{2b - 2}{1} = \frac{-2a - 3b - 1}{-1}.$$

Solving the system results:

$$(a, b) = (-3, 3).$$

(iii) *Find the foci.*

From the reduced equation obtained in section 1 we know (with the usual notation) that $a^2 = 1, b^2 = 4$. Therefore:

$$c = \sqrt{a^2 + b^2} = \sqrt{5},$$

and the foci in the new reference are:

$$F_1 = (\sqrt{5}, 0)_{R''}, \quad F_2 = (-\sqrt{5}, 0)_{R''}.$$

It remains to pass them to the starting reference; For this we calculate the reference change formulas.

The center is the point (a, b) verifying:

$$(a, b, 1)A = (0, 0, h) \iff 2b - 2 = 0, \quad 2a + 3b - 3 = 0 \iff (a, b) = (0, 1).$$

We calculate the eigenvectors of T. Associated with $\lambda_1 = 4$:

$$(T - 4Id)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff 2x - y = 0 \iff (x, y) \in \mathcal{L}\{(1, 2)\}$$

Associated with $\lambda_2 = -1$:

$$(T + Id)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x + 2y = 0 \iff (x, y) \in \mathcal{L}\{(2, -1)\}$$

The normalized eigenvectors are:

$$\frac{(1,2)}{\|(1,2)\|} = \frac{1}{\sqrt{5}}(1,2), \qquad \frac{(2,-1)}{\|(2,-1)\|} = \frac{1}{\sqrt{5}}(2,-1).$$

And finally the reference change equation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x'' \\ y'' \end{pmatrix}.$$

We change the reference foci:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \pm\sqrt{5} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \pm \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The foci are therefore:

$$F_1 = (1,3), \qquad F_2 = (-1,-1).$$

7.– *In the affine plane the family of conics is considered:*

$$x^2 + 2axy + 2y^2 + 2x - 6ay + 1 = 0, \qquad , \qquad a \in \mathbb{R}.$$

(i) *Classify the conic based on the parameter a.*

We classify according to the signs of the determinants of the associated matrices and quadratic terms.

$$A = \begin{pmatrix} 1 & a & 1 \\ a & 2 & -3a \\ 1 & -3a & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & a \\ a & 2 \end{pmatrix}$$

So:

$$|A| = -16a^2, \qquad |T| = 2 - a^2$$

To make the table we study when the determinants are canceled to know the limit cases. We see that $|A| = 0$ if $a = 0$ and $|T| = 0$ if $a = +\sqrt{2}$ or $a = -\sqrt{2}$.

So:

- If $a < -\sqrt{2}$, then $|A| < 0$ and $|T| < 0$. It is a hyperbola.
- If $a = -\sqrt{2}$, $|A| < 0$ and $|T| = 0$. It is a parable.
- If $-\sqrt{2} < a < 0$, $|A| < 0$ and $|T| > 0$. It is a real ellipse.
- If $a = 0$, $|A| = 0$ and $|T| > 0$, they are complex lines intersecting at a real point.
- If $0 < a < \sqrt{2}$, $|A| < 0$ and $|T| > 0$. It is a real ellipse.
- If $a = \sqrt{2}$, then $|A| < 0$ and $|T| = 0$. It is a parable.
- If $a > \sqrt{2}$, then $|A| < 0$ and $|T| < 0$. It is a hyperbola.

(ii) *For $a = 1$ calculate the center of the conic.*

For $a = 1$ it is a real ellipse. The center is the point (p, q) verifying that:

$$A(p,q,1)^t = (0,0,h)^t \text{where } h \text{ is any value}$$

We get the equations:

$$p + q + 1 = 0, \qquad p + 2q - 3 = 0$$

where $(p,q)=(-5,4)$.

(iii) *For $a=\sqrt{2}$ compute the distance between the vertex and the focus.*

For $a=\sqrt{2}$ the conic is a parabola. To find the distance between vertex and focus we find the reduced equation in the form:

$$x''^2 = 2py''$$

The distance between vertices and focus is $p/2$.

To find the reduced equation we begin by calculating the eigenvectors of T:

$$|T-\lambda Id| = 0 \iff \lambda^2 - 3\lambda = 0 \iff \lambda = 3 \text{ or } \lambda = 0.$$

The reduced equation is of the form:

$$3x''^2 - 2cy' = 0$$

with

$$c = \sqrt{\frac{-|A|}{3}} = \sqrt{\frac{32}{3}} = \frac{4\sqrt{6}}{3}.$$

It remains then:

$$3x''^2 - 2\frac{4\sqrt{6}}{3}y' = 0 \iff x''^2 - 2\frac{4\sqrt{6}}{9}y' = 0.$$

Hence $p = \frac{4\sqrt{6}}{9}$ and the requested distance is $p/2 = \frac{2\sqrt{6}}{9}$.

(iv) *¿For what values of a is the eccentricity of the conic greater than 1?.*

The eccentricity is greater than 1 for hyperbolas, that is, when $a<-\sqrt{2}$ or $a>\sqrt{2}$.

8.– *In the affine plane the conic of the equation is considered:*

$$x^2 + 2xy + y^2 - 4x - 1 = 0$$

(i) *Classify the conic and find its reduced equation.* (0.6 points)

To classify the conic we calculate the determinants of its associated matrix and matrix of quadratic terms:

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

where: $det(A) = -4$ and $det(T) = 0$. It is then a parable.

reduced equation is of the form:

$$\lambda x'^2 + 2dy' = 0$$

with λ the nonzero eigenvalue of T and $d^2 = \frac{-|A|}{\lambda}$ (we will assign to d the sign opposite of that of λ).

We calculate the eigenvalues of T as roots of the characteristic polynomial:

$$|T-\lambda Id| = 0 \iff det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = 0 \iff \lambda^2 - 2\lambda = 0.$$

the result is $\lambda_1 = 2$ and $\lambda_2 = 0$.

So $d = -\sqrt{\frac{-|A|}{\lambda}} = -\sqrt{\frac{4}{2}} = -\sqrt{2}$. The reduced equation is:

$$2x'^2 - 2\sqrt{2}y' = 0.$$

In canonical form :

$$x'^2 = 2\frac{\sqrt{2}}{2}y'$$

such that the parameter p of the parabola is $p = \dfrac{\sqrt{2}}{2}$.

(ii) *Find its eccentricity, asnotas , and the distance between a focus and the vertex closest to it.* (0.4 points)

Since it is an abolic pair, the eccentricity is 1 and it has no asymptotes. The distance between its single focus and its single vertex is $\dfrac{p}{2} = \dfrac{\sqrt{2}}{2}$.

