Part II

Matrices and Determinants.

1. Matrices.

1 Basic definitions.

Definition 1.1 *A* **matrix** *A of dimension $m \times n$ is a set of numbers (scalars) of a field $\mathbb{K}$ arranged in a rectangular formation with m rows and n columns.*

When $m = n$ the matrix is called **square matrix***; in other case* **rectangular matrix***.*

We will usually denote a matrix with capital letters, while for its elements we will use lowercase letters. Specifically, the element at the i-th row and the j-th column of a matrix A will be denoted by a_{ij} or exceptionally, by $(A)_{ij}$:

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

The set of all matrices with m rows and n columns defined on a field $\mathbb{K}$ will be denoted by $\mathcal{M}_{m\times n}(\mathbb{K})$. If we have previously fixed the field, we will simply use $\mathcal{M}_{m\times n}$.

Two matrices are equal when they have the same dimension and the corresponding elements are equal.

2 Operations with matrices.

2.1 Addition of matrices.

Given two matrices A, B with the same dimension and defined over the same field, the sum of A and B is the matrix

$$C = A + B, \quad \text{with} \quad c_{ij} = a_{ij} + b_{ij}, \quad i = 1, \dots, m; \quad j = 1, \dots, n.$$

Therefore matrix addition is an internal operation on $\mathcal{M}_{m\times n}(\mathbb{K})$. As a consequence of the group structure of the field $\mathbb{K}$, matrix addition satisfies the following properties:

1. **Commutative:**
$$A + B = B + A,$$
for any $A, B \in \mathcal{M}_{m\times n}(\mathbb{K})$.
2. **Associative:**
$$A + (B + C) = (A + B) + C,$$
for any $A, B, C \in \mathcal{M}_{m\times n}(\mathbb{K})$.
3. **Neutral element:** The matrix $\Omega \in \mathcal{M}_{m\times n}(\mathbb{K})$ where all elements are zero, is the neutral element of the addition, that is:
$$\Omega = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad \text{and} \quad A + \Omega = \Omega + A = A,$$
for any $A \in \mathcal{M}_{m\times n}(\mathbb{K})$.
4. **Additive symmetric element:** For any matrix $A \in \mathcal{M}_{m\times n}(\mathbb{K})$, there exists a matrix $-A = (-a_{ij})$, satisfying
$$A + (-A) = \Omega.$$

Therefore the set $\mathcal{M}_{m\times n}(\mathbb{K})$ under matrix addition is an abelian group.

2.2 Multiplication of a matrix by a scalar.

Given a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{K})$ and a scalar $\alpha \in \mathbb{K}$ we define the matrix product of α by A as:

$$B = \alpha \cdot A, \quad \text{with} \quad b_{ij} = \alpha \cdot a_{ij}, \quad i = 1, \dots, m; \quad j = 1, \dots, n.$$

Multiplication of a matrix by a scalar satisfies‘ the following properties:

1. $1 \cdot A = A$, for any $A \in \mathcal{M}_{m\times n}(\mathbb{K})$.
2. $\alpha \cdot (\beta \cdot A) = (\alpha\beta) \cdot A$, for any $A \in \mathcal{M}_{m\times n}(\mathbb{K})$ and $\alpha, \beta \in \mathbb{K}$.
3. $(\alpha + \beta) \cdot A = \alpha \cdot A + \beta \cdot A$, for any $A \in \mathcal{M}_{m\times n}(\mathbb{K})$ and $\alpha, \beta \in \mathbb{K}$.
4. $\alpha \cdot (A + B) = \alpha \cdot A + \alpha \cdot B$, for any $A, B \in \mathcal{M}_{m\times n}(\mathbb{K})$ and $\alpha \in \mathbb{K}$.

2.3 Product of matrices.

The product of two matrices will be defined if **the number of columns in the first matrix is equal to the number of rows in the second matrix**.

Thus, given two matrices $A \in \mathcal{M}_{m\times n}(\mathbb{K})$ and $B \in \mathcal{M}_{n\times p}(\mathbb{K})$ we define the product matrix $C = A \cdot B \in M_{m\times p}$ as:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \quad i = 1, \ldots, m; \quad j = 1, \ldots, p.$$

Note that the element in the i, j position of the product matrix is obtained by using the i-th row of the first matrix and the j-th column of the second matrix.

$$\begin{pmatrix} & \vdots & & \\ \boxed{a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}} \\ & \vdots & & \end{pmatrix} \begin{pmatrix} \cdots & \boxed{\begin{matrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{matrix}} & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \boxed{a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

Some properties of the matrix product are:

1. **Associative:**
$$A \cdot (B \cdot C) = (A \cdot B) \cdot C,$$
for any $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{n\times p}$, $C \in \mathcal{M}_{p\times q}$.
Proof: We have:
$$[A \cdot (B \cdot C)]_{ij} = \sum_{k=1}^{n} a_{ik} (B \cdot C)_{kj} = \sum_{k=1}^{n} a_{ik} \sum_{l=1}^{p} b_{kl} c_{lj} = \sum_{k=1}^{n} \sum_{l=1}^{p} a_{ik} b_{kl} c_{lj},$$
and on the other hand:
$$[(A \cdot B) \cdot C]_{ij} = \sum_{k=1}^{p} (A \cdot B)_{ik} c_{kj} = \sum_{k=1}^{p} \left(\sum_{l=1}^{n} a_{il} b_{lk} \right) c_{kj} = \sum_{k=1}^{p} \sum_{l=1}^{n} a_{il} b_{lk} c_{kj}.$$
We see that both expressions coincide (the roles of the indices i and k appear interchanged).
2. **Identity element:** The **identity matrix of size** r, $I_r \in \mathcal{M}_{r\times r}(\mathbb{K})$ is:
$$I_r = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$
It satisfies:
$$A \cdot I_n = I_m \cdot A = A$$
for any $A \in \mathcal{M}_{m\times n}(\mathbb{K})$.
The **Kronecker delta** is a symbol depending on two integers i and j defined as:
$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$
With this notations, the identity matrix has entries equal to the Kronecker delta.
3. **Distributive:**
$$A \cdot (B + C) = A \cdot B + A \cdot C, \quad \text{with } A \in \mathcal{M}_{m\times n}, \quad B, C \in \mathcal{M}_{n\times p}$$
and
$$(A + B) \cdot C = A \cdot C + B \cdot C, \quad \text{with } A, B \in \mathcal{M}_{m\times n}, C \in \mathcal{M}_{n\times p}.$$

An important remark is,

The product of matrices is NOT COMMUTATIVE

that is, in general $A \cdot B \neq B \cdot A$.

2.3.1 Inverse of a matrix.

Definition 2.1 *A square matrix $A \in \mathcal{M}_{n\times n}(\mathbb{K})$ has an* **inverse** *if there exists a matrix $B \in \mathcal{M}_{n\times n}(\mathbb{K})$ satisfying:*

$$A \cdot B = B \cdot A = I_n.$$

In this case A is said to be **regular** *or* **invertible** *or* **nonsingular** *and the inverse matrix is denoted by A^{-1}.*

When A has no inverse, it is said to be **singular***.*

Let us see some properties:

1. If the inverse of A exists, it is unique:
Proof: Suppose there are matrices $B, C \in \mathcal{M}_{n\times n}(\mathbb{K})$ verifying:
$$A \cdot B = B \cdot A = I_n \text{ and } A \cdot C = C \cdot A = I_n.$$
Then:
$$B = I_n \cdot B = (C \cdot A) \cdot B = C \cdot (A \cdot B) = C \cdot I_n = C.$$
2. If $A, B \in \mathcal{M}_{n\times n}(\mathbb{K})$ are invertible, then:
$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}.$$
Proof: We have:
$$(A \cdot B) \cdot (B^{-1} \cdot A^{-1}) = A \cdot (B \cdot B^{-1}) \cdot A^{-1} = A \cdot I_n \cdot A^{-1} = A \cdot A^{-1} = I_n.$$
$$(B^{-1} \cdot A^{-1}) \cdot (A \cdot B) = B^{-1} \cdot (A^{-1} \cdot A) \cdot B = B^{-1} \cdot I_n \cdot B = B^{-1} \cdot B = I_n.$$

We will see later several ways to calculate the inverse of a matrix.

2.3.2 Power of a matrix.

Given an integer k and a square matrix $A \in \mathcal{M}_{n\times n}(\mathbb{K})$ we define the k-th power of A as:

$$A^k = \begin{cases} \underbrace{A\cdot A\cdot\ldots\cdot A}_{k\text{-times}} & \text{if } k>0. \\ I_n & \text{if } k=0. \\ \underbrace{A^{-1}\cdot A^{-1}\cdot\ldots\cdot A^{-1}}_{(-k)\text{-veces}} & \text{if } k<0 \text{ (and } A \text{ is regular).} \end{cases}$$

