CN - 2023/2024 PR1 - 01

1) Identificando coeficientes

a) $P_2(x) = a_0 + a_1 x + a_2 x^2$

$$\left.\begin{array}{l} P_2(1) = \ln(1) = 0 \\ P_2(2) = \ln(2) = c \\ P_2(1/2) = \ln(1/2) = -c \end{array}\right\} \Leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 1/2 & 1/4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ c \\ -c \end{Bmatrix}$$

b) $P_4(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

$$\left.\begin{array}{l} P_4(1) = \ln(1) = 0 \\ P_4(2) = \ln(2) = c \\ P_4(1/2) = \ln(1/2) = -c \\ P_4(4) = \ln(4) = 2c \\ P_4(1/4) = \ln(1/4) = -2c \end{array}\right\} \Leftrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 1/2 & 1/4 & 1/8 & 1/16 \\ 1 & 4 & 16 & 64 & 256 \\ 1 & 1/4 & 1/16 & 1/64 & 1/256 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ c \\ -c \\ 2c \\ -2c \end{Bmatrix}$$

2) Lagrange

a) $P_2(x) = \sum_{i=0}^{2} \ln(x_i)\, L_i(x)$; $L_i(x) = \prod_{\substack{j=0,2 \\ j \neq i}} \dfrac{(x - x_j)}{(x_i - x_j)}$

$$P_2(x) = \overbrace{\ln(1)}^{0} \frac{(x-2)(x-1/2)}{(1-2)(1-1/2)} + \overbrace{\ln(2)}^{c} \frac{(x-1)(x-1/2)}{(2-1)(2-1/2)} + \overbrace{\ln(1/2)}^{-c} \frac{(x-1)(x-2)}{(1/2-1)(1/2-2)}$$

b) $P_4(x) = \sum_{i=0}^{4} \ln(x_i)\, L_i(x)$; $L_i(x) = \prod_{\substack{j=0,4 \\ j \neq i}} \dfrac{(x - x_j)}{(x_i - x_j)}$

$$P_4(x) = \overbrace{\ln(1)}^{0} \frac{(x-2)(x-1/2)(x-4)(x-1/4)}{(1-2)(1-1/2)(1-4)(1-1/4)}$$

$$+ \overbrace{\ln(2)}^{c} \frac{(x-1)(x-1/2)(x-4)(x-1/4)}{(2-1)(2-1/2)(2-4)(2-1/4)} + \overbrace{\ln(1/2)}^{-c} \frac{(x-1)(x-2)(x-4)(x-1/4)}{(1/2-1)(1/2-2)(1/2-4)(1/2-1/4)}$$

$$+ \overbrace{\ln(4)}^{2c} \frac{(x-1)(x-2)(x-1/2)(x-1/4)}{(4-1)(4-2)(4-1/2)(4-1/4)} + \overbrace{\ln(1/4)}^{-2c} \frac{(x-1)(x-2)(x-1/2)(x-4)}{(1/4-1)(1/4-2)(1/4-1/2)(1/4-4)}$$

3) Newton

i	x_i	$f(x_i)$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$	$f[\,,\,,\,,\,,\,]$
0	1	0	c	$-\frac{2}{3}c$	$\frac{1}{7}c$	$-\frac{4}{15}c$
1	2	c	$\frac{4}{3}c$	$-\frac{5}{21}c$	$\frac{12}{35}c$	
2	$1/2$	$-c$	$\frac{6}{7}c$	$-\frac{88}{105}c$		
3	4	$2c$	$\frac{16}{15}c$			
4	$1/4$	$-2c$				

$$P_2(x) = 0 + c(x-1) - \frac{2}{3}c(x-1)(x-2)$$

$$P_4(x) = P_2(x) + \frac{1}{7}c(x-1)(x-2)(x-\tfrac{1}{2}) - \frac{4}{15}c(x-1)(x-2)(x-\tfrac{1}{2})(x-4)$$

Calculamos $P_2(3)$, $P_4(3)$

1) Identificando coeficientes

Hay que resolver dos sistemas de ecuaciones lineales (3x3 y 5x5), lo que es poco práctico. Además este tipo de sistemas están mal condicionados por lo que se producen errores en los coeficientes que dan lugar a resultados poco fiables.