(iii) *Calculate the tangents to the conic that pass through the point* $(-1, 2)$. (0.5 points)

The points of tangency of the lines sought are the intersection of the polar line of $(-1, 2)$ with respect to the conic. Such a polar line is:

$$\begin{pmatrix} -1 & 2 & 1 \end{pmatrix} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \iff -x + y + 1 = 0 \iff x - y - 1 = 0.$$

We intersect with the conic solving the system formed by its equations:

$$\begin{cases} x - y - 1 = 0 \\ x^2 + 2xy + y^2 - 4x - 1 = 0 \end{cases}$$

Solving for y in the first equation $y = x - 1$ and substituting in the second:

$$x^2 + 2x(x-1) + (x-1)^2 - 4x - 1 = 0 \iff 4x^2 - 8x = 0$$

Where from:

$$x = 0, \quad y = x - 1 = -1 \text{and we obtain the point} R = (0, -1).$$
$$x = 2, \quad y = x - 1 = 1 \text{and we obtain the point} S = (2, 1).$$

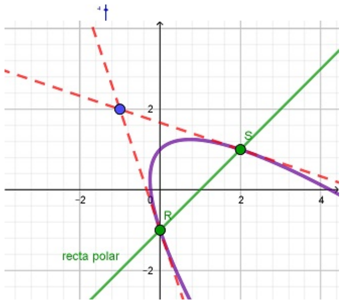
The corresponding tangents are:

- At the point $R = (0, -1)$:

$$\begin{pmatrix} 0 & -1 & 1 \end{pmatrix} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \iff -3x - y - 1 \iff 3x - 3y + 1 = 0.$$

- At the point $S = (2, 1)$:

$$\begin{pmatrix} 2 & 1 & 1 \end{pmatrix} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \iff x + 3y - 5 = 0.$$



(iv) *Calculate the equation of a conic that has the same tangents as the given curve at the points* $(0, -1)$ *and* $(2, 1)$ *and passes through the origin.* (1 point)

We consider the bundle of conics that has the same tangents as the given curve at the points $(0,-1)$ and $(2,1)$, which are precisely the points of tangency from the previous section.

To generate it we need two conics that satisfy this condition. One can be the given conic itself and the other the double line that joins them, that is, the polar line already calculated squared.

The beam remains:

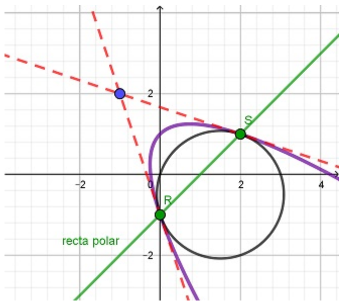
$$x^2 + 2xy + y^2 - 4x - 1 + \lambda(x - y - 1)^2 = 0$$

We impose that the sought conic passes through the origin $(0,0)$:

$$0^2 + 2 \cdot 0 \cdot 0 + 0^2 - 4 \cdot 0 - 1 + \lambda(0 - 0 - 1)^2 = 0 \iff \lambda = 1.$$

The requested curve is:

$$x^2 + 2xy + y^2 - 4x - 1 + (xy - 1)^2 = 0 \iff 2x^2 + 2y^2 - 6x + 2y = 0 \iff x^2 + y^2 - 3x + y = 0.$$



9.– *In the affine plane the conic of equation is considered:*

$$x^2 + 2kxy + y^2 + 2ky = 0$$

(i) *Classify the conic based on the values of k.* (0.5 points)

To classify the conic we study the signs of the determinants of the associated matrix and of the matrix of quadratic terms:

$$A = \begin{pmatrix} 1 & k & 0 \\ k & 1 & k \\ 0 & k & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix}.$$

Where $det(A) = -k^2$ and $det(T) = 1 - k^2$. We see where they cancel to see the values of k limit in which a sign change can occur:

$$-k^2 = 0 \iff k = 0, \qquad\qquad 1 - k^2 = 0 \iff k = \pm 1.$$

So it remains:

k	$\lvert T\rvert$	$\lvert A\rvert$	Type of conic
$k < -1$	$-$	$-$	hyperbola
$k = -1$	0	$-$	parabola
$-1 < k < 0$	$+$	$-$	ellipse (note that $a_{11} > 0$)
$k = 0$	$+$	0	imaginary lines intersecting at a real point
$0 < k < 1$	$+$	$-$	ellipse
$k = 1$	0	$-$	parabola
$k > 1$	$-$	$-$	hyperbola

(ii) *For $k = 2$ and $k = -1$ find the center, the axes, the asymptotes and the eccentricity.* (1 point)

We start with $k=2$. We saw that it is a hyperbola . Then:

$$A=\begin{pmatrix}1&2&0\\2&1&2\\0&2&0\end{pmatrix},\qquad T=\begin{pmatrix}1&2\\2&1\end{pmatrix}.$$

The center is the point (r,s) verifying:

$$A\begin{pmatrix}r\\s\\1\end{pmatrix}=\begin{pmatrix}0\\0\\h\end{pmatrix}\iff r+2s=0,\quad 2r+s+2=0$$

Solving is $(r,s)=(-4/3,2/3)$.

The axes are the polar lines of the eigenvectors of T associated with nonzero eigenvalues. The characteristic polynomial of T is:

$$|T-\lambda Id|=(1-\lambda)^2-2^2=(3-\lambda)(-1-\lambda).$$

Therefore the eigenvalues are $\lambda_1=-1$ and $\lambda_2=3$. The eigenvectors remain:

- Associated with $\lambda_1=-1$:

$$(T+Id)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\iff 2x+2y=0\quad\Rightarrow\quad S_{-1}=\mathcal{L}\{(1,-1)\}.$$

- Associated with $\lambda_2=3$:

$$(T-3Id)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\iff -2x+2y=0\quad\Rightarrow\quad S_3=\mathcal{L}\{(1,1)\}.$$

Its polar lines are the axes:

$$(1\quad -1\quad 0)\,A\begin{pmatrix}x\\y\\1\end{pmatrix}=0\iff x-y+2=0.$$

$$(1\quad 1\quad 0)\,A\begin{pmatrix}x\\y\\1\end{pmatrix}=0\iff 3x+3y+2=0.$$

The intotes are the polar lines of the asymptotic directions. These are vectors (p,q) satisfying:

$$(p\quad q)\,t\begin{pmatrix}p\\q\end{pmatrix}=0\iff p^2+4pq+q^2=0\iff p=-2q\pm\sqrt{(2q)^2-q^2}=(-2\pm\sqrt{3})q$$

Taking $q=1$ we are left with the asymptomatic addresses ǿtic $(-2+\sqrt{3},1)$ and $(-2-\sqrt{3},1)$. The asymptotes are their corresponding polar lines:

$$(-2+\sqrt{3}\quad 1\quad 0)\,A\begin{pmatrix}x\\y\\1\end{pmatrix}=0\iff \sqrt{3}x+(-3+2\sqrt{3})y+2=0.$$

$$(-2-\sqrt{3}\quad 1\quad 0)\,A\begin{pmatrix}x\\y\\1\end{pmatrix}=0\iff \sqrt{3}x+(3+2\sqrt{3})y-2=0.$$

For the eccentricity we write the hyperbola in reduced form as:

$$\lambda_1x'^2+\lambda_2y'^2+d=0,\qquad d=det(A)/det(K)$$

We take as λ_1 the eigenvalue with the same sign as $det(A)$; in this case negative. We are left:

$$-x'^2 + 3y'^2 + \frac{4}{3} = 0$$

Simplifying:

$$\frac{x'^2}{4/3} - \frac{y'^2}{4/9} = 1 \quad \Rightarrow \quad a^2 = 4/3, \quad b^2 = 4/9.$$

Where from:

$$eccentricity = \frac{c}{a} = \frac{\sqrt{a^2+b^2}}{a} = \frac{4/3}{2/\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

For $k=-1$ we know that it is an abola pair. In that case :

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The eccentricity is 1 and it has neither center nor intotes.

