For each $k \in \mathbb{R}$ the quadratic form $w : \mathbb{R}^3 \to \mathbb{R}$ is considered,

$$w(x, y, z) = x^2 - 4z^2 + 4xy + 2xz + 2kyz.$$

1. *Classify the quadratic form in terms of k, also indicating its rank and signature.*

   To classify it, we will calculate the associated matrix F_C with respect to the canonical basis. Then we will diagonalize it by congruence (which is equivalent to changing the basis of the associated matrix) and based on the signs of the diagonal we will have the classification.

   To find F_C we incorporate as its entries the coefficients of the equation: those on the diagonal as they are, and those outside of it divided by two.

$$F_C = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & k \\ 1 & k & -16 \end{pmatrix}$$

   We diagonalize by doing the same row and column operations.

$$F_C \overset{H_{21}(-2)}{\longrightarrow} \overset{H_{31}(-1)}{\longrightarrow} \overset{\mu_{21}(-2)}{\longrightarrow} \overset{\mu_{31}(-1)}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & k-2 \\ 0 & k-2 & -17 \end{pmatrix} \overset{H_{32}((k-2)/4)}{\longrightarrow} \overset{\mu_{32}((k-2)/4)}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -17 + \frac{(k-2)^2}{4} \end{pmatrix}$$

   The classification of the quadratic form depends on the signs that appear on the diagonal. To see how they change as a function of k, we study as critical points those values of k for which

$$-17 + \frac{(k-2)^2}{4} = 0 \iff (k-2)^2 = 4 \cdot 17 \iff k = 2 \pm 2\sqrt{17}$$

   We then distinguish the following cases:

k	Signature	Range	Classification
$k < 2 - 2\sqrt{17}$	$(2, 1)$	3	Indefinite and non-degenerate
$k = 2 - 2\sqrt{17}$	$(1, 1)$	2	Indefinite and degenerate
$2 - 2\sqrt{17} < k < 2 + 2\sqrt{17}$	$(1, 2)$	3	Indefinite and non-degenerate
$k = 2 + \sqrt{17}$	$(1, 1)$	2	Indefinite and degenerate
$k > 2 + \sqrt{17}$	$(2, 1)$	3	Indefinite and non-degenerate

2. *Find a basis of conjugate vectors.*

   A basis B is a basis of conjugate vectors if and only if the associated matrix relative to that basis F_B is diagonal. Since in the previous section we have already diagonalized , this is equivalent to a change of basis:

$$F_B = M_{CB}^t F_C M_{CB}$$

   The matrix M_{CB} can be calculated by performing on the identity the same column operations as we did in the diagonalization process . Its columns will give the required basis.

$$Id \overset{\mu_{21}(-2)}{\longrightarrow} \overset{\mu_{31}(-1)}{\longrightarrow} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{\mu_{32}((k-2)/4)}{\longrightarrow} \begin{pmatrix} 1 & -2 & -k/2 \\ 0 & 1 & (k-2)/4 \\ 0 & 0 & 1 \end{pmatrix} = M_{CB}$$

   We obtain the basis

$$B = \{(1, 0, 0), (-2, 1, 0), (-k/2, (k-2)/4, 1)\}$$

3. *For those cases in which the quadratic form is undefined and of rank 2, give the implicit equation relative to the canonical basis of the two planes into which the set of self-conjugate vectors decomposes .*

   We have seen in the first section that there are two cases in which the quadratic form is undefined and of range 2: $k = 2 \pm 2\sqrt{17}$. In both cases, the diagonalized matrix obtained is:

$$F_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   We will work in principle with this matrix, that is, in the basis B to calculate and handle the self-conjugate vectors . If $(x', y', z')_B$ denotes the coordinates of a vector relative to basis B, then:

$$w(x', y', z') = (\, x' \quad y' \quad z' \,)\, F_B \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = (\, x' \quad y' \quad z' \,) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = x'^2 - 4y'^2.$$