2) Lagrange

$$P_2(3) = \cdots = \frac{2}{3}c$$

$$P_4(3) = \cdots = \frac{57}{21}c$$

3) Newton

$$P_2(3) = 0 + c(3-1) - \frac{2}{3}c(3-1)(3-2) = \frac{2}{3}c$$

$$P_4(3) = P_2(3) + \frac{1}{7}c(3-1)(3-2)(3-\tfrac{1}{2}) - \frac{4}{15}c(3-1)(3-2)(3-\tfrac{1}{2})(3-4) = \frac{57}{21}c$$

Luego:
$$\begin{cases} c = \ln(2) \simeq 0.69314718 \\ P_2(3) = 2/_3\, c \simeq 0.46209812 \\ P_4(3) = 57/_{21}\, c \simeq 0.18813995 \cdot 10^1 \end{cases}$$

$$\ln(3) \simeq 0.10986123 \cdot 10^1 \Rightarrow \begin{cases} \simeq 57,94\% \text{ de error en } P_2(3) \\ \simeq -71,25\% \text{ de error en } P_4(3) \end{cases}$$

Acotaciones

$$\left.\begin{array}{l} R_2(x) = \ln(x) - P_2(x) = \dfrac{f^{(3}(\mu)}{3!}(x-1)(x-2)(x-1/_2) \\ \left.\begin{array}{l} f^{(3}(\mu) = 2/\mu^3 \\ \mu \in [1/_2, 3] \end{array}\right\} \Rightarrow |f^{(3}(\mu)| \leq \dfrac{2}{(1/_2)^3} = 16 \end{array}\right\} \Rightarrow$$

$$\boxed{|R_2(x)| \leq \frac{16}{3!}\left|(3-1)(3-2)(3-1/_2)\right| \simeq 13,333333}$$

$$R_4(x) = \ln(x) - P_4(x) = \frac{f^{(5}(\mu)}{5!}(x-1)(x-2)(x-1/_2)(x-4)(x-1/_4)$$

$$\left.\begin{array}{l} f^{(5}(\mu) = 24/\mu^5 \\ \mu \in [1/_4, 4] \end{array}\right\} \Rightarrow |f^{(5}(\mu)| \leq \frac{24}{(1/_4)^5} = 24576$$

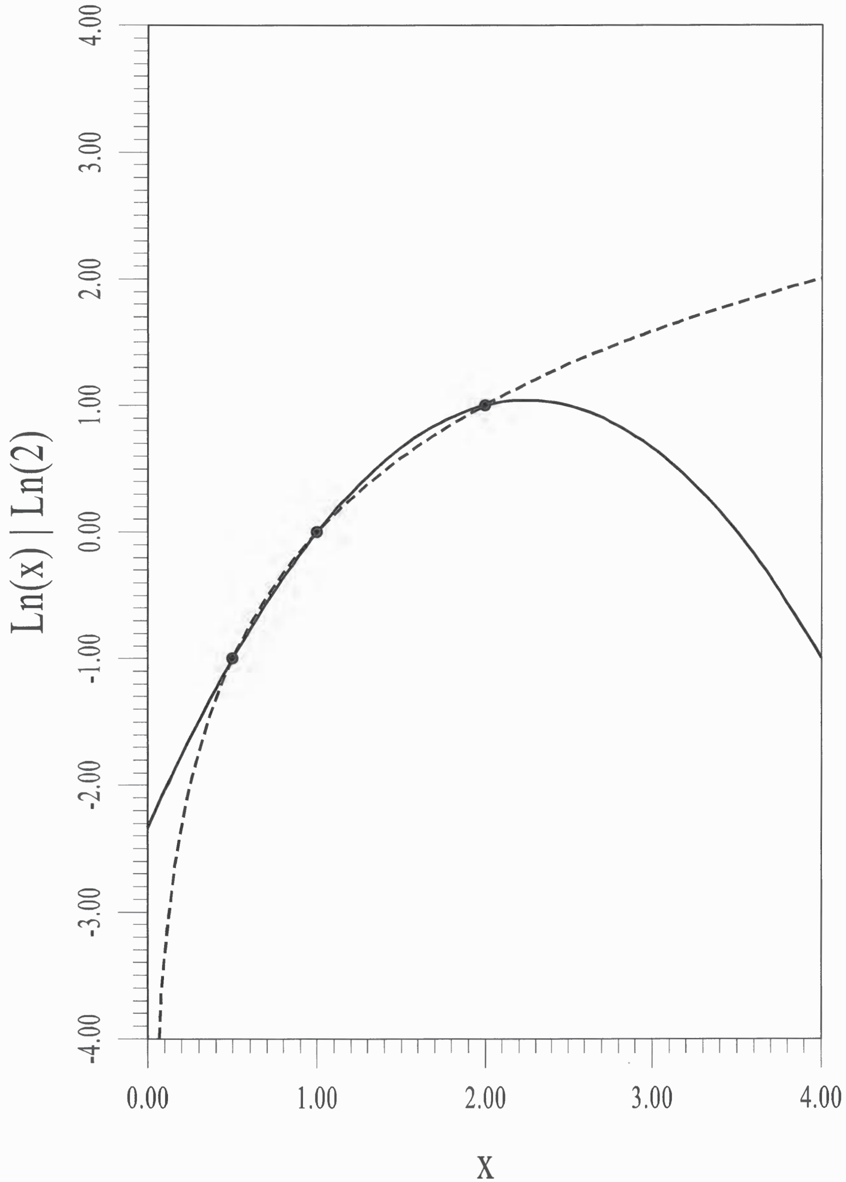
$$\boxed{|R_4(x)| \leq \frac{24576}{5!}\left|(3-1)(3-2)(3-1/_2)(3-4)(3-1/_4)\right| = 2816}$$