The axis is the polar line of the eigenvector of T associated with the only nonzero eigenvalue of T. The characteristic polynomial of T is:

$$|T - \lambda Id| = (1-\lambda)^2 - 1^2 = \lambda(\lambda - 2).$$

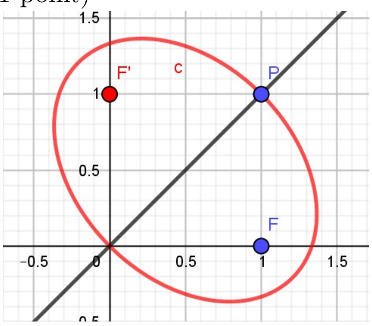
The non-zero eigenvalue of T is $\lambda_1 = 2$ and its associated eigenvector :

$$(T - 2\cdot Id)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff -x - y = 0 \quad \Rightarrow \quad S_2 = \mathcal{L}\{(1,-1)\}.$$

and its polar line:

$$\begin{pmatrix} 1 & -1 & 0 \end{pmatrix} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \iff 2x - 2y + 1 = 0.$$

(iii) *Calculate the equation of an ellipse that has a focus at the point $F(1,0)$, an axis is the line $xy = 0$ and passes through the point $(1,1)$.* (1 point)



We will calculate the second focus F' as symmetric of the first with respect to the axis. Then we will use the definition of an ellipse as a locus of points in the plane whose sum of distances to the foci is constant.

Let $F' = (x_0, y_0)$. Because it is symmetrical to the first with respect to the line $xy = 0$ it fulfills:

i) $\dfrac{F+F'}{2} \in axis \quad \Rightarrow \quad \dfrac{x_0+1}{2} - \dfrac{y_0}{2} = 0 \iff x_0 - y_0 + 1 = 0$.

ii) $\vec{FF'} \perp axis$, that is, $(x_0 - 1, y_0)$ is parallel to the normal vector $(1,-1)$ of the axis. Hence: $x_0 + y_0 - 1 = 0$.

Solving gives $F' = (x_0, y_0) = (0,1)$.

The equation of the ellipse is then:

$$\sqrt{(x-1)^2+y^2}+\sqrt{x^2+(y-1)^2}=constant$$

To find the constant we impose that the conic passes through the given point $(1,1)$:

$$\sqrt{(1-1)^2+1^2}+\sqrt{1^2+(1-1)^2}=cte \quad\Rightarrow\quad constant=2.$$

We obtain:

$$\sqrt{(x-1)^2+y^2}+\sqrt{x^2+(y-1)^2}=2 \quad\Rightarrow\quad \sqrt{(x-1)^2+y^2}=2-\sqrt{x^2+(y-1)^2}$$

Raising both members to the square:

$$x^2-2x+1+y^2=4+x^2+y^2-2y+1-4\sqrt{x^2+(y-1)^2} \quad\Rightarrow\quad 2\sqrt{x^2+(y-1)^2}=x-y+2$$

Raising again:

$$4x^2+4y^2-8y+4=x^2+y^2-2xy+4x-4y+4$$

and simplifying:

$$3x^2+2xy+3y^2-4x-4y=0.$$

10.– *The family of conics dependent on the parameter $a\in R$ is considered :*

$$x^2+8xy-ay^2-2x-2ay=0$$

a) *Classify conics in terms of a.*

The matrices associated with the conic and quadratic terms are respectively:

$$A=\begin{pmatrix}1 & 4 & -1\\ 4 & -a & -a\\ -1 & -a & 0\end{pmatrix}, \qquad T=\begin{pmatrix}1 & 4\\ 4 & -a\end{pmatrix}.$$

We classify the conic based on the determinants of A and T:

$$det(A)=-a^2-9a=-a(a-9), \qquad det(T)=-a-16$$

We distinguish the cases taking as limit values of a those in which the determinants are canceled and therefore capable of being the frontier of intervals with different signs:

	$\lvert T\rvert$	$\lvert A\rvert$	Classification
$a<-16$	$+$	$-$	ellipse
$a=-16$	0	$-$	parable
$-16<a<0$	$-$	$-$	hyperbola
$a=0$	$-$	0	real intersecting lines
$0<a<9$	$-$	$+$	hyperbola
$a=9$	$-$	0	real intersecting lines
$a>9$	$-$	$-$	hyperbola

b) *For $a=-1$ find the distance between its two foci.*

For $a=-1$ it is a hyperbola.

reduced equation of the form:

$$\frac{x''^2}{a'^2}-\frac{y''^2}{b'^2}=1$$

The focal length in that case will be $2c' = 2\sqrt{a'^2 + b'^2}$.

For the reduced equation we start by calculating the eigenvalues of T:

$$|T - \lambda Id| = 0 \iff (1-\lambda)^2 - 4^2 = 0 \iff (\lambda+3)(\lambda-5) = 0.$$

The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 5$ (we put the negative first because $det(A) = -10 < 0$). The reduced equation is of the form:

$$-3x''^2 + 5y''^2 + d = 0 \qquad \text{with} d = \frac{|A|}{\lambda_1\lambda_2} = \frac{-10}{-15} = \frac{2}{3}.$$

Is left over:

$$-3x''^2 + 5y''^2 + \frac{2}{3} = 0 \iff 3x''^2 - 5y''^2 = \frac{2}{3} \iff \frac{3x''^2}{2/3} - \frac{5y''^2}{2/3} = 1$$

And finally:

$$\frac{x''^2}{2/9} - \frac{y''^2}{2/15} = 1.$$

The focal length is:

$$2\sqrt{\frac{2}{9} + \frac{2}{15}} = 2\sqrt{\frac{10+6}{45}} = \frac{8\sqrt{45}}{45} = \frac{8\sqrt{5}}{15}.$$

c) *For the conics of the family that break down into a pair of intersecting lines, find such lines.*

The cases are $a = 0$ and $a = 9$.

For $a = 0$ the given equation becomes:

$$x^2 + 8xy - 2x = 0 \iff x(x + 8y - 2) = 0$$

The two straight lines are therefore $x = 0$ and $x + 8y - 2 = 0$.

