LINEAR ALGEBRA II **Exercises Unit II. Chapters 1 and 2**

Introduction to Euclidean spaces. Orthogonality. (Academic Year 2022–2023)

1.– Check which of the following bilinear forms are scalar products:

(a) $f : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f((x, y), (x', y')) = xx' + 3xy' - yx' + 2yy'$

(b) $f : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f((x_1, x_2), (y_1, y_2)) = x_1y_2 - x_2y_1$

(c) $f : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad f((x, y, z), (x', y', z')) = xx' + xy' + xz' + yx' + 2yy' + 3yz' + zx' + 3zy' + 9zz'$

(d) $f : \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}, \quad f(p(x), q(x)) = \int_0^1 p(x)q(x)dx$

2.– For the scalar product in item (c) of exercise 1, obtain:

(a) The Gram matrix of the dot product.

(b) $(1, 0, 1) \cdot (1, 1, 1)$.

(c) $\|(1, 0, 1)\|$.

(d) The angle formed by the vectors $(1, 0, 1)$ and $(1, 1, 1)$.

3.– For the scalar product in item (d) of exercise 1, obtain:

(a) The Gram matrix of the dot product.

(b) $< 1 - x, \, x - x^2 >$.

(c) $\|1 - x\|$.

(d) The angle formed by the vectors (polynomials) $1 - x$ and $x - x^2$.

4.– For the scalar product in item (c) of exercise 1:

(a) Check if the vectors $(1, 0, 0)$ and $(2, -1, -1)$ are orthogonal.

(b) Obtain an orthogonal basis of $\mathbb{R}^3$.

(c) Obtain an orthonormal basis of $\mathbb{R}^3$.

5.– With respect to the usual dot product of $\mathbb{R}^3$ and using Gram-Schmidt method, calculate an orthogonal basis and an orthonormal basis of the subspace:

$$U = \mathcal{L}\{(1, 0, 1), (1, 1, 1)\}$$

6.– With respect to the scalar product in item (c) of exercise 1 and using Gram-Schmidt method, calculate an orthogonal basis and an orthonormal basis of the subspace:

$$U = \mathcal{L}\{(1, 0, 1), (1, 1, 1)\}$$

7.– With the usual scalar product, calculate the projection of the vector $(2, 1, 0)$ onto the vector subspace $U = \mathcal{L}\{(1, 0, 1), (1, 1, 1)\}$

8.– With the usual scalar product and relative to the canonical basis, calculate the matrix associated to the projection map on the subspace $U = \mathcal{L}\{(1,0,1),(1,1,1)\}$

9.– Given the matrix $A = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$,

(a) Calculate the eigenvalues of A.

(b) Calculate the eigenvectors of A.

(c) Compute an orthonormal basis of eigenvectors of A.

(d) Compute an orthogonal matrix P (i. e. satisfying $P^t = P^{-1}$) such that $D = P^{-1}AP$ is a diagonal matrix.

Solutions.

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1. (a) No (it is not symmetric). (b) No (it is not symmetric). (c) Yes. (d) Yes.

2. (a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 9 \end{pmatrix}$. (b) 16. (c) $2\sqrt{3}$. (d) $arcos\frac{4\sqrt{66}}{33}$.

3. (a) $\begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$. (b) $\frac{1}{12}$. (c) $\frac{1}{\sqrt{3}}$. (d) $arcos\frac{\sqrt{10}}{4}$.

4. (a) Yes. (b) $\{(1,0,0),(-1,1,0),(1,-2,1)\}$. (c) $\{(1,0,0),(-1,1,0),(\frac{1}{2},-1,\frac{1}{2})\}$.

5. Orthogonal basis: $\{(1,0,1),(0,1,0)\}$. Orthonormal basis: $\{(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}),(0,1,0)\}$.

6. Orthogonal basis: $\{(1,0,1),(-\frac{1}{3},1,-\frac{1}{3})\}$. Orthonormal basis: $\{(\frac{\sqrt{3}}{6},0,\frac{\sqrt{3}}{6}),(-\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{2},-\frac{\sqrt{6}}{6})\}$.

7. $(1,1,1)$.

8. $\begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$.

9. (a) $\lambda_1 = 2, \quad a.\ m. = 2, \qquad \lambda_2 = -2, \quad a.\ m. = 1.$

(b) $S_2 = \mathcal{L}\{(1,1,0),(0,0,1)\}$, $S_{-2} = \mathcal{L}\{(1,-1.0)\}$.

(c) $\{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0),(0,0,1),(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0)\}$.

(d) $P = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$.