From this definition it is clear that:

$$A^k\cdot A^l = A^{k+l}$$

However, since the matrix product **is not commutative**, in general:

$$A^kB^k \neq (A\cdot B)^k$$

2.4 Transpose matrix.

Definition 2.2 *Given a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{K})$ we define its* **transpose matrix** *as a matrix $A^t \in \mathcal{M}_{n\times m}(\mathbb{K})$ that is obtained from A by swapping its rows whit its columns:*

$$(A^t)_{ij} = (A)_{ji} \qquad i=1,\ldots,n; \quad j=1,\ldots,m.$$

Some properties of transposition are:

1. $(A^t)^t = A$, for any $A \in \mathcal{M}_{m\times n}(\mathbb{K})$.
2. $(A+B)^t = A^t + B^t$, for any $A, B \in \mathcal{M}_{m\times n}(\mathbb{K})$.
3. $(\alpha A)^t = \alpha A^t$, for any $A \in \mathcal{M}_{m\times n}(\mathbb{K})$, $\alpha \in \mathbb{K}$.
4. $(A\cdot B)^t = B^t\cdot A^t$, for any $A \in \mathcal{M}_{m\times n}(\mathbb{K})$, $B \in \mathcal{M}_{n\times p}(\mathbb{K})$.
   Proof: The following holds:
   $$(A\cdot B)^t_{ij} = (A\cdot B)_{ji} = \sum_{k=1}^n a_{jk}b_{ki} = \sum_{k=1}^n (B^t)_{ik}(A^t)_{kj} = (B^t\cdot A^t)_{ij}$$
5. $(A^t)^{-1} = (A^{-1})^t$, for any $A \in \mathcal{M}_{n\times n}(\mathbb{K})$.
   Proof: It is sufficient to note that
   $$A^t\cdot(A^{-1})^t = (A^{-1}\cdot A)^t = I^t = I; \qquad (A^{-1})^t\cdot A^t = (A\cdot A^{-1})^t = I^t = I.$$

2.5 Trace of a matrix.

Definition 2.3 *Given a square matrix $A \in \mathcal{M}_{n\times n}$ the sum of the elements of its diagonal is called the trace of the matrix, that is:*

$$trace(A) = tr(A) = \sum_{k=1}^n a_{kk}.$$

The trace of a matrix satisfies the following properties:

1. If $A, B \in \mathcal{M}_{n\times n}(K)$ then $tr(A+B) = tr(A) + tr(B)$.
   Proof: This follows easily because the terms of diagonal of $A+B$ are obtained by adding the terms of the diagonal of A to the terms of the diagonal of B. Formally, if $D = A+B$:
   $$tr(A+B) = \sum_{k=1}^n d_{kk} = \sum_{k=1}^n (a_{kk}+b_{kk}) = (\sum_{k=1}^n a_{kk}) + (\sum_{k=1}^n b_{kk}) = tr(A) + tr(B)$$
2. If $A \in \mathcal{M}_{n\times n}(K)$ and $\alpha \in K$ then $tr(\alpha A) = \alpha tr(A)$.
   Proof: Once again, we only have to consider that the terms of the diagonal of (αA) are obtained by multiplying by α the terms of the diagonal of A. Formally, if $D = \alpha A$:
   $$tr(\alpha A) = \sum_{k=1}^n d_{kk} = \sum_{k=1}^n (\alpha a_{kk}) = \alpha\sum_{k=1}^n a_{kk} = \alpha tr(A)$$
3. If $A \in \mathcal{M}_{n\times n}(K)$ then $tr(A) = tr(A^t)$.
   Proof: It is immediate that the elements of the diagonal of a matrix and its transpose coincide, since they remain at the same position by swapping rows and columns.
4. If $A, B \in \mathcal{M}_{n\times n}(K)$ then $tr(BA) = tr(AB)$.
   Proof: First of all let us remember that the product of matrices is not commutative, that is, in general $AB \neq BA$. However, what the property tell us is that although the two matrices obtained by changing the order of the product could be different, their traces are the same.
   Let us first calculate the trace of AB. Let $D = AB$.
   $$tr(AB) = \sum_{k=1}^n d_{kk} = \sum_{k=1}^n\sum_{m=1}^n a_{km}b_{mk}$$
   Now, the trace of BA. Let $D' = BA$.
   $$tr(BA) = \sum_{k=1}^n d'_{kk} = \sum_{k=1}^n\sum_{m=1}^n b_{km}a_{mk} = \sum_{m=1}^n\sum_{k=1}^n a_{mk}b_{km}$$
   We see that they coincide.

5. If $A, C \in \mathcal{M}_{n\times n}(K)$ and C is invertible then $tr(C^{-1}AC) = tr(A)$.
   It is sufficient to apply the previous property to $C^{-1}A$ and C:
$$tr((C^{-1}A)C) = tr(C(C^{-1}A)) = tr(IA) = tr(A)$$
   (where I is the identity matrix of order n.)