For $a = 9$ the equation remains:

$$x^2 + 8xy - 9y^2 - 2x - 18y = 0.$$

The algebraic decomposition is not so obvious. We proceed as follows:

- We calculate the center (p, q) of the conic:

$$A \begin{pmatrix} p \\ q \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$$

where h is any real value. Is left over:

$$p + 4b - 1 = 0, \quad 4p - 9q - 9 = 0.$$

From where $(p, q) = (9/5, -1/5)$.

- We intersect any straight line with the conic. For example $x = 0$:

$$\left.\begin{array}{r} x = 0 \\ x^2 + 8xy - 9y^2 - 2x - 18y = 0 \end{array}\right\} \iff (x,y) = (0,0) \text{ or } (x,y) = (0,-2)$$

- The lines we are looking for are those that join the center with the two points found:

The line that joins $(9/5, -1/5)$ and $(0, 0)$ is:

$$\frac{x-0}{9/5} = \frac{y-0}{-1/5} \iff x + 9y = 0.$$

The line that joins $(9/5, -1/5)$ and $(0, -2)$ is:

$$\frac{x-0}{9/5} = \frac{y+2}{-1/5+2} \iff x - y - 2 = 0.$$

11.– *Find the equations of*

(a) *a parabola knowing that it passes through the points $P = (0, 3)$, $Q = (2, 6)$ and has the straight line $x - y + 1 = asitsaxis0$.*

We will use the fact that the axis is a line of symmetry to find the symmetrics P' and Q' of the points P and Q.

Then we will build the bundle of conics that pass through four points.

Finally, we select the parabolas from the bundle, imposing that the matrix of quadratic terms has a null determinant.

Let $P' = (a, b)$ be the symmetric of $P = (0, 3)$ with respect to the axis $x - y + 1 = 0$ (which has $(1, 1)$ as its direction vector) .

- It is true that $\dfrac{P + P'}{2} \in axis$, that is, $\dfrac{a}{2} - \dfrac{b+3}{2} + 1 = 0$.

- The vector $\vec{P}P'$ is perpendicular to the axis. Therefore $\vec{P}P' \cdot (1, 1) = 0$ that is, $a + b - 3 = 0$.

Solving, we obtain $P' = (2, 1)$.

Let $Q' = (c, d)$ be the symmetric of $Q = (2, 6)$ with respect to the axis $x - y + 1 = 0$ (which has $(1, 1)$ as its direction vector) .

- It is true that $\dfrac{Q + Q'}{2} \in axis$, that is, $\dfrac{c+2}{2} - \dfrac{d+6}{2} + 1 = 0$.

- The vector $\vec{Q}Q'$ is perpendicular to the axis. Therefore $\vec{Q}Q' \cdot (1, 1) = 0$ that is, $c - 2 + d - 6 = 0$.

Solving, we obtain $Q' = (5, 3)$.

We now form the conic bundle that passes through P, P', Q, Q'. We build a first conic with the lines PP' and QQ:

- Line PP', joins $P = (0, 3)$ and $P' = (2, 1)$. It remains: $\dfrac{x-0}{2-0} = \dfrac{y-3}{1-3}$. Simplifying: $x + y - 3 = 0$.

- Line QQ', joins $Q = (2, 6)$ and $Q' = (5, 3)$. It remains: $\dfrac{x-2}{5-2} = \dfrac{y-6}{3-6}$. Simplifying: $x + y - 8 = 0$.

The second conic with the lines PQ' and $P'Q$.

- Line PQ', joins $P = (0, 3)$ and $Q' = (5, 3)$. It is $y = 3$, that is, $y - 3 = 0$.

- Line $P'Q$, joins $P' = (2, 1)$ and $Q' = (2, 6)$. It is $x = 2$, that is, $x - 2 = 0$.

The bundle of conics remains:

$$(x + y - 3)(x + y - 8) + \lambda(x - 2)(y - 3) = 0$$

Operating:

$$x^2 + (2 + \lambda)xy + y^2 - (11 + 3\lambda)x - (11 + 2\lambda y) + 24 - 6\lambda = 0.$$

The matrix of quadratic terms is:

$$T = \begin{pmatrix} 1 & (2+\lambda)/2 \\ (2+\lambda)/2 & 1 \end{pmatrix}$$

The determinant is zero if:

$$1 - \left(\frac{2+\lambda}{2}\right)^2 = 0 \iff \lambda = 0 \text{ or } \lambda = -4$$

If $\lambda = 0$, the conic remains $(x+y-3)(x+y-8) = 0$, which is clearly degenerate (product of two lines) and therefore is not a parabola.

If $\lambda = -4$ remains:

$$x^2 - 2xy + y^2 + x - 3y = 0$$

The matrix A of the conic is:

$$A = \begin{pmatrix} 1 & -2 & 1/2 \\ -2 & 1 & -3 \\ 1/2 & -3/2 & 0 \end{pmatrix}$$

It has a nonzero determinant and therefore it is the requested parabola.

(b) *the conic whose center is $C(1,1)$ and such that $y = 1$ is an axis and the polar of the point $(2,2)$ is the line $x + y - 3 = 0$.*

Since one axis is $y = 1$, we know that one eigenvector of the associated matrix T has the direction $(1,0)$ and the other (perpendicular) the $(0,1)$. Therefore the matrix of the conic is of the form :

$$A = \begin{pmatrix} a & 0 & d \\ 0 & b & e \\ d & e & c \end{pmatrix}$$

Since the center is $(1,1)$, it is verified:

$$(1,1,1)A = (0,0,t) \quad \Rightarrow \quad \begin{cases} a + d = 0 & \Rightarrow \quad d = -a \\ b + e = 0 & \Rightarrow \quad e = -b \end{cases}$$

Now we impose the polarity condition:

$$(2,2,1)A = (1,1,-3) \quad \Rightarrow \quad \begin{cases} 2a + d = 1 \\ 2b + e = 1 \\ 2d + 2e - c = -3 \end{cases}$$

Solving the system formed by the 5 equations is:

$$a = 1; \quad b = 1; \quad c = 1; \quad d = -1; \quad e = -1$$

That is, the conic is:

$$x^2 + y^2 - 2x - 2y + 1 = 0 \iff (x-1)^2 + (y-1)^2 = 1$$

(the circle with center $(1,1)$ and radius 1).

