LINEAR ALGEBRA II **Problem Set 2.1-2.2**

Euclidean spaces. Orthogonality (Academic Year 2022–2023)

1.– We consider a basis $\{\bar{e}_1,\ \bar{e}_2,\ \bar{e}_3\}$ of a 3-dimensional Euclidean space, such that the modulus of both $\bar{e}_1$ and $\bar{e}_3$ is 2 and $\bar{e}_2$ is 1, and besides, the angle formed by $\bar{e}_1$ and $\bar{e}_3$ is 90^o and the one formed both by $\bar{e}_1$ and $\bar{e}_2$ and by $\bar{e}_2$ and $\bar{e}_3$ is 60^o. We also consider the vectors

$$\bar{a} = \bar{e}_1 - \bar{e}_2$$
$$\bar{b} = \bar{e}_1 - 2\bar{e}_2 + \bar{e}_3$$

(a) Obtain the Gram matrix relative to the basis $\{\bar{e}_1,\ \bar{e}_2,\ \bar{e}_3\}$

(b) Obtain the modulus of $\bar{a}$ and $\bar{b}$.

(c) Find the scalar product of $\bar{a}$ and $\bar{b}$.

(d) Find a vector with module 2, which forms an angle of 90^o with $\bar{a}$ and one of 60^o with $\bar{b}$.

2.– We consider on $\mathbb{R}^3$ the scalar product whose Gram matrix with respect to the canonical basis is:

$$G_C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

(i) Give a pair of vectors from the canonical basis that are orthogonal.

(ii) Given $U = \mathcal{L}\{(1,1,1),(1,1,2),(2,2,3)\}$ compute a basis of its orthogonal subspace $U^{\perp}$.

(iii) Find an orthonormal basis of $\mathbb{R}^3$.

(iv) Calculate the angle formed by the vectors $(1,0,0)$ and $(0,1,0)$.

(Final exam, May 2021)

3.– We consider on the vector space $\mathbb{R}^3$ a scalar product whose Gram matrix with respect to the canonical basis is:

$$G_C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Calculate the associated matrix with respect to the canonical basis of the orthogonal projection mapping onto the vector subspace:

$$V = \mathcal{L}\{(1,0,0),(0,1,0)\}$$

(Final exam, June 2019)

4.– On the vector space $\mathbb{R}^3$ we consider a scalar product whose Gram matrix with respect to the base $B = \{(1,0,0),(1,1,0),(1,1,1)\}$ is:

$$G_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Given the vectors $\vec{v} = (1,0,1)$, $\vec{u} = (0,1,1)$ compute $\vec{u} \cdot \vec{v}$, $\|u\|$, $\|v\|$ and the angle formed by them.

(Final exam, July 2019)

5.– Let $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be a scalar product satisfying the following conditions:

- The vector subspaces $\mathcal{L}\{(1,0,1)\}$ and $\mathcal{L}\{(1,1,0),(0,0,1)\}$ are orthogonal.

- The vectors $(1,1,0)$ and $(0,0,1)$ form an angle of $\pi/3$.

- The three considered vectors are unitary.

(i) Find the Gram matrix of the scalar product with respect to the canonical basis.

(ii) Given $U = \mathcal{L}\{(1,1,1),(1,1,2),(2,2,3)\}$ compute a basis of its orthogonal subspace $U^\perp$, relative to the given scalar product.

(Final exam, May 2018)

6.– Let $f : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ a bilinear form whose matrix relative to the canonical basis is

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

(i) Prove that the bilinear form f defines a scalar product.

(ii) Obtain an orthonormal basis of $U = \mathcal{L}\{(1,0,1),(1,1,0),(3,2,1)\}$ relative to the scalar product f.

(Final exam, July 2021)

7.– Consider the map

$$f : \mathcal{M}_{2\times 2}(\mathbb{R}) \times \mathcal{M}_{2\times 2}(\mathbb{R}) \to \mathbb{R}, \quad f(A,B) = trace(AB^t)$$

(i) Prove that f is a scalar product defined on the vector space $\mathcal{M}_{2\times 2}(\mathbb{R})$.

(ii) Calculate the Gram matrix of f with respect to the canonical basis.

(iii) If $U = \mathcal{L}(Id)$, find a basis of the orthogonal subspace $U^\perp$.

(Final exam, July 2018)

8.– Given the symmetric matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ find an orthogonal matrix P (i. e. $P^{-1} = P^t$) such that $P^{-1}AP$ is a diagonal matrix.

9.– On $\mathbb{R}^3$ we consider a bilinear form f whose associated matrix relative to the canonical basis is:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}$$

(i) Prove that it is a scalar product.

(ii) With respect to the scalar product defined by f:

(ii.a) Find the associated matrix of the orthogonal projection onto $\mathcal{L}\{(1,0,0)\}$ relative to the canonical basis .

(ii.b) Find an orthogonal basis of the subspace $U = \{(x,y,z) \in \mathbb{R}^3 | x + y + z = 0\}$.

(Final exam, July 2022)

10.– Let $\mathcal{P}_1(\mathbb{R})$ be the vector space of all polynomials with degree less than or equal to 1. We consider the bilinear form $f : \mathcal{P}_1(\mathbb{R}) \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ whose associated matrix with respect to the canonical basis is:

$$F_C = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}.$$

(i) Prove that f is a scalar product.

(ii) With respect to the scalar product defined by f:

(ii.a) Give two polynomials that form an orthonormal basis of $\mathcal{P}_1(\mathbb{R})$.

(ii.b) Find the angle formed by the polynomials $p(x) = 1 + x$ and $q(x) = 1 - x$.

(Final exam, May 2022)

11.– Consider the real vector space $\mathbb{R}^3$ endowed with the ordinary scalar product. Find the matrix F (relative to the canonical basis) of a symmetric endomorphism f of $\mathbb{R}^3$, knowing that the kernel of f is the subspace $\mathcal{L}\{(1,1,1)\}$ and 3 is a double eigenvalue of f.

(Final exam, September 2002)

12.– Let $f : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a bilinear form whose associated matrix relative to the canonical basis is:

$$F_C = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 0 \\ 1 & 0 & a \end{pmatrix}$$

(i) For which values of a is f a scalar product?

(ii) Find a so that the vectors $(1,0,1)$ and $(0,3,-1)$ are orthogonal with respect to the scalar product defined by f.

(iii) For $a = 4$ and relative to the scalar product defined by f, give an orthonormal basis and calculate the angle formed by the vectors $(1,0,0)$ and $(0,1,0)$.

(Final exam, May 2015)

13.– We consider on $\mathbb{R}^3$ the scalar product given by the Gram matrix

$$G_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}.$$

Fin the matrix associated to the orthogonal projection on the subspace:

$$U = \{(x, y, z) \in \mathbb{R}^3 |\, x = 0, \quad y + 2z = 0\}$$

relative to the canonical basis. What is the orthogonal projection of $(1, 1, 1)$ onto the subspace U?

(Final exam, July 2015)

14.– Find the Gram matrix (relative to the canonical basis) of a scalar product which satisfies:

- The vectors $(1, 0)$ and $(0, 1)$ form an angle of 60 degrees.

- $\|(1, 1)\| = \sqrt{3}$.

- $B = \{(1, 0), (1, -2)\}$ is an orthogonal basis.

(Final exam, July 2017)