Observamos que las cotas son correctas, pero poco ajustadas por lo que no son muy útiles.
En todo caso, que las cotas vaticinen posibles errores de tal magnitud indica que los polinomios interpoladores se alejarán previsiblemente de la función $f(x)$. Luego algo falla.

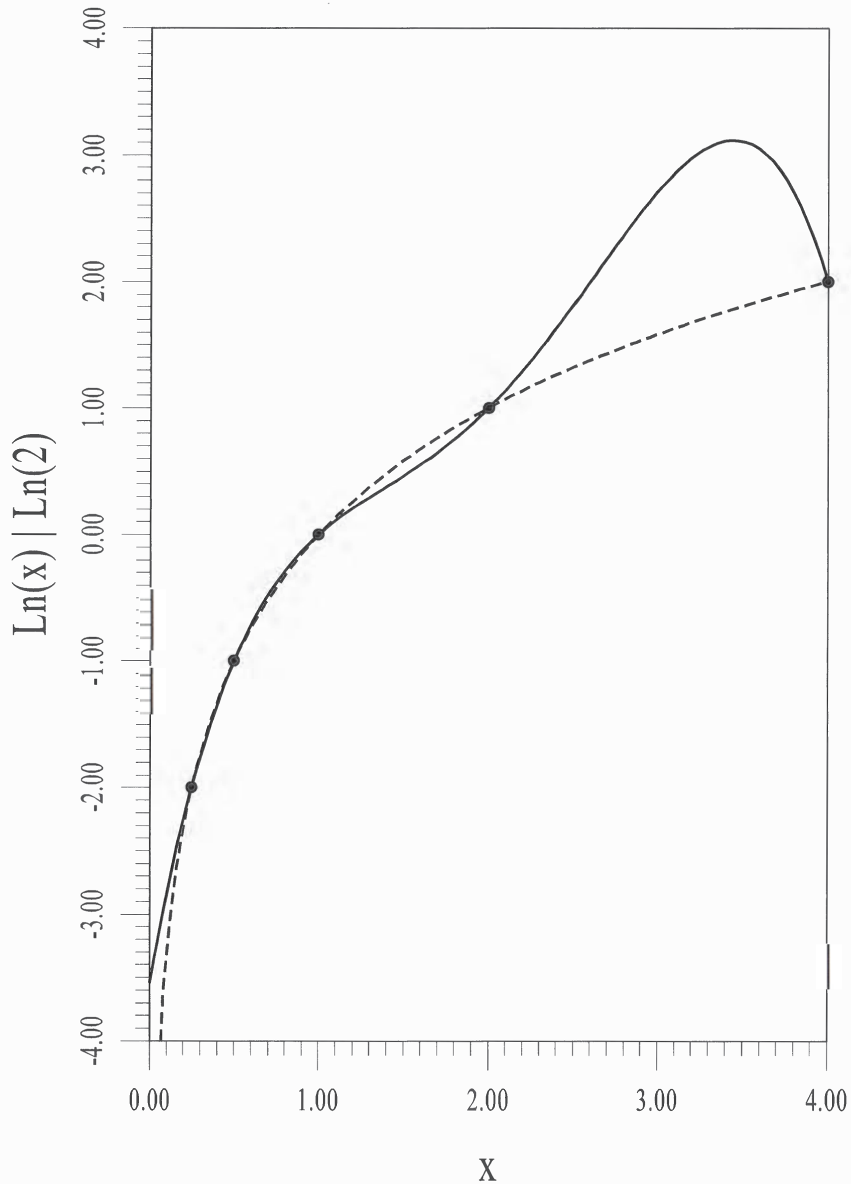
(VÉANSE GRÁFICOS EN PÁGINA SIGUIENTE)