(c) *the equation of an ellipse with center the origin, which has the point $F(1,1)$ as its focus and passes through the point $(1,-1)$.*

Given that the center of the conic is a center of symmetry of the same, the symmetrical F' of F with respect to the origin would be the other focus; equivalently the origin is the midpoint of the two foci:

$$(0,0) = \frac{(1,1) + F'}{2} \iff F' = (-1,-1)$$

Now we use the characterization of the ellipse as a locus, that is, as a set of points whose sum of distances to the foci is constant:

$$\sqrt{(x-1)^2 + (y-1)^2} + \sqrt{(x+1)^2 + (y+1)^2} = k$$

To find the constant k we impose that the conic pass through the point $(x,y) = (1,-1)$:

$$k = \sqrt{(1-1)^2 + (-1-1)^2} + \sqrt{(1+1)^2 + (-1+1)^2} = 4.$$

The equation remains:

$$\sqrt{(x-1)^2+(y-1)^2}+\sqrt{(x+1)^2+(y+1)^2}=4.$$

It only remains to operate to simplify the expression.

$$\begin{aligned}
&\sqrt{(x-1)^2+(y-1)^2}=4-\sqrt{(x+1)^2+(y+1)^2}\\
&(x-1)^2+(y-1)^2=4^2+(x+1)^2+(y+1)^2-8\sqrt{(x+1)^2+(y+1)^2}\\
&x^2-2x+1-y^2-2y+1=16+x^2+2x+1+y^2+2y+1-8\sqrt{(x+1)^2+(y+1)^2}\\
&8\sqrt{(x+1)^2+(y+1)^2}=4x+4y+16\\
&2\sqrt{(x+1)^2+(y+1)^2}=x+y+4\\
&4(x^2+2x+1+y^2+2y+1)=x^2+y^2+16+2xy+8x+8y\\
&3x^2-2xy+3y^2-8=0.
\end{aligned}$$

(d) *Find the equation of a hyperbola that passes through the origin, has the line $x-2y-1=0$ as an asymptote and one of its axes is the line $x-y-1=0$.*

The axis is an axis of symmetry of the conic. Therefore we can calculate the other asymptote as symmetric of the given one . Once we have the two asymptotes we will form the bundle of conics and we will obtain the sought conic imposing that it passes through the origin.

To find the symmetry of the asymptote we take into account that one of its points is the intersection of it and the axis of symmetry:

$$\left.\begin{array}{r} x-2y-1=0\\ x-y-1=0 \end{array}\right\} \quad\Rightarrow\quad (x,y)=(1,0).$$

To find another point of the sought line, we find the symmetric of any point of the given asymptote. We take for example the point $A=(-1,-1)$ that verifies the equation $x-2y-1=0$. Its symmetric $A'=(a,b)$ satisfies:

- The midpoint of A, A' belongs to the axis of symmetry:

$$\frac{A+A'}{2}=(\frac{a-1}{2},\frac{b-1}{2})\text{belongs to the line} xy-1=0,$$

where from,

$$a-b=2.$$

- The vector AA' is perpendicular to the axis of symmetry, from where:

$$(a+1,b+1)(1,1)=0 \iff a+b=-2.$$

We get $A'=(a,b)=(0,-2)$. The sought asymptote is the line that joins the points $(1,0)$ and $(0,-2)$:

$$\frac{x}{1}-\frac{y}{2}=1 \iff 2x-y-2=0.$$

The bundle of conics known to the two asymptotes is:

$$(x-2y-1)(2x-y-2)+c=0.$$

Assuming that it passes through the origin, we find the value of c:

$$(-1)(-2)+c=0 \quad\Rightarrow\quad c=-2.$$

The requested conic remains:

$$(x-2y-1)(2x-y-2)-2=0,$$

operand:

$$2x^2-5xy+2y^2-4x+5y=0.$$

(e) *a parabola passing through the points* $P=(0,2)$, $Q=(1,0)$ *and such that the line joining* P *and* Q *is the polar line of the point* $(0,0)$.

Method I: Let us remember that the polar line of a point O with respect to a conic joins the points of tangency of the external tangents to the conic passing through O. In our case this means that the lines joining $O=(0,0)$ with P and Q respectively, are tangent to the parabola.

We will use the bundle of known conics two tangents and the points of tangency.

The line OP has the equation $x=0$; the line OQ has the equation $y=0$; the line PQ is:

$$\frac{x-0}{1-0}=\frac{y-2}{0-2} \iff 2x+y-2=0.$$

The beam remains:

$$\lambda xy+(2x+y-2)^2=0 \iff 4x^2+y^2+(4+\lambda)xy-8x-4y+4=0.$$

To find the parameter we impose that the conic be of the parabolic type, that is, that the determinant of the matrix of quadratic terms is null.

$$\begin{vmatrix} 4 & (4+\lambda)/2 \\ (4+\lambda)/2 & 1 \end{vmatrix}=0 \iff \frac{\lambda^2}{4}+2\lambda=0.$$

We obtain:

- $\lambda=0$, but then substituting in the bundle we would only have the double line $(2x+y-2)^2=0$ which is not a parabola.

-$\lambda=-8$. Then we are left with the equation:

$$4x^2+y^2-4xy-8x-4y+4=0.$$

The associated array is:

$$\begin{pmatrix} 4 & -2 & -4 \\ -2 & 1 & -2 \\ -4 & -2 & 4 \end{pmatrix}.$$

Its determinant is non-zero, therefore it is a non-degenerate conic and ultimately the sought-after parabola.

Method II: The matrix associated with the sought conic is:

$$A=\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}.$$

We impose the conditions of the statement.

Since the points P and Q belong to the conic we have the equations:

$$\begin{aligned} (0,2,1)A(0,2,1)^t=0 &\iff 4d+4e+f=0. \\ (1,0,1)A(1,0,1)^t=0 &\iff a+2c+f=0. \end{aligned} \qquad (I)$$

Furthermore, the polar line of the point $(0,0)$ is the line PQ, $2x+y-2=0$:

$$(0,0,1)A(x,y,1)^t = 0 \quad \equiv \quad 2x+y-2=0 \iff cx+ey+f \quad \equiv 2x+y-2=0.$$

We deduce that:

$$\frac{c}{2}=\frac{e}{1}=\frac{f}{-2} \iff c=2e, \quad f=-2e.$$

Substituting in (I) we get:

$$d=-e/2, \qquad a=-2e.$$

And the associated array:

$$A=\begin{pmatrix} -2e & b & 2e \\ b & -e/2 & e \\ 2e & e & -2e \end{pmatrix}.$$

We impose that the determinant of the matrix of quadratic terms is null so that the conic is of the parabolic type:

$$\begin{vmatrix} -2e & b \\ b & -e/2 \end{vmatrix} = 0 \iff e^2=b^2.$$

We have two solutions:

$$e=b \text{ or } e=-b.$$

If we assume $e=b=2$, the associated matrix becomes:

$$A=\begin{pmatrix} -4 & 2 & 4 \\ 2 & -1 & 2 \\ 4 & 2 & -4 \end{pmatrix}$$

with $det(A)\neq 0$. It is the parable sought:

$$-4x^2-y^2+4xy+8x+4y-4=0 \iff 4x^2+y^2-4xy-8x-4y+4=0.$$

If we assume $e=-b=2$, the associated matrix becomes:

$$A=\begin{pmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{pmatrix}$$

with $det(A)=0$. It is degenerate and is not a parable.

(f) *a conic whose axis is the line $x-2y=0$, is tangent to $x=3$ and passes through the points $(3,1)$ and $(4,1)$.*

Since it is tangent to $x=3$ and passes through $A=(3,1)$ (which satisfies the equation of that line) the point of tangency is precisely A. So:

- Using the axis and for symmetry we will obtain a second tangent and a second point of tangency.

