Part I

Introductory notions.

1. Sets and functions.

1 Sets.

1.1 Definition and notation.

Definition 1.1 *A* **set** *is a collection of objects called* **elements**.

We will use the following notation:

1. "$\exists$" will mean "*there exists*".
2. "$\exists^*$" will mean "*there exists a unique*".
3. $:=$" will mean "*equal by definition*".
4. Sets will be represented with capital letters $A, B, C, \ldots$.
5. Elements will be represented with lowercase letters $a, b, c, \ldots$.
6. If a is an element of a set A we will write $a \in A$ (or $A \ni a$).
7. If a is NOT an element of a set A we will write $a \notin A$ (or $A \not\ni a$).
8. If all the elements of a set A are in B, we will write $A \subset B$ (or $B \supset A$) and say that A is a *subset* of B. It is clear that:
$$A = B \iff A \subset B \text{ and } B \subset A.$$
Because of this, the usual method to check that two sets A and B are equal is to verify first that $A \subset B$ and then that $B \subset A$.
9. We will denote by $\emptyset$ the *empty set*, that is, the set having no elements.
10. Given a set A we will denote by $\mathcal{P}(A)$ the set of all subsets of A (it is called **power set of** A).
11. The number of elements of a finite set A is called the **cardinal of** A and it is denoted by $\#A$.

1.2 Operations with sets.

Definition 1.2 *Given two sets A and B, the* **union** *of A and B is the set $A \cup B$ formed by all the elements of A and B:*
$$A \cup B = \{x | \, x \in A \text{ or } x \in B\}.$$

This definition can be extended to n sets (and in general to an arbitrary family of sets):
$$A_1 \cup A_2 \cup \ldots \cup A_n = \{x | \, x \in A_1 \text{ or } x \in A_2 \text{ or } \ldots \text{ or } x \in A_n\}.$$

Definition 1.3 *Given two sets A and B the* **intersection** *$A \cap B$ is the set of all those elements which are common to both A and B.*
$$A \cap B = \{x | \, x \in A \, , \, x \in B\}.$$

Again, this definition cam be extended to n sets (and in general to an arbitrary family of sets):
$$A_1 \cap A_2 \cap \ldots \cap A_n = \{x | \, x \in A_1 \, , \, x \in A_2 \, , \, \ldots \, , x \in A_n\}.$$

Definition 1.4 *Given two sets A and X such that $A \subset X$, the* **complement** *of A with respect to X is the set of elements in X that are not in A.*
$$X \setminus A = \{x \in X | \, x \notin A\}.$$

Some properties of these operations are:

1. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
Proof:
$$x \in (A \cup B) \cap C \iff \left\{ \begin{array}{c} x \in A \cup B \\ \text{and} \\ x \in C \end{array} \right\} \iff \left\{ \begin{array}{c} x \in A \text{ or } x \in B \\ \text{and} \\ x \in C \end{array} \right\} \iff$$
$$\iff \left\{ \begin{array}{c} x \in A \text{ and } x \in C \\ \text{or} \\ x \in B \text{ and } x \in C \end{array} \right\} \iff x \in (A \cap C) \cup (B \cap C).$$
2. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
3. $X \setminus (X \setminus A) = A$.
4. $A \subset B \quad \Rightarrow \quad X \setminus B \subset X \setminus A$.
5. **De Morgan's laws.** Let A, B be two sets contained in another set X. Then:

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B).$$
The complement of the union of two sets is the same as the intersection of their complements.