So:

$$\begin{aligned} Self-conj(w) &= \{(x',y',z')_B | w(x',y',z')=0\} = \{(x',y',z')_B | x'^2-4y'^2=0\} = \\ &= \{(x',y',z')_B | (x'-2y')(x'+2y')=0\} = \\ &= \{(x',y',z')_B | x'-2y'=0\} \cup \{(x',y',z')_B | x'+2y'=0\}. \end{aligned}$$

We see that they break down into two planes. We are going to express their equations with respect to the canonical basis. Matricially they can be written as:

$$\begin{pmatrix} 1 & -2 & 0 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_B = 0 \iff \begin{pmatrix} 1 & -2 & 0 \\ 1 & 2 & 0 \end{pmatrix} M_{BC} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_C = 0$$

where $M_{BC} = M_{CB}^{-1}$, and M_{CB} is in turn the change-of-basis matrix calculated in part (ii), for $k = 2 \pm 2\sqrt{17}$. A convenient way to find the inverse is to perform on the identity (and in opposite order) the inverses of the column operations that we did in (ii):

$$Id \overset{\mu_{32}(\mp\sqrt{17}/2)}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mp\sqrt{17}/2 \\ 0 & 0 & 1 \end{pmatrix} \overset{\mu_{21}(2)\mu_{31}(1)}{\longrightarrow} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & \mp\sqrt{17}/2 \\ 0 & 0 & 1 \end{pmatrix} = M_{BC}$$

Then the change of basis is

$$\begin{pmatrix} 1 & -2 & 0 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & \mp\sqrt{17}/2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_C = 0 \iff \begin{pmatrix} 1 & 0 & 1 \pm \sqrt{17} \\ 1 & 4 & 1 \mp \sqrt{17} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_C = 0$$

So the self-conjugate vectors are

- For $k = 2 + 2\sqrt{17}$:

$$Self-conj(w) = \{(x,y,z) \in \mathbb{R}^3 | x + (1+\sqrt{17})z = 0\} \cup \{(x,y,z) \in \mathbb{R}^3 | x + 4y + (1-\sqrt{17})z = 0\}.$$

- For $k = 2 - 2\sqrt{17}$:

$$Self-conj(w) = \{(x,y,z) \in \mathbb{R}^3 | x + (1-\sqrt{17})z = 0\} \cup \{(x,y,z) \in \mathbb{R}^3 | x + 4y + (1+\sqrt{17})z = 0\}.$$

4. *For $k = 2$ give the implicit equations of a subspace $U \subset \mathbb{R}^3$ of the maximum possible dimension, such that $w(\vec{u}) < 0$ for all nonzero $\vec{u} \in U$.*

As we saw in theory , the maximum dimension of a subspace over which the quadratic form is negative for any nonzero vector is just the number of minus signs that appear in the signature of the quadratic form. In this case we saw in (i) that $sign(w) = (1, 2)$ and therefore such dimension is 2.

To find the equations of the subspace once again we work with the diagonalized associated matrix calculated in the first section:

$$F_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -17 \end{pmatrix}$$

so that:

$$w(x',y',z') = x'^2 - 4y'^2 - 17z'^2$$

the nonzero vectors $(0, y', z')_B$ satisfy $w(0,y',z') = -4y'^2 - 17z'^2 < 0$ and therefore $x' = 0$ defines a subspace of dimension 2 with the required characteristics.

We change its equation to the canonical basis in the same way we have done in the previous section:

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_B = 0 \iff \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} M_{BC} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_C = 0$$

where $M_{BC} = M_{CB}^{-1}$, where M_{CB} is the change of basis matrix calculated in section (ii), for $k = 2$. Is left over:

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} M_{BC} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_C = 0 \iff \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_C = 0$$

Operating we obtain the implicit equation :

$$x + 2y + z = 0.$$

5. *For $k = 4$ give the implicit equations of a subspace $V \subset \mathbb{R}^3$ of the maximum possible dimension, such that $w(\vec{v}) > 0$ for all $\vec{v} \in U$ not null.*

Analogously to what was seen in the previous section, the maximum dimension of a subspace over which the quadratic form is positive for any non-zero vector is just the number of plus signs that appear in the signature of the quadratic form. In this case we saw in (i) that $sign(w) = (1, 2)$ and therefore such dimension is 1.