- We will build the bundle of known conics two tangents and the points of tangency.

- We will impose that the sought conic passes through $(-2,1)$.

The symmetric of the point $A(3,1)$ with respect to the line $x-2y=0$ is a point $A'(a,b)$ such that:

- The vector $\vec{A}A'=(a-3,b-1)$ is perpendicular to the axis of symmetry or equivalently parallel to its normal vector:

$$\frac{a-3}{1}=\frac{b-1}{-2} \iff 2a+b-7=0.$$

- The midpoint of A and A' belongs to the axis $x-2y=0$:

$$\frac{3+a}{2}-2\cdot\frac{b+1}{2}=0 \iff a-2b+1=0.$$

Solving the system formed by both equations yields $A' = (\frac{13}{5}, \frac{9}{5})$.

The intersection B of the given tangent with the axis is:

$$\begin{cases} 0 = x - 2y \\ 0 = x - 3 \end{cases} \iff B = (x, y) = (3, \frac{3}{2}).$$

The symmetric of the tangent is the line that joins A' and B:

$$\frac{x - \frac{13}{5}}{3 - \frac{13}{5}} = \frac{y - \frac{9}{5}}{\frac{3}{2} - \frac{9}{5}} \iff 3x + 4y - 15 = 0.$$

We form the beam described above. The first conic is the product of the two tangents :

$$(x - 3)(3x + 4y - 15) = 0$$

The second is the double line that joins the points of tangency $(3, 1)$ and $(13/5, 9/5)$:

$$\frac{x - 3}{\frac{13}{5} - 3} = \frac{y - 1}{\frac{9}{5} - 1} \iff 2x + y - 7 = 0.$$

The beam turns out:

$$\lambda(x - 3)(3x + 4y - 15) + (2x + y - 7)^2 = 0$$

We impose that it passes through the point $(4, 1)$:

$$\lambda(4 - 4)(3 \cdot 4 + 4 \cdot 1 - 15) + (2 \cdot 4 + 1 - 7)^2 = 0 \iff \lambda = -4.$$

The requested conic has the equation:

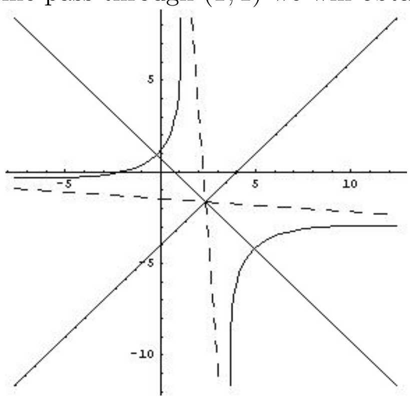
$$(2x + y - 7)^2 - 4(x - 3)(3x + 4y - 15) = 0$$

Simplifying:

$$8x^2 + 12xy - y^2 - 34y - 68x + 131 = 0.$$

(g) *a conic with vertex at the point $V(1, 1)$, passing through the point $(2, 4)$ and such that the lines $x + y - 2 = 0$ and $x = 2$ are tangent to it.*

The line $x + y - 2 = 0$ passes through the vertex $V(1, 1)$ and therefore is tangent to the conic at that point. The perpendicular to it is an axis of the conic. It will allow us to calculate the symmetric of the other tangent. We will have two tangents and points of tangency; Using a bundle of conics and imposing that the requested conic pass through $(1, 1)$ we will obtain the requested curve.



1) We first calculate the axis. The normal vector of $x+y-2=0$ is $(1,1)$. Therefore, the axis is the straight line that passes through $V(1,1)$ and has $(1,1)$ as its direction vector:

$$\frac{x-1}{1}=\frac{y-1}{1} \iff x-y=0.$$

2) We calculate the symmetric of the line $x=2$ by the line $xy=0$. To do this, we first intersect the two lines:

$$x=2, \quad x-y=0 \quad \iff (x,y)=(2,2).$$

We now calculate the symmetric of the point $P(2,4)$ by the line $xy=0$. Such a symmetric $P'(p,q)$ satisfies:

$$\frac{P+P'}{2} \in \text{Axis of symmetry} \qquad P'-P \perp \text{Axis}$$

Where we get:

$$\frac{2+p}{2}-\frac{4+q}{2}=0, \qquad (p-2,q-4)\cdot(1,1)=0$$

and solving $(p,q)=(4,2)$.

The symmetric line is the one that joins $(2,2)$ and $(4,2)$, that is, the line $y=2$.

3) We form the bundle of known conics two tangents and the points of tangency $x=2$ tangent in $(2,4)$ and $y=2$ tangent in $(4,2)$. The line joining the points of tangency is:

$$x+y=6.$$

The beam remains:

$$\lambda(x-2)(y-2)+(x+y-6)^2=0$$

4) Finally we impose that it passes through the point $(1,1)$:

$$\lambda(1-2)(1-2)+(1+1-6)^2=0 \iff \lambda=-16.$$

The conic remains:

$$(x+y-6)^2-16(x-2)(y-2)=0$$

Simplifying:

$$x^2+y^2-14xy+20x+20y-28=0.$$

(h) *an ellipse knowing that one of its foci is at the point* $(-4,2)$ *, the farthest vertex of it is the point* $(2,-1)$ *and the eccentricity is worth* $1/2$.

If we denote by a,b,c respectively the lengths of the semi-major and minor axis and the distance from the focus to the center, from the given data we deduce that:

$$a+c=d(F,V), \qquad \frac{c}{a}=e=\frac{1}{2},$$

where $F=(-4,2)$ and $V=(2,-1)$. Operating we get:

$$d(F,V)=\sqrt{(-4-2)^2+(2-(-1)))^2}=3\sqrt{5}$$

and hence $a=2\sqrt{5}$ and $c=\sqrt{5}$.

The other focus F' is at a distance $2c$ on the line that joins F and V in the direction $\vec{FV}$, that is,

$$F'=F+\frac{\vec{FV}}{\|FV\|}\cdot 2c.$$

Operating we have:

$$\vec{F}V = (2,-1)-(-4,2) = (6,-3)\text{y}\|\vec{F}V\| = \sqrt{6^2+(-3)^2} = 3\sqrt{5}.$$

where from:

$$F' = (-4,2) + \frac{(6,-3)}{3\sqrt{5}} \cdot 2\sqrt{5} = (0,0).$$

Finally, to obtain the equation of the ellipse, we use its characterization as a locus of points whose sum of distances to the foci is constant. We also know that such a constant is $2a$. The equation will then be:

$$\sqrt{(x-(-4))^2+(y-2)^2} + \sqrt{x^2+y^2} = 4\sqrt{5}.$$

Simplifying it remains:

$$16x^2 + 19y^2 + 4xy + 60x - 30y - 225 = 0.$$

(1.2 points)

(i) *the parabola C such that: the line of equation $x+y-2=0$ is the tangent to C at the vertex; C passes through the origin of coordinates; and the polar line of the point $(2,1)$ with respect to C is parallel to the OX axis.*

The eigenvector associated with the nonzero eigenvalue of the matrix T (quadratic part) of the parabola has the same direction as the tangent at the vertex, that is, $(1,-1)$. Bearing in mind that $det(T)=0$, except for a scalar product, we deduce that T is of the form:

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and the associated array A:

$$A = \begin{pmatrix} 1 & -1 & d \\ -1 & 1 & e \\ d & e & f \end{pmatrix}$$

Since the origin belongs to the conic, we deduce that $f=0$.