p10

INTERPOLACION PURA SOBRE 3 PUNTOS



INTERPOLACION PURA SOBRE 5 PUNTOS



CN - 2023/2024 PR1 - 02

a) $P_{i+1}(x) = a_i(x-b_i)P_i(x) + c_i P_{i-1}(x)$ $\quad \left\{ \begin{array}{l} a_i = 1 \\ b_i = \dfrac{\langle xP_i, P_i\rangle}{\langle P_i, P_i\rangle} \\ c_i = -\dfrac{a_i}{a_{i-1}} \dfrac{\langle P_i, P_i\rangle}{\langle P_{i-1}, P_{i-1}\rangle} \end{array} \right.$ $\Rightarrow$

$\left.\begin{array}{l}\langle f, g\rangle = \sum\limits_{i=0,2} f(x_i)\, g(x_i) \\ x_0 = 0,\; x_1 = 1,\; x_2 = 2\end{array}\right\} \Rightarrow \langle f,g\rangle = f(0)g(0)+f(1)g(1)+f(2)g(2)$

$\Rightarrow P_0(x) = 1$; $b_0 = \dfrac{\langle xP_0, P_0\rangle}{\langle P_0, P_0\rangle}$; $\langle P_0,P_0\rangle = 1^2+1^2+1^2 = 3$

$\langle xP_0, P_0\rangle = 0\cdot1^2 + 1\cdot1^2 + 2\cdot1^2 = 3$

$P_1(x) = (x-b_0)P_0(x)$; $b_1 = \dfrac{\langle xP_1, P_1\rangle}{\langle P_1, P_1\rangle}$; $\langle P_1,P_1\rangle = (0-1)^2+(1-1)^2+(2-1)^2 = 2$

$= x-1$ $\qquad \langle xP_1,P_1\rangle = 0\cdot(0-1)^2+1\cdot(1-1)^2+2(2-1)^2 = 2$

$c_1 = -\dfrac{\langle P_1, P_1\rangle}{\langle P_0, P_0\rangle}$

$P_2(x) = (x-b_1)P_1(x) + c_1 P_0(x)$; $\langle P_2,P_2\rangle = (1/3)^2+(-2/3)^2+(1/3)^2 = 2/3$