To find the equations of the subspace once again we work with the diagonalized associated matrix calculated in the first section:

$$F_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -16 \end{pmatrix}$$

so that:

$$w(x', y', z') = x'^2 - 4y'^2 - 16z'^2$$

It is clear that if $y' = z' = 0$ then the nonzero vectors $(x', 0, 0)_B$ satisfy $w(x', 0, 0) = x'^2 > 0$ and therefore $y' = z' = 0$ define a subspace of dimension 1 with the required characteristics.

We change its equation to the canonical basis in the way we have done in the previous section:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_B = 0 \quad \Longleftrightarrow \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M_{BC} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_C = 0$$

where $M_{BC} = M_{CB}^{-1}$, where M_{CB} is the change of basis matrix calculated in section (ii), for $k = 2$. We obtain

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M_{BC} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_C = 0 \quad \Longleftrightarrow \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_C = 0$$

Operating we obtain the implicit equations :

$$\begin{array}{l} x - \frac{y}{2} = 0 \\ z = 0 \end{array}$$

6. *For $k = 0$ give non-zero vectors $\vec{u}, \vec{v}, \vec{w}$ such that $w(\vec{u}) > 0$, $w(\vec{v}) = 0$ and $w(\vec{w}) < 0$.*

One could work directly with the expression of w in the canonical basis to obtain the requested vectors. But once again it is more comfortable to work in the diagonalized form :

$$F_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -16 \end{pmatrix}$$

so that:

$$w(x', y', z') = x'^2 - 4y'^2 - 16z'^2$$

So it is immediate that:

- A vector satisfying

$$w(\vec{u}) > 0$$

is $u = (1, 0, 0)_B$. We change it to the canonical basis:

$$M_{CB} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

- A vector satisfying

$$w(\vec{v}) = 0$$

is $u = (2, 1, 0)_B$. We change it to the canonical basis:

$$M_{CB} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

- A vector satisfying

$$w(\vec{w}) < 0$$

is $u = (0,1,0)_B$. We change it to the canonical basis:

$$M_{CB}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1 & -2 & 0\\0 & 1 & -1/2\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-2\\1\\0\end{pmatrix}.$$

7. *The function $w(x,y,z)$ is bounded from below (respectively from above) if there exists some constant M such that $w(x,y,z) > M$ (respectively $w(x,y,z) < M$) for every vector $(x,y,z) \in \mathbb{R}^3$. ¿Is there any value of k for which w is bounded from above or below?.*

We saw in the first section that the quadratic form is INDEFINITE for any value of k. ThIS means that there always exists a vector u_+ such that $w(u_+) > 0$ and a vector u_- such that $w(u_-) < 0$.

Let us see that then the function $w(x,y,z)$ cannot be bounded from above nor from below. We will use the property of quadratic forms:

$$w(\lambda u) = \lambda^2 w(u)$$

So if there were an upper bound $M > 0$ it would have to be true that $w(u) < M$ for all $u \in \mathbb{R}^3$. But since $w(u_+) > 0$ and

$$w(\lambda u_+) = \lambda^2 w(u_+)$$

We see that $w(\lambda u_+)$ is as big as we want when $\lambda \to \infty$. For example if we take $\lambda = \sqrt{2M/w(u_+)}$,

$$w(\lambda u_+) = \lambda^2 w(u_+) = 2M > M$$

contradicting the assumed bound.

Similarly, if there were a lower bound $M < 0$, it would have to be true that $w(u) > M$ for all $u \in \mathbb{R}^3$. But since $w(u_-) < 0$ and

$$w(\lambda u_-) = \lambda^2 w(u_-).$$

We see that $w(\lambda u_-)$ which is as large (in negative) as we want as $\lambda \to \infty$. For example if we take again $\lambda = \sqrt{2M/w(u_-)}$,

$$w(\lambda u_-) = \lambda^2 w(u_-) = 2M < M$$

contradicting the assumed bound.