We now apply that the polar line at the point $(2,1)$ is parallel to the OX axis. It means that:

$$(2,1,1)A(x,y,1)^t \text{is parallel to } y=0 \quad \Rightarrow \quad 2-1+d=0 \quad \Rightarrow \quad d=-1$$

We have that, the equation of the conic is:

$$x^2+y^2-2xy-2x-2ey=0 \iff (x-y)^2-2x+2ey=0$$

To calculate the coefficient e, we apply that $x+y-2$ is tangent to the parabola. Its intersection with the conic must be a single point. The generic point of this line is $(\lambda, 2-\lambda)$. We substitute in the conic:

$$(2\lambda-2)^2-2\lambda+2e(2-\lambda)=0 \iff 4\lambda^2+(-2e-10)\lambda+(4+4e)=0$$

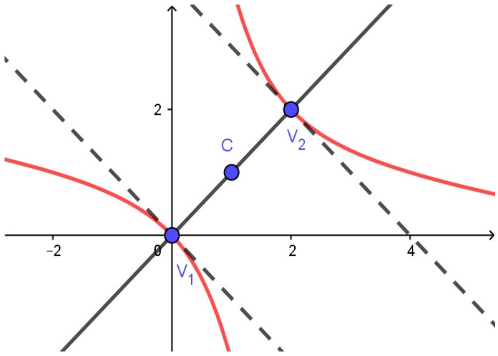
For there to be only one solution, the discriminant must be zero:

$$(-2e-10)^2-16(4+4e)=0 \quad \Rightarrow \quad e=3$$

The requested equation is:

$$x^2+y^2-2xy-2x+6y=0$$

12.– *Find the equation of a hyperbola knowing that its center is $(1,1)$, it has a vertex at $(0,0)$ and it passes through the point $(4,1)$.*



We will develop the following idea:

1) Since the conic is symmetric about the center we can calculate the second vertex.

2) In the vertices the tangents are perpendicular to the axis that is the line that joins them. Therefore we can use the bundle of conics known two points and their tangents in them .

3) Finally, we will impose that the conic passes through the point $(4,1)$.

1) We have $C=(1,1)$ and $V_1=(0,0)$. If V_2 is the second vertex:

$$\frac{V_1+V_2}{2}=C \quad \Rightarrow \quad V_2=2C-V_1=(2,2).$$

2) The pencil known two points and two tangents is generated by the product of the two tangents and the double line that joins the points of tangency.

The tangents at the vertices are perpendicular to the axis and therefore their normal vector is the one that joins the center and vertices: $\vec{V_1C}=(1,1)$. They are lines of the form $x+y+d=0$.

- If we impose that it pass through $V_1=(0,0)$, $0+0+d=0$ remains, that is, $d=0$ and the line is $x+y=0$.

- If we impose that it pass through $V_2=(2,2)$, $2+2+d=0$ remains, that is, $d=-4$ and the straight line is $x+y-4=0$.

The line that joins the points of tangency, that is, the vertices $V_1=(0,0)$ and $V_2=(2,2)$ is:

$$\frac{x-0}{2-0}=\frac{y-0}{2-0} \iff x-y=0.$$

The corresponding bundle of conics remains:

$$(x+y)(x+y-4)+\lambda(x-y)^2=0.$$

3) Now we impose that it passes through the point $(4,1)$:

$$(4+1)(4+1-4)+\lambda(4-1)^2=0 \iff \lambda=-5/9$$

Substituting the value of $\lambda=-5/9$ into the bundle results in:

$$(x+y)(x+y-4)-\frac{5}{9}(xy)^2=0 \iff 9(x+y)(x+y-4)-5(x-y)^2=0$$

and simplifying:

$$x^2+y^2+7xy-9x-9y=0.$$

13.– *In a bundle of conics generated by two conics that are not of parabolic type , what is the maximum number of parabolas that can be?.*

A bundle of conics is formed by taking linear combinations of the generating conic equations:

$$((x, y, 1)A_1(x, y, 1)^t) + \lambda((x, y, 1)A_2(x, y, 1)^t) = 0.$$

(by using a single parameter we omit one of the generating conics; but we already know that these are not a parabola).

The condition for them to be parabolas is that the determinant of the matrix of quadratic terms is null. But this determinant, being of order two and not always vanishing (because there is at least one conic that is not a parabola), gives us an equation of degree one or two in the variable λ. So at most there are one or two solutions. The maximum number of parables that can be is two.

14.– *Find the equation of a hyperbola with vertices at the points $(0, 0)$ and $V = (2, 2)$ and an asymptote perpendicular to the line $2x + y = 0$.*

We will follow the following steps:

- The center is the midpoint of the vertices and the axis is the line that joins them.

- One of the asymptotes is perpendicular to the given one and passing through the center.

- We calculate the symmetric of the asymptote with respect to the axis to obtain a second asymptote. A point of it we know is the center. The other will be symmetric about the axis of a point of the first asymptote.

- Finally we propose the bundle of known conics as two asymptotes and we impose that the sought conic pass through one of the vertices.

The center is:

$$C = \frac{(0, 0) + (2, 2)}{2} = (1, 1).$$

The axis of the line that joins $(0, 0)$ and $(2, 2)$:

$$\frac{x - 0}{2 - 0} = \frac{y - 0}{2 - 0} \iff x - y = 0.$$

A line perpendicular to $2x + y = 0$ is of the form $x - 2y + c = 0$. Imposing that $(1, 1)$ pass through the center, $c = 1$ remains. That is, the line $x - 2y + 1 = 0$.

We take any point of the given asymptote $x - 2y + 1 = 0$. For example $P = (-1, 0)$. Its symmetric $P' = (a, b)$ with respect to the axis fulfills:

$$\frac{P + P'}{2} \in axis \iff \frac{a - 1}{2} - \frac{b}{2} = 0 \iff a - b = 1.$$

and

$$\vec{PP'} \perp axis \iff (a + 1, b) \perp (1, 1) \iff a + b = -1$$

We get $P' = (a, b) = (0, -1)$.

