Linear Algebra II — Final Exam

Single exercise (3 hours) **July 3, 2023**

1.– Let $\mathcal{P}_1(\mathbb{R})$ be the vector space of all polynomials with degree less or equal to 1. Consider the mapping:

$$f : \mathcal{P}_1(\mathbb{R}) \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}, \qquad f(p(x), q(x)) = p(1)q(1) - p'(1)q'(1)$$

(i) Show that f is a symmetric bilinear form.

(ii) Classify f, also indicating its rank and signature.

(iii) Give two polynomials which form a basis of conjugate vectors with respect to f.

(iv) Calculate the set of self-conjugate vectors. If this set can be expressed as the union of two subspaces of dimension 1, give a generator of each of them.

(v) Find $w(1 + 2x)$ where w is the quadratic form associated with f.

(1.3 points)

2.– We consider the Euclidean space $\mathbb{R}^3$ with the scalar product with respect to which the basis $B = \{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$ is orthonormal.

(i) Find the Gram matrix of this scalar product with respect to the canonical basis.

(ii) Let $U = \mathcal{L}\{(1, 0, 0)\}$. Find the parametric equations of $U^{\perp}$.

(iii) Find the orthogonal projection of the vector $(2, 1, 0)$ onto the subspace U.

(1.1 points)

3.– In the Euclidean space $\mathbb{R}^3$ we consider the endomorphism $t : \mathbb{R}^3 \to \mathbb{R}^3$ whose associated matrix with respect to the canonical basis is:

$$T_C = \begin{pmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{pmatrix}$$

Show that t is an orthogonal transformation. Classify it and describe it geometrically.

(1 point)

4.– Let F be the matrix associated to the quadratic form w. Justify the truth or falsehood of the following assertions:

(i) If $\text{trace}(F) > 0$ then w is positive definite.

(ii) If $\text{trace}(F) = 0$ and $\det(F) \neq 0$ then w is indefinite.

(iii) If $\det(F) < 0$ then w cannot be positive definite.

(iv) If $\det(F) < 0$ then w is negative definite.

(1.2 points)

5.– In the affine Euclidean space $\mathbb{R}^3$ find the equations of a symmetry with respect to a plane which transforms the point $(0, 0, 1)$ into the point $(2, 2, 1)$.

(1 point)

6.– In the affine Euclidean space $\mathbb{R}^2$ consider the lines $r : x + y = 1$ and $s : x = 3$ and the point $P = (2, 2)$.

(i) Find the equation of a line t which contains P in such a way that P is the midpoint of the points of intersection of t with the lines r and s.

(ii) Find the equations of a homothety which transforms the point P into the origin and transforms any shape into another one whose area is four times the initial one.

(1.2 points)

7.– In the affine plane for each $k \in \mathbb{R}$ consider the conic with equation

$$x^2 - 2kxy + y^2 + 2y + 1 = 0$$

(i) Classify the conic in terms of the values of k.

(ii) For $k = 2$ find its center, its axes and its eccentricity.

(iii) Show that the OY axis is tangent to the conic at a point P which does not depend on the value of k. Find P.

(1.3 points)

8.– Find the equation of a conic such that its eccentricity is $e = 2$, the point $(4, 2)$ is a focus, and the line $2x + y - 5 = 0$ is one of its axes.

(1.3 points)

9.– Given the quadric surface of equation:

$$x^2 + 2xz + 4yz + 4x + 2y + 2z = 0.$$

classify the surface and sketch a drawing of it.

(0.6 points)
