Linear Algebra I **Final Exam**

Single exercise (3 hours) **June 30, 2023**

1.– With letters from the word INTERNET,

(i) How many 8-letter words can be formed?

(ii) How many 7-letter words? How many 6-letter ones? (1.1 points)

2.– Given $n \in \mathbb{N}$, the matrix $P_n \in \mathcal{M}_{n\times n}(\mathbb{R})$ is defined as:

$$(P_n)_{ij} = \begin{cases} i \text{ if } i = j \\ n \text{ if } i \neq j \end{cases}$$

(i) Write down the matrix P_4 and find its determinant.

(ii) For any $n \geq 2$, find $\det(P_n)$.

(iii) For which values of n is P_n congruent with the identity matrix? (1.1 points)

3.– Let A, B and C be three $n \times n$ invertible matrices. Find trace(M), where

$$M = (A \cdot B \cdot C)^{-1} - (B^t \cdot C)^{-1} + C^{-1} \cdot [(B^{-1} + C^t)^t - (A \cdot B)^{-1}]$$

(0.7 points)

4.– Given the matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{pmatrix}$

(i) Decide whether A is row equivalent to the identity matrix.

(ii) Give a matrix of rank 2 that is NOT row equivalent to A.

(iii) Give a diagonal matrix congruent to A.

(iv) Is there a diagonal matrix which is column equivalent to A? Justify your answer. (1.2 points)

5.– For each $a \in \mathbb{R}$ we consider the following subsets of $\mathbb{R}^4$:

$$U = \{(x, y, z, t) \in \mathbb{R}^4 | x + y - z - t = 0,\ x + y + z + t = 0\}, \quad V = \mathcal{L}\{(1, 1, 0, 0), (1, 1, a, 1), (0, 0, 2, 1)\}$$

(i) Obtain $dim(U), dim(V), dim(U + V)$ and $dim(U \cap V)$ in terms on the values of a.

(ii) For $a = 2$, decompose the vector $(2, 2, 3, 0)$ as the sum of a vector in U and another one in V. Is the decomposition unique?

(iii) For $a = 1$ give the implicit equations of a vector subspace W such that V and W are supplementary. (1.4 points)

6.– Justify the truth or falsehood of the following assertions:

(i) There is a base of $\mathcal{M}_{2\times 2}(\mathbb{R})$ formed by four invertible matrices.

(ii) If $U, V \subset \mathbb{R}^n$ are vector subspaces, $dim(U) = dim(V) = 70$ and $dim(U \cap V) = 40$ then $n \geq 100$.

(iii) $\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z \geq 0\}$ is a vector subspace of $\mathbb{R}^3$.

(iv) If $A, B \in \mathcal{M}_{4\times 4}(\mathbb{R})$ and $\text{rank}(AB) = 4$ then $\text{rank}(A) = 4$. (1.2 points)

7.– Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of all polynomials with degree less or equal to 2. Let $f : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be an endomorphism such that

$$f(1) = 2, \quad \ker(f) = \{p(x) \in \mathcal{P}_2(\mathbb{R}) | p(1) = 0\}$$

(i) Find the matrix associated to f relative to the canonical basis.

(ii) Find the implicit equations of $\text{Im}(f)$ relative to the canonical basis.

(ii) Find $f((x + 1)^2)$. (1.1 points)

8.– Consider the endomorphism

$$f : \mathbb{R}^4 \to \mathbb{R}^4, \quad f(x, y, z, t) = (x + 2y + t,\ 2x + y + t,\ -z + 2t, t)$$

(i) Find its eigenvalues and eigenvectors.

(ii) Give a basis B and a diagonal matrix D such that $F_B = D$, if they exist.

(iii) Compute $\text{trace}(F_C^9)$.

(iv) Let us define $g : \mathbb{R}^4 \to \mathbb{R}^2$, $g(x, y, z, t) = (x + y + z + t,\ x + y - z - t)$. Find the matrix associated to $g \circ f$ with respect to the canonical bases of $\mathbb{R}^4$ and $\mathbb{R}^2$. (1.4 points)

9.– Consider the linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x + y,\ x + y)$ Can f be the projection mapping onto some subspace along another one?. Justify your answer. (0.8 points)