The second asymptote is therefore the straight line that joins the center $(1, 1)$ with $(0, -1)$:

$$\frac{x - 0}{1 - 0} = \frac{y + 1}{1 + 1} \iff 2x - y - 1 = 0$$

The corresponding bundle of known conics two asymptotes is:

$$(x - 2y + 1)(2x - y - 1) + d = 0$$

We impose that it passes through the vertex $(0, 0)$:

$$(0 - 2 \cdot 0 + 1)(2 \cdot 0 - 0 - 1) + d = 0 \iff d = 1.$$

The conic remains:

$$(x - 2y + 1)(2x - y - 1) + 1 = 0 \iff 2x^2 - 5xy + 2y^2 + x + y = 0.$$

15.– *Find the equation of a conic that passes through the point* $(3, 2)$, *has the line* $xy = 0$ *as an asymptote and* $x - 2y + 1 = 0$ *as its axis.*

The axis of the conic is an axis of symmetry . Therefore we will follow the following path:

- the symmetric of the asynote is the second asynote .

- we will form the known beam two asymptotes.

- we will impose that it passes through the point $(3, 2)$.

To find the symmetric of the asymptote, we first take into account that it has to pass through the intersection of the y-axis of the given asymptote:

$$\begin{cases} 0 = x - y \\ 0 = x - 2y + 1 \end{cases} \quad \Rightarrow \quad (x, y) = (1, 1).$$

Then we take any point of the asymptote and calculate its symmetry. A point of the asymptote is for example $A = (0, 0)$. The symmetric $A' = (a, b)$ will satisfy:

- $\dfrac{A + A'}{2} \in axis$; that is, $(a/2, b/2) \in axis \iff (a/2) - 2(b/2) + 1 = 0 \iff a - 2b + 2 = 0.$

- $\vec{AA'} \perp axis$; i.e. $\vec{AA'} || normal_{axis} \iff (a, b)||(1, -2) \iff \begin{vmatrix} a & b \\ 1 & -2 \end{vmatrix} = 0 \iff 2a + b = 0.$

Solving $A' = (a, b) = (-2/5, 4/5)$.

The second asymptote is the line joining $(1, 1)$ and $(-2/5, 4/5)$:

$$\frac{x - 1}{(-2/5) - 1} = \frac{y - 1}{4/5 - 1} \iff x - 7y + 6 = 0.$$

The bundle of conics given the two asymptotes is:

$$(x - y)(x - 7y + 6) + \lambda = 0.$$

We force it to pass through $(3, 2)$:

$$(3 - 2)(3 - 7 \cdot 2 + 6) + \lambda = 0 \quad \Rightarrow \quad \lambda = 5.$$

The requested equation is:

$$(x - y)(x - 7y + 6) + 5 = 0 \iff x^2 - 8xy + 7y^2 + 6x - 6y + 5 = 0.$$

16.– *Find the equation of a conic given that its center is the point* $(1, 2)$, *it is tangent to the line* $x+y-2 = 0$ *at the point* $(2, 0)$ *y passes through the origin.*

We will develop the following idea:

- Given that a conic is symmetric with respect to its center we can know another tangent "doubling" by symmetry the given one .

- Then we can form the bundle of cncias known two tangents and the two points of tangency.

- Finally we impose that the sought conic passes through the origin.

We begin by finding the symmetric (p, q) of the point of tangency $(2, 0)$ with respect to the center $(1, 2)$. It must be fulfilled:

$$\frac{(p, q) + (2, 0)}{2} = (1, 2) \quad \Rightarrow \quad (p, q) = 2(1, 2) - (2, 0) = (0, 4).$$

Then we find the symmetric line of $x + y - 2 = 0$ with respect to the center $(1, 2)$. But the symmetry of a line with respect to a point is parallel to the original line. Therefore, the line sought is the line parallel to $x + y - 2 = 0$ that passes through the point $(0, 4)$ calculated before. It will be of the form $x + y + d = 0$. Enforcing it to pass through $(0, 4)$:

$$0 + 4 + d = 0 \quad \Rightarrow \quad d = -4$$

The line becomes $x + y - 4 = 0$.

The bundle of known conics two tangents and the points of tangency is generated by the conic formed by the product of both tangents and the double line that joins the points of tangency.

The line that joins the points of tangency $(2, 0)$ and $(0, 4)$ is:

$$\frac{x}{2} + \frac{y}{4} = 1 \iff 2x + y - 4 = 0.$$

The beam remains:

$$(x + y - 2)(x + y - 4) + \lambda(2x + y - 4)^2 = 0$$

We impose that it passes through the origin:

$$(0 + 0 - 2)(0 + 0 - 4) + \lambda(2 \cdot 0 + 0 - 4)^2 = 0 \iff \lambda = -1/2.$$

The equation remains:

$$(x + y - 2)(x + y - 4) - \frac{1}{2}(2x + y - 4)^2 = 0.$$

Operating and simplifying:

$$2x^2 - y^2 - 4x + 4y = 0.$$

17.– *Find the equation of an ellipse with center point $(1, 2)$, a focus at $(2, 4)$ and also knowing that the distance between the two vertices located on the minor axis is 4 .*

Using the fact that the conic is symmetric with respect to the center, we will calculate the second focus as symmetric of the first.

Later we will use the definition of the ellipse as locus: the sum of distances from points to foci is constant equal to $2a$.

Finally we will take into account that $a^2 = b^2 + c^2$ where b is the minor radius of the ellipse and c is the distance from the center to the focus.

So, $C = (1, 2)$, $F = (2, 4)$, F' is the symmetric of F with respect to C:

$$\frac{F + F'}{2} = C \iff F' = 2C - F = (2, 4) - (2, 4) = (0, 0).$$

Now from the given data $2b = 4$, that is, $b = 2$ and $c = d(F', C) = \sqrt{(1 - 0)^2 + (2 - 0)^2} = \sqrt{5}$. Therefore:

$$a = \sqrt{b^2 + c^2} = \sqrt{4 + 5} = 3.$$

Finally the equation of the ellipse is:

$$d(F, (x, y)) + d(F', (x, y)) = 2a \iff \sqrt{(x - 2)^2 + (y - 4)^2} + \sqrt{(x - 0)^2 + (y - 0)^2} = 6$$

Operating:

$$\sqrt{(x-2)^2+(y-4)^2} = 6 - \sqrt{(x^2+y^2)}$$
$$(x-2)^2+(y-4)^2 = 36 + x^2 + y^2 - 12\sqrt{x^2+y^2}$$
$$x^2 - 4x + 4 + y^2 - 8y + 16 = 36 + x^2 + y^2 - 12\sqrt{x^2+y2}$$
$$12\sqrt{x^2+y^2} = 4x + 8y + 16$$
$$3\sqrt{x^2+y^2} = x + 2y + 4$$
$$9(x^2+y^2) = (x+2y+4)^2$$
$$9x^2 + 9y^2 = x^2 + 4y^2 + 16 + 4xy + 8x + 16y$$
$$8x^2 - 4xy + 8y^2 - 8x - 16y - 16 = 0$$
$$2x^2 - xy + 2y^2 - 2x - 4y - 4 = 0$$