$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B).$$
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Proof: Let us prove that $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$. For any element $x \in X$ we have:

$$x \in X \setminus (A \cup B) \iff x \notin A \cup B \iff x \notin A \text{ and } x \notin B \iff \left\{ \begin{array}{c} x \in (X \setminus A) \\ \text{and} \\ x \in (X \setminus B) \end{array} \right\} \iff x \in (X \setminus A) \cap (X \setminus B)$$

Definition 1.5 *Given two sets A and B, the* **Cartesian product** *$A \times B$ is the set of all ordered pairs in which the first element belongs to A and the second belongs to B:*

$$A \times B = \{(a, b) |\, a \in A\,,\, b \in B\}.$$

This definition can be generalized to n sets:

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) |\, a_1 \in A_1\,,\, a_2 \in A_2\,, \ldots\,,\, a_n \in A_n\}.$$

When all sets are equal, the Cartesian product is denoted by:

$$A^n = \underbrace{A \times A \times \ldots \times A}_{n \text{ times}}.$$

2 Correspondences.

2.1 Basic definitions.

Definition 2.1 *Given two sets A and B, a* **correspondence** *between A and B is a subset F of the Cartesian product $A \times B$. Moreover:*

1. *A is called the* **initial set** *of the correspondence.*
2. *B is called the* **final set** *of the correspondence.*
3. *If $(a, b) \in F$ we will say that b is an* **image** *of a and a is an* **origin** *of b.*
4. *The* **domain** *of F is the set formed by all the origins of the elements of B:*

$$\textit{Domain of } F = \{a \in A |\, \exists y \in B \textit{ with } (a, y) \in F\}.$$

5. *The* **range** *of F is the set formed by all the images of the elements of A:*

$$\textit{Range of } F = \{b \in B |\, \exists x \in A \textit{ with } (x, b) \in F\}.$$

Definition 2.2 *Given two sets A and B and a correspondence F between A and B,* **the inverse correspondence** *F^{-1} between B and A is defined as follows:*

$$F^{-1} = \{(b, a) \in B \times A |\, (a, b) \in F\}.$$

The following relations hold:

- Initial set of F=Final set of F^{-1}.
- Final set of F=Initial set of F^{-1}.
- Domain of F=Range of F^{-1}.
- Range of F=Domain of F^{-1}.

Definition 2.3 *Given three sets A, B, C and two correspondences, F from A to B and G from B to C, we can define a* **correspondence** *$H = G \circ F$ from A to C as:*

$$H = \{(a, c) \in A \times C |\, \exists b \in B \textit{ with } (a, b) \in F \textit{ and } (b, c) \in G\}.$$

2.2 Functions.

2.2.1 Definition.

Definition 2.4 *A* **function** *or* **map** *from the set A to the set B is a correspondence on $A \times B$ satisfying the following conditions:*

1. *The domain is the entire set A.*

$$\forall a \in A, \quad \exists b \in B |\, (a, b) \in F.$$

2. *Each element of A has a unique image.*

$$\forall a \in A, \quad \exists^* b \in B |\, (a, b) \in F.$$

Both conditions are equivalent to any element of A having a unique image.

$$\forall a \in A, \quad \exists^* b \in B |\, (a, b) \in F.$$

We will denote a funcion F from A to B by $f : A \longrightarrow B$; when $(a, b) \in F$ we will write $f(a) = b$. If A' is a subset of A, $f(A')$ will be the set of all images of the elements of A':

$$f(A') = \{f(a) |\, a \in A'\}.$$

2.2.2 Classification of functions.

Given two sets A and B, we distinguish several types of functions from A to B:

1. A function is called **injective** when each element of the range has a unique origin. To verify the injectivity of a function, any of the following equivalent conditions can be used:

$$\boxed{f \text{ injective}} \iff \boxed{f(x) = f(y) \quad \Rightarrow \quad x = y} \iff \boxed{x \neq y \quad \Rightarrow \quad f(x) \neq f(y)}$$

2. A function is called **surjective** when the range is equal to the entire set B:

$$\boxed{f \text{ surjective}} \iff \boxed{f(A) = B} \iff \boxed{\forall b \in B, \quad \exists a \in A |\, f(a) = b}$$

3. A function is called **bijective** when it is injective and surjective, that is, any element of B has a unique origin:

$$\boxed{f \text{ bijective}} \iff \boxed{\begin{array}{l} f \text{ injective} \\ f \text{ surjective} \end{array}} \iff \boxed{\forall b \in B, \quad \exists^* a \in A |\, f(a) = b}$$

2.2.3 Inverse function.

Given a function f from A to B, the inverse correspondence of f always exists, but sometimes it is not a function. Note that:

$$\begin{array}{rl} f^{-1} \text{ function} & \iff \forall b \in B, \quad \exists^* a \in A |\, (b, a) \in F^{-1} \iff \\ & \iff \forall b \in B, \quad \exists^* a \in A |\, (a, b) \in F \iff \\ & \iff \forall b \in B, \quad \exists^* a \in A |\, f(a) = b \iff \\ & \iff \left\{ \begin{array}{l} f \text{ injective} \\ f \text{ surjective} \end{array} \right. \end{array}$$

From this:

Proposition 2.5 *Let f be a function from A to B. The inverse correspondence f^{-1} is a function if and only if f is bijective.*

As a consequence of this, we can only compute the inverse function of a bijective function. In this case this inverse function will also be bijective.

2.2.4 Composition of functions.

Definition 2.6 *Given three sets A, B, C, and two functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$ we define the* **function f composed with** *g as:*

$$(a, c) \in g \circ f \iff \exists b \in B |\, (a, b) \in f \text{ and } (b, c) \in g,$$

or equivalently,

$$(g \circ f) : A \longrightarrow C \qquad (g \circ f)(a) = g(f(a)).$$

The equivalence between both definitions is immediate; it is sufficient to note that:

$$(a, b) \in f \text{ and } (b, c) \in g \iff b = f(a) \text{ and } c = g(b) \iff c = g(f(a))$$

The second definition makes it clear that $g \circ f$ is actually a function: since f and g are functions each element of A has a unique image defined in C.

Let $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $h : C \longrightarrow D$ be three functions. Let us see some properties of the composition:

1. The **identity function** on A, $id_A : A \longrightarrow A$ is defined to be a function satisfying $id_A(a) = a$ for all $a \in A$. It is the identity element for the composition:
$$id_B \circ f = f; \qquad f \circ id_A = f.$$
2. The **associative** property is satisfied, that is:
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
3. The composition of functions is **NOT commutative**.
4. If $f : A \longrightarrow B$ is bijective, $f \circ f^{-1} = id_B$ and $f^{-1} \circ f = id_A$.
5. f and g injective $\quad\Rightarrow\quad g \circ f$ is injective.
Proof:
$$(g \circ f)(x) = (g \circ f)(y) \quad\Rightarrow\quad g(f(x)) = g(f(y)) \overset{g \text{ injct.}}{\Rightarrow} f(x) = f(y) \overset{f \text{ injct.}}{\Rightarrow} x = y.$$
6. f and g surjective $\quad\Rightarrow\quad g \circ f$ is surjective.
Proof:
$$(g \circ f)(A) = g(f(A)) \overset{f \text{ surjective}}{=} g(B) \overset{g \text{ surjective}}{=} C.$$
7. f and g bijective $\quad\Rightarrow\quad g \circ f$ bijective.
8. $g \circ f$ injective $\quad\Rightarrow\quad f$ injective.
Proof:
$$f(x) = f(y) \quad\Rightarrow\quad g(f(x)) = g(f(y)) \overset{(g \circ f) \text{ injct.}}{\Rightarrow} x = y.$$
9. $g \circ f$ surjective $\quad\Rightarrow\quad g$ surjective.
Proof:
$$f(A) \subset B \quad\Rightarrow\quad g(f(A)) \subset g(B) \overset{g \circ f \text{ surj.}}{\Rightarrow} C \subset g(B) \overset{g(B) \subset C}{\Rightarrow} g(B) = C.$$
10. $g \circ f$ bijective $\quad\Rightarrow\quad f$ injective and g surjective.