$= (x-1)(x-1) - 2/3$

$= x^2 - 2x + 1/3$

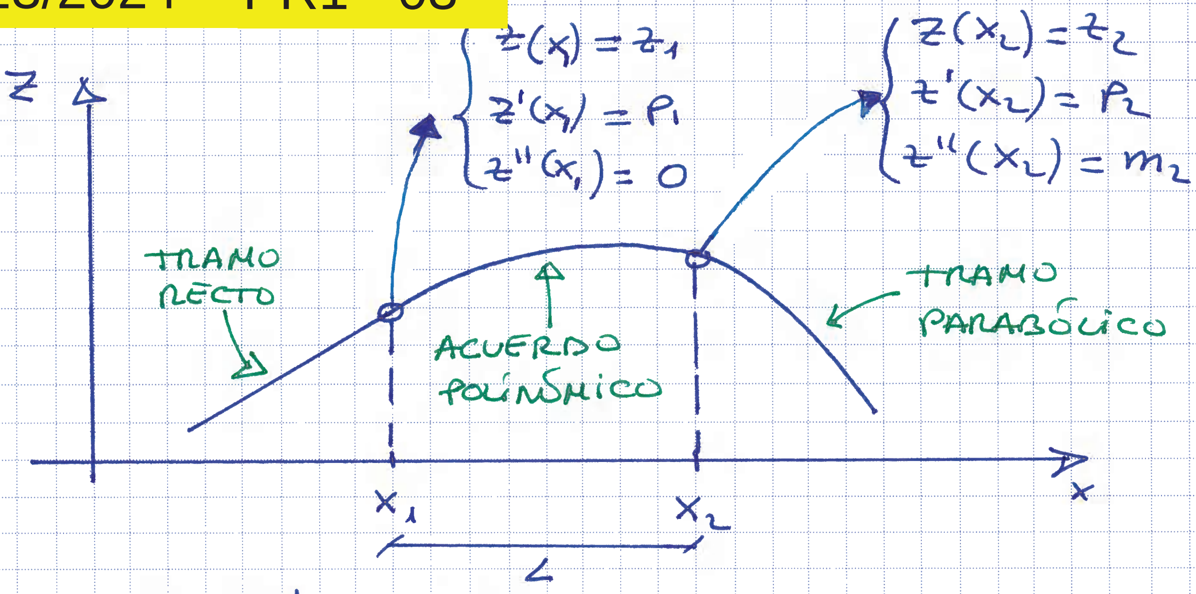
b) $f_0 = 1$, $f_1 = 2$, $f_2 = 4$

$\left\{ \begin{array}{l} Q_0(x) = \alpha_0 P_0(x) \quad ; \alpha_0 = \dfrac{\langle P_0, f\rangle}{\langle P_0, P_0\rangle} \; ; \langle P_0, f\rangle = 1\cdot1+1\cdot2+1\cdot4 = 7 \\ Q_1(x) = Q_0(x) + \alpha_1 P_1(x) \quad ; \alpha_1 = \dfrac{\langle P_1, f\rangle}{\langle P_1, P_1\rangle} \; ; \langle P_1, f\rangle = (-1)\cdot1 + 0\cdot2 + 1\cdot4 = 3 \\ Q_2(x) = Q_1(x) + \alpha_2 P_2(x) \; ; \alpha_2 = \dfrac{\langle P_2, f\rangle}{\langle P_2, P_2\rangle} \; ; \langle P_2, f\rangle = 1/3\cdot1 + (-2/3)\cdot2 + 1/3\cdot4 = 1/3 \end{array} \right.$

Luego: $\left\{ \begin{array}{l} Q_0(x) = 7/3 \\ Q_1(x) = 7/3 + 3/2\,(x-1) = 5/6 + 3/2\,x \\ Q_2(x) = (5/6 + 3/2\,x) + 1/2\,(x^2 - 2x - 1/3) = 1 + 1/2\,x + \frac{1}{2}x^2 \end{array} \right.$

Observamos que $Q_2(x)$ pasa por los tres puntos.

c) Cuando el número de puntos es mayor las ecuaciones normales son mal-condicionadas y el error es inaceptable. En este caso se puede hacer,



a) La ecuación del acuerdo será $z(x) = P_n(x)$

Debe verificar:

$$\left.\begin{array}{lcl} P_n(x_1) = z_1 & ; & P_n(x_2) = z_2 \; ; \\ P'_n(x_1) = p_1 & ; & P'_n(x_2) = p_2 \; ; \\ P''_n(x_1) = 0 & ; & P''_n(x_2) = m_2 \; ; \end{array}\right\} \Rightarrow \text{6 condiciones}$$

Luego $n = 5$, $P_5(x) = \sum_{i=0,5} a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$

b)
$$\begin{cases} P_5(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 \\ P'_5(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 \\ P''_5(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 & x_2^5 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 \\ 0 & 1 & 2x_2 & 3x_2^2 & 4x_2^3 & 5x_2^4 \\ 0 & 0 & 2 & 6x_1 & 12x_1^2 & 20x_1^3 \\ 0 & 