LINEAR ALGEBRA II — Problem Set 1.1

Bilinear mappings and quadratic forms (Academic year 2022–2023)

1.– Check whether each one of the following mappings is bilinear or not. For those which turn out to be, give their associated matrix relative to the corresponding canonical basis. Find out also which of the bilinear forms are either symmetric or antisymmetric.

(a) $f : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f((x_1, x_2), (y_1, y_2)) = 2x_1y_2 - 3x_1y_1$

(b) $g : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad g((x_1, x_2), (y_1, y_2)) = x_1x_2 + y_1y_2$

(c) $h : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad h((x_1, x_2, x_3), (y_1, y_2, y_3)) = 5x_1y_1 + 4x_1y_2 + 1 - x_2y_1 + 5x_3y_1$

(d) $l : \mathcal{M}_{2\times2}(\mathbb{R}) \times \mathcal{M}_{2\times2}(\mathbb{R}) \longrightarrow \mathbb{R}, \quad l(A, B) = \mathrm{tr} AB$

(e) $m : \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}, \quad m(p, q) = p(1)q(-1) - p(-1)q(1)$

2.– Given the quadratic form $w : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $w(x, y) = x^2 + 4xy + 3y^2$:

(i) Classify it, and give its rank and signature.

(ii) Find a basis of conjugate vectors.

(iii) Find all self-conjugate vectors, expressing them in the most simple way possible (give the result in terms of the canonical basis).

(iv) Obtain the matrix associated to w relative to the basis

$$B = \{(1, 1), (1, -1)\}$$

(v) If f is the symmetric bilinear form associated to w, find $f((2, 1), (1, 3))$.

3.– Given the quadratic form $w : \mathbb{R}^3 \longrightarrow \mathbb{R}$,

$w(x, y, z) = x^2 + 2xy + 2xz + 2yz + z^2$.

(i) Obtain the matrix associated to w relative to the canonical basis. Classify this quadratic form and give its rank and signature.

(ii) Obtain a basis of conjugate vectors.

(iii) Find all self-conjugate vectors expressing them, if possible, as the union of two planes.

(iv) If f is the symmetric bilinear form associated to w, obtain $f((1, 0, 1), (0, 1, 0))$.

(Final exam, July 2018)

4.– Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of all polynomials with degree less or equal 2. Consider the mapping

$$f : \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}, \qquad f(p(x), q(x)) = \int_{-1}^{1} p(x)q(x)dx - 2p(0)q(0).$$

(i) Show that f is a symmetric bilinear form.

(ii) Obtain the matrix associated to f relative to the canonical basis.

(iii) Give a set of polynomials that constitutes a basis of conjugate vectors.

(iv) Find all self-conjugate polynomials.

(v) Classify the quadratic form asociated to f.

(Final Exam, July 2012)

5.– On $\mathbb{R}^3$ we define a quadratic form w whose matrix relative to the canonical basis is

$$F_C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & a \\ 1 & a & 2 \end{pmatrix}$$

(i) Classify the quadratic form in terms of a, indicating its rank and signature.

(ii) For $a = 1$ compute a basis of conjugate vectors.

(iii) For $a = 0$ obtain all self-conjugate vectors.

(iv) Find the values of a for which the bilinear form associated to w is a scalar product.

(Final Exam, July 2017)

6.– On $\mathbb{R}^3$ we define the quadratic form given by:

$$w(x, y, z) = ax^2 + y^2 + 4xy + 2axz.$$

(i) Classify it in terms of a, indicating its rank and signature in each case.

(ii) For $a = 0$ find all self-conjugate vectors, expressing them as the union of two planes.

(Partial exam, February 2015)

7.– We define the following quadratic form for each value of $k \in \mathbb{R}$:

$$w : \mathbb{R}^3 \to \mathbb{R}, \qquad w(x, y, z) = x^2 + 2kxy - z^2 + 2yz$$

(i) Classify w in terms of k, indicating its rank and signature.

(ii) For $k = 1$ find all self-conjugate vectors. Express the result in the simplest way and relative to the canonical basis.

(iii) Give one vector which is self-conjugate for any value of k.

(iv) Find k such that the vectors $(1, 0, 0)$ and $(0, 1, 0)$ are conjugate.

(v) Is there any value of k for which there aren't any non-null self-conjugate verctors?

(Final Exam, June 2020)

8.– A certain quadratic form $w : \mathbb{R}^3 \to \mathbb{R}$ satisfies

- $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ is a basis of conjugate vectors.
- The vector $(0, 1, 1)$ is in the kernel of this quadratic form.
- $w(1, 1, 0) = 2$.

(i) Obtain the matrix associated to w relative to the canonical basis.

(ii) Classify w and give its rank and signature.

(iii) Find all self-conjugate vectors. If possible, express them as the union of two planes, giving generators for each one of them.

(iv) If f is the symmetric bilinear form associated to w, find $f((1, 1, 0), (1, 2, -1))$.

(Final Exam, July 2021)

9.– Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of all polynomials with degree less or equal 2 and real coefficients. Consider the bilinear form:

$$\phi : \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}, \quad \phi(p(x), q(x)) = \int_0^1 p(x)q'(x)dx + \int_0^1 p'(x)q(x)dx$$

(i) Show that ϕ is symmetric.

(ii) Find the matrix associated to ϕ relative to the canonical basis.

(iii) Find the rank and signature of the quadratic form associated to ϕ.

(iv) Find a basis of polynomials of $\mathcal{P}_2(\mathbb{R})$, relative to which the matrix associated to ϕ is diagonal.

Notation: $p'(x), q'(x)$ denote respectively the derivatives of $p(x), q(x)$.

(Final Exam, May 2011)

10.– Let $w : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a quadratic form . We know that

- $\{(1,1,0),(0,1,0),(1,1,1)\}$ is a basis of conjugate vectors.
- The rank of w is 1.
- $(0,1,0)$ is a self-conjugate vector.
- $w(1,2,0) = 1$.

i) Find the matrix associated to w relative to the canonical basis.

ii) Classify w.

iii) Find all self-conjugate vectors of w.

(Final Exam, July 2013)

11.– Let $f : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a symmetric bilinear form. We know that

- $ker(f) = \mathcal{L}\{(1,0,1)\}$.
- The vectors $(1,0,0)$ y $(1,1,1)$ are conjugate relative to f.
- If w is the quadratic form associated to f, $w(1,0,0) = w(1,1,1) = 2$.

(i) Find the matrix associated to F relative to the canonical basis.

(ii) Classify the quadratic form w indicating its rank and signature.

(iii) Give a basis of conjugate vectors for w.

(iv) Find all self-conjugate vectors.

(Final Exam, June 2019)

12.– Let $w : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a quadratic form and $u, v \in \mathbb{R}^n$ satisfying $w(u) = 1, w(v) = -1$. Show that $\{u, v\}$ are linearly independent. Is necessarily w an indefinite quadratic form?

(Final Exam, June 2009)

13.– In each one of the following items, give a nondiagonal matrix associated to a quadratic form w on $\mathbb{R}^3$ which also satisfies the indicated condition (justify your answers).

(i) w is positive definite.

(ii) w is negative semidefinite.

(iii) w is indefinite and nondegenerate.

(iv) w is indefinite and degenerate.

(Final Exam, July 2022)

14.– Let $w : \mathbb{R}^3 \to \mathbb{R}$ be a nondegenerate quadratic form on $\mathbb{R}^3$ and let F_C be its associated matrix relative to the canonical basis. Justify the truth or falsehood of the following asssertions:

(i) If every diagonal entry of F_C is positive then w is positive definite.

(ii) If at least one diagonal entry of F_C is zero then w is indefinite.

(iii) F_C^2 is the matrix associated to a definite positive quadratic form.

(iv) If $F_C = Id$ then $f((x_1, x_2, x_3)_B, (y_1, y_2, y_3)_B) = x_1y_1+x_1y_3+2x_2y_2+x_3y_1+3x_3y_3$ can be the expression of a symmetric bilinear form associated to w, relative to some basis B.

(v) $f((x, y, z), (x', y', z')) = xx' + xy' + yx' + yy' + zz'$ can be the expression of a symmetric bilinear form associated to w.

(Final Exam, July 2020)

15.– Analyze and justify the truth or falsehood of the following assertions :

(i) If w is a quadratic form on $\mathbb{R}^2$ whose rank is 1 then it cannot be indefinite..

(ii) If $A \in \mathcal{M}_{3\times3}(\mathbb{R})$ is the matrix associated to a quadratic form w and it satisfies $a_{11} = 1$, $a_{22} = 1$ and $a_{33} = 0$ then w is positive semidefinite.

(iii) If $A \in \mathcal{M}_{3\times3}(\mathbb{R})$ is the matrix associated to a quadratic form w and it satisfies $a_{11} = 1$, $a_{22} = 1$ and $a_{33} = -1$ then w is indefinite.

(iv) $\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$ and $\begin{pmatrix} 3 & 6 \\ 6 & 5 \end{pmatrix}$ can be two matrices associated to the same quadratic form relative to different basis.

(v) If $A \in \mathcal{M}_{2015\times2015}(\mathbb{R})$ is the matrix associated to a negative semidefinite quadratic form, then $det(A) < 0$.

(Partial exam, May 2015)
