2. Combinatorics.

Combinatorics is the branch of mathematics concerned with counting how many structures can be assembled by combining a finite number of elements under certain rules.

1 Product rule.

If we make a series of choices in k stages, such that at the i-th stage we have n_i possibilities, the total number of possible choices is:

$$n_1 \cdot n_2 \cdot \ldots \cdot n_k.$$

Equivalently, if $A_1, A_2, \ldots, A_k$ are nonempty finite sets, the number of elements of the Cartesian product $A_1 \times A_2 \times \ldots \times A_k$ is:

$$\#(A_1 \times A_2 \times \ldots \times A_k) = \#A_1 \cdot \#A_2 \cdot \ldots \cdot \#A_k.$$

2 Variations (arrangements).

2.1 Variations with repetition.

Variations (or arrangements) with repetition of n elements taken p at a time are the number of ways of selecting p items from a collection of n items, such that the order of selection matters and the repetition of items is allowed.

$$VR_{n,p} = \underbrace{n \cdot n \cdot \ldots \cdot n}_{p \text{ times}} = n^p.$$

2.2 Variations without repetition.

Variations (or arrangements) without repetition of n elements taken p at a time are the number of ways of selecting p items from a collection of n items, such that the order of selection matters and the repetition of items is NOT allowed.

$$V_{n,p} = \underbrace{n \cdot (n-1) \cdot \ldots \cdot (n-p+1)}_{p \text{ factors}} = \frac{n!}{(n-p)!}.$$

3 Permutations.

Permutations of n elements are the different ways of ordering those items. This is the same as the variations without repetition of n elements taken n at a time:

$$P_n = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 = n!$$

3.1 Permutations with repetition.

Given a set of n objects such that there are n_1 identical objects of type 1, n_2 identical objects of type 2, ..., and n_k identical objects of type k, **permutations with repetition** are the different ways of ordering those items (the elements of each type are indistinguishable from each other):

$$PR(n; n_1, n_2, \ldots, n_k) = \frac{n!}{n_1! n_2! \ldots n_k!}.$$

4 Combinations.

4.1 Binomial coefficients.

Definition 4.1 *Given two nonnegative integers n, p, with $n \geq p$, the* **binomial coefficient "n choose p"** *is defined as:*

$$\binom{n}{p} = \frac{n!}{p!(n-p)!}.$$

Some properties of binomial coefficients are:

1. $\binom{n}{0} = \binom{n}{n} = 1.$
2. $\binom{n}{p} = \binom{n}{n-p}.$
3. $\binom{n}{p} + \binom{n}{p-1} = \binom{n+1}{p}.$

Proof:

$$\begin{aligned}\binom{n}{p}+\binom{n}{p-1} &= \frac{n!}{p!(n-p)!}+\frac{n!}{(p-1)!(n-p+1)!}=\\ &= \frac{n!}{(p-1)!(n-p)!}\left(\frac{1}{p}+\frac{1}{n-p+1}\right)=\\ &= \frac{n!}{(p-1)!(n-p)!}\left(\frac{n+1}{p(n-p+1)}\right)=\\ &= \frac{(n+1)!}{p!(n-p+1)!}=\binom{n+1}{p}.\end{aligned}$$

The name of *binomial coefficient* comes from the fact that these can be used to give a formula for the power of a sum of two terms.

Theorem 4.2 (Binomial Theorem)

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^k\,b^{n-k}\qquad (a,\ b\in\mathbb{R},\ \ n\in\mathbb{N}).$$

Proof:

We proceed by induction:

- For $n=1$ it is clear that:

$$(a+b)^1=\binom{1}{0}a^1b^0+\binom{1}{1}a^0b^1.$$

- Assume that the formula holds for $n-1$ and let us prove it for n:

$$\begin{aligned}(a+b)^n &= (a+b)(a+b)^{n-1}=(a+b)(\textstyle\sum_{k=0}^{n-1}\binom{n-1}{k}a^k\,b^{n-1-k})=\\ &= (\textstyle\sum_{k=0}^{n-1}\binom{n-1}{k}a^{k+1}\,b^{n-1-k})+(\sum_{k=0}^{n-1}\binom{n-1}{k}a^k\,b^{n-k})=\\ &= (\textstyle\sum_{k=1}^{n}\binom{n-1}{k-1}a^k\,b^{n-k})+(\sum_{k=0}^{n-1}\binom{n-1}{k}a^k\,b^{n-k})=\\ &= b^n+\textstyle\sum_{k=1}^{n-1}(\binom{n-1}{k-1}+\binom{n-1}{k})a^kb^{n-k}+a^n=\\ &= b^n+\textstyle\sum_{k=1}^{n-1}\binom{n}{k}a^kb^{n-k}+a^n=\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}.\end{aligned}$$

4.2 Combinations without repetition.

Combinations of n elements taken p at a time are the number of ways of selecting p items from a collection of n items, such that the order of selection does not matter. The repetition of items is NOT allowed. Equivalently, it is the number of subsets with p elements of a set with n elements.

$$C_{n,p}=\binom{n}{p}=\frac{V_{n,p}}{p!}=\frac{n!}{p!(n-p)!}.$$

4.3 Combinations with repetition

Combinations with repetition of n elements taken p at a time are the number of ways of selecting p items from n types of items, such that the order of selection does not matter. The repetition of types of items is allowed.

$$CR_{n,p}=\binom{n+p-1}{p}.$$

5 Summary.

Groups with p elements choosen from n items.		
Criteria.	REPETITION ALLOWED	REPETITION NOT ALLOWED
ORDER DOES MATTER	**Variations with rep.** $VR_{n,p}=n^p$	**Variations without rep.** $V_{n,p}=\frac{n!}{(n-p)!}$
ORDER DOES NOT MATTER	**Combinations with rep.** $CR_{n,p}=\binom{n+p-1}{p}$	**Combinations without rep.** $C_{n,p}=\binom{n}{p}$