LINEAR ALGEBRA II — Problem Set 2.3

Orthogonal transformations — (Course 2022–2023)

1.– Consider the Euclidean vector space $\mathbb{R}^3$ and a fixed orthonormal basis in it. Obtain the matrix expression, relative to this basis, of:

(a) the orthogonal symmetry with respect to the subspace $\mathcal{L}\{(1,1,1)\}$;

(b) the orthogonal symmetry with respect to the subspace $\mathcal{L}\{(1,1,1),(2,0,1)\}$;

(c) the rotation of 60^o around the semi-axis that contains $(1,1,1)$ (considering in $\mathbb{R}^3$ the orientation corresponding to the starting basis).

2.– In the Euclidean vector space $\mathbb{R}^3$ find the equations of a rotation of angle 90^o and semi-axis generated by the vector $(3,0,4)$.

(Final exam, July 2017)

3.– On $\mathbb{R}^3$ we consider the scalar product whose Gram matrix with respect to the canonical basis is:

$$G = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Find the associated matrix with respect to the canonical basis of a rotation of 36.87^o, with a semi-axis generated by the vector $(1,0,0)$ and considering the positive orientation given by the canonical basis. **Observation:** $sin(36.87^o) = \dfrac{3}{5}$.

(Extraordinary exam, December 2008)

4.– In the Euclidean space $\mathbb{R}^3$ with the usual scalar product, we consider the endomorphism $t : \mathbb{R}^3 \to \mathbb{R}^3$ whose associated matrix relative to the canonical basis is:

$$T_C = \begin{pmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \end{pmatrix}$$

(i) Prove that t is a orthogonal transformation.

(ii) Classify t indicating, if applicable, the angle and axis of rotation and/or the plane of symmetry.

(Final exam, July 2021)

5.– In the Euclidean space $\mathbb{R}^2$ with the usual scalar product, we consider the endomorphism $t : \mathbb{R}^2 \to \mathbb{R}^2$, $t(x,y) = \left(\dfrac{3}{5}x + \dfrac{4}{5}y, \dfrac{4}{5}x - \dfrac{3}{5}y\right)$. Prove that t is an orthogonal transformation and describe it geometrically providing the angle of rotation or the axis of symmetry (whichever is appropriate).

(Final exam, July 2018)

6.– In the vector space $\mathbb{R}^3$ we consider the usual scalar product and the positive orientation given by the canonical basis.

(i) The matrix of an endomorphism relative to the canonical basis is

$$T_C = \begin{pmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix}$$

Prove that it is an orthogonal transformation and classify it, describing what it does geometrically.

(ii) Calculate the matrix associated with a symmetry with respect to the plane with equation $x + y - z = 0$.

(iii) Is it possible to obtain the previous symmetry as the composition of two conveniently chosen rotations? Justify your answer.

(Final exam, July 2020)

7.– Decide whether the following assertions are true or false:

(i) If T is the matrix associated with an orthogonal transformation t in $\mathbb{R}^3$ and $trace(T) = 2$ then t is a rotation of 60^o.

(ii) If T is the matrix associated with an orthogonal transformation t in $\mathbb{R}^3$ and $trace(T) = 0$ then t is a rotation of 120^o.

(iii) In $\mathbb{R}^3$ the composition of a symmetry with respect to a line with a symmetry with respect to a plane is a rotation .

(iv) If t is an orthogonal transformation on $\mathbb{R}^3$ and $t(1, 0, 1) = (-1, 0, -1)$ then it is an inverse transformation.

(Final exam, June 2021)

8.– Justify whether the following assertions are true or false:

(i) If T is the matrix of an orthogonal transformation in $\mathbb{R}^2$ and $trace(T) = 1$ then T is a rotation.

(ii) If T is the matrix of a rotation in $\mathbb{R}^2$ then it has no real eigenvalues.

(iii) If we consider the Euclidean space $\mathbb{R}^3$ with the usual conditions (usual scalar product and positive orientation given by the canonical basis), $T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the matrix of a symmetry with respect to the plane $x - y = 0$. (0.5 points)

(iv) On the Euclidean space $\mathbb{R}^3$ (usual conditions) an orthogonal transformation t is defined. Gauss holds that it is a rotation of 90 degrees and Euler says that it is a rotation of -90 degrees. Can they both be right?

(Final exam, June 2020)

9.– Let T be the matrix associated with an orthogonal transformation t in the Eculidean space $\mathbb{R}^3$. Knowing that $det(T) < 0$, $t(1, 1, 2) = (-1, -1, -2)$ and $trace(T) = 1$, classify and describe geometrically the transformation t.

(Final exam, May 2018)

10.– In $\mathbb{R}^2$ with the usual scalar product, find the associated matrices with respect to the canonical basis of all possible inverse orthogonal transformations that take the line $x + y = 0$ into the line $x - y = 0$.

(Final exam, June 2021)

11.– In the Euclidean space $\mathbb{R}^3$ with the usual scalar product we consider the planes $\pi_1 : x+y-z = 0$ and $\pi_2 : 2x - y + z = 0$. Find the equations of a rotation that takes the plane π_1 into the plane π_2.

12.– In $\mathbb{R}^2$ with respect to the usual scalar product, consider a linear transformation $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ whose associated matrix with respect to the canonical basis is

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

(i) Find a and b so that t is a symmetry with respect to a line.

(ii) For each of the values of a and b obtained in the previous section, calculate the axis of symmetry.
(Final exam, May 2017)

13.– In the Euclidean space $\mathbb{R}^3$ we consider the linear mapping $t : \mathbb{R}^3 \to \mathbb{R}^3$ whose associated matrix with respect to canonical basis is

$$T_C = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ b & 0 & a \end{pmatrix}$$

(i) Study for which values of a and b, t is an orthogonal transformation.

(ii) For each of the previous cases, classify it and describe it geometrically.

(Final exam, May 2022)

14.– Justify the truth or falsehood of the following statements.

(i) The trace of the matrix associated with a symmetry with respect to a line in the plane is 0.

(ii) If the matrix associated to an orthogonal transformation in the plane has trace 0 then it is a symmetry with respect to a line.

(iii) In a three-dimensional space, the symmetry with respect to a line is an inverse transformation.

(iv) In a three-dimensional space, the composition of a symmetry with respect to a point and a symmetry with respect to a plane is always a rotation.

(Final exam, June 2019)