3 Special matrices.

3.1 Symmetric and antisymmetric matrices.

Definition 3.1 *Given a square matrix $A \in \mathcal{M}_{n\times n}(\mathbb{K})$ we say that A is:*

- **symmetric** *when $A = A^t$.*
- **antisymmetric** *when $A = -A^t$.*

From this definition it is easy to check the following properties:

1. All the diagonal entries in an antisymmetric are zeros.
   Proof: If A is antisymmetric:
$$a_{ii} = -a_{ii} \quad \Rightarrow \quad 2a_{ii} = 0 \quad \Rightarrow \quad a_{ii} = 0$$
   for any $i = 1, \ldots, n$.
2. The only matrix that is both symmetric and antisymmetric is the zero matrix Ω.
   Proof: If A is both symmetric and antisymmetric we have:
$$a_{ij} \overset{A\text{ symm.}}{=} a_{ji} \overset{A\text{ antisymm.}}{=} -a_{ij} \quad \Rightarrow \quad 2a_{ij} = 0 \quad \Rightarrow \quad a_{ij} = 0$$
   for any $i, j = 1, \ldots, n$.
3. Any square matrix has a unique decomposition as the sum of a symmetric matrix and an antisymmetric matrix.
   Proof: Any square matrix $A \in \mathcal{M}_{n\times n}(\mathbb{K})$ can be written as
$$A = S + H, \quad \text{with} \quad \begin{cases} S = \frac{1}{2}(A + A^t) \\ \\ H = \frac{1}{2}(A - A^t) \end{cases}$$
   where S is symmetric and H is antisymmetric.
   Furthermore, if A had another descomposition $A = S' + H'$ with S' symmetric and H' antisymmetric
$$S + H = A = S' + H' \quad \Rightarrow \quad S - S' = H - H',$$
   so $S - S' = H - H'$ would be symmetric and antisymmetric. From this $S - S' = H - H' = \Omega$ and both decompositions are the same.

3.2 Diagonal matrices.

Definition 3.2 *A* **diagonal matrix** *is a square matrix in which the entries outside the main diagonal are all zero.*

$$D = \begin{pmatrix} d_{11} & 0 & \ldots & 0 \\ 0 & d_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_{nn} \end{pmatrix}.$$

Equivalently:

$$D \text{ diagonal} \iff d_{ij} = 0, \quad \forall i, j = 1, \ldots, n, \quad i \neq j$$

For any pair of matrices $A, B \in \mathcal{M}_{n\times n}(\mathbb{K})$ and $\alpha \in \mathbb{K}$ it holds:

$$A, B \text{ diagonal} \quad \Rightarrow \quad \alpha A,\ A^t,\ A^{-1} \text{ (if exists)},\ A + B,\ A \cdot B \text{ diagonal}$$

3.3 Triangular matrices

Definition 3.3 *An* **upper triangular matrix** *is a square matrix whose all elements below the main diagonal are zero.*

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{pmatrix}.$$

Equivalently:

$$A \textbf{ upper triangular} \iff a_{ij} = 0, \quad \forall i, j = 1, \ldots, n, \quad i > j.$$

For any pair of matrices $A, B \in \mathcal{M}_{n\times n}(\mathbb{K})$ and $\alpha \in \mathbb{K}$ the following holds:

$$A, B \begin{array}{c}\text{upper}\\ \text{triangular}\end{array} \quad \Rightarrow \quad \alpha A,\ A^{-1} \text{ (if it exists)},\ A + B,\ A \cdot B \begin{array}{c}\text{upper}\\ \text{triangular}\end{array}$$

Definition 3.4 *A* **lower triangular matrix** *is a square matrix all of whose elements above the main diagonal are zero.*

$$A = \begin{pmatrix} a_{11} & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}.$$

Equivalently:

$$A \textbf{ lower triangular} \iff a_{ij} = 0, \quad \forall i, j = 1, \ldots, n, \quad i < j.$$

For any pair of matrices $A, B \in \mathcal{M}_{n\times n}(\mathbb{K})$ and $\alpha \in \mathbb{K}$

$$A, B \begin{array}{c}\text{lower}\\ \text{triangular}\end{array} \quad \Rightarrow \quad \alpha A,\ A^{-1} \text{ (if it exists)},\ A + B,\ A \cdot B \begin{array}{c}\text{lower}\\ \text{triangular}\end{array}$$

3.4 Orthogonal matrices.

Definition 3.5 *A real square matrix A is said to be an* **orthogonal matrix** *if its transpose is equal to its inverse matrix:*

$$A^{-1} = A^t$$