Linear Algebra I — Final Exam

Single exercise (3 hours) **January 20, 2023**

1.– Ten friends rent a minibus with 12 passenger seats, distributed into 4 rows with 3 seats each. One of the friends will be the driver, but only three of them have the required license.

(i) In how many ways can they sit in the minibus?

(ii) In how many of them none of the 4 rows is totally empty?

(1 point)

2.– For any given $n \in \mathbb{N}$ we define the matrix $P_n \in \mathcal{M}_{n\times n}(\mathbb{R})$ as follows:

$$(P_n)_{ij} = \begin{cases} 0 \text{ if } i+j = n+1 \\ 1 \text{ if } i+j \neq n+1 \end{cases}$$

(i) Write the matrix P_4 and obtain its determinant.

(ii) For $n \geq 2$ arbitrary, find $det(P_n)$ and $trace(P_n)$.

(1 point)

3.– Given the matrices $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & -3 \\ 0 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$, obtain an invertible matrix Y such that $YA = B$, if it exists.

(1 point)

4.– Given the matrices $A = \begin{pmatrix} 1 & a \\ 2 & b \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$ analyze for which values of a and b there exists an invertible matrix $X \in M_{2\times 2}(\mathbb{R})$ such that $XAX^t = B$, and obtain the matrix X in these cases.

(1 point)

5.– Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of all polynomials with degree less than or equal to 2. Consider the following subspaces:

$$U = \{p(x) \in \mathcal{P}_2(\mathbb{R}) \,|\, degree(p(x)) \leq 1\}, \quad V = \{p(x) \in \mathcal{P}_2(\mathbb{R}) | p'(0) = 0\}, \quad W = \mathcal{L}\{1 + x^2\}$$

(i) Are U and W complementary subspaces? Decompose, if possible, the polynomial $1 + 2x^2$ as a sum of a polynomial in U and another one in W. Is this decomposition unique?

(ii) Are U and V complementary subspaces? Decompose, if possible, the polynomial $1 + 2x^2$ as a sum of a polynomial in U and another one in V. Is this decomposition unique?

(1.3 points)

6.– Consider the mapping

$$f : \mathcal{M}_{2\times 2}(\mathbb{R}) \to \mathbb{R}^2, \quad f(A) = (trace(A), trace(A + A^t))$$

(i) Show that f is linear.

(ii) Find the matrix associated to f relative to the canonical basis of $\mathcal{M}_{2\times 2}(\mathbb{R})$ and $\mathbb{R}^2$.

(iii) Obtain $f(Id)$ making use of the matrix computed in (ii).

(iv) Obtain the parametric and implicit equations of $ker(f)$ relative to the canonical basis of $\mathcal{M}_{2\times 2}(\mathbb{R})$.

(v) Prove that the following is a basis of $\mathcal{M}_{2\times 2}$,

$$B = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

(vi) Obtain the matrix associated to f relative to the bases B of $\mathcal{M}_{2\times 2}$ and $B' = \{(0,1),(1,1)\}$ of $\mathbb{R}^2$.

(1.3 points)

7.– Decide and justify whether the following assertions are true or false:

(i) If a 2×2 matrix has two different eigenvalues, then it is diagonalizable by similarity.

(ii) If two matrices are congruent, then they are row equivalent.

(iii) The set $\{A \in \mathcal{M}_{2\times 2}(\mathbb{R}) | rank(A) < 2\}$ is a vector subspace of $\mathcal{M}_{2\times 2}(\mathbb{R})$.

(iv) If $f : \mathbb{R}^4 \to \mathbb{R}^4$ is a linear mapping such that $f(1,0,0,1) = f(1,0,0,0)$, then $dim(Im(f)) < 4$.

(1.2 points)

8.– In the vector space $\mathbb{R}^4$ we consider the following vector subspaces:

$$U = \{(x,y,z,t) \in \mathbb{R}^4 | x+y+z-t = 0,\ x+y-z+t = 0,\ z-t = 0\}, \quad V = \mathcal{L}\{(1,0,1,1),(1,0,2,2),(1,0,0,0)\}$$

(i) Obtain $dim(U), dim(V), dim(U+V)$ and $dim(U \cap V)$.

(ii) Obtain the parametric and implicit equations of $U \cap V$ relative to the canonical basis.

(iii) Find the parametric and implicit equations of V relative to the basis

$$B = \{(1,1,0,0),(1,-1,0,0),(0,0,1,1),(0,0,1,-1)\}.$$

(iv) Find the implicit equations of a subspace W such that V and W are complementary subspaces.

(1 point)

9.– For each $a \in \mathbb{R}$ we define the matrix $A = \begin{pmatrix} a & 0 & 0 \\ 1 & 1 & 2 \\ -a & 2 & 1 \end{pmatrix}$:

(i) Analyze in terms of the values of a whether the matrix A is diagonalizable and/or triangularizable by similarity.

(ii) For $a = -1$, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(iii) Find the value of a for which $trace(A^5) = 210$.

(1.2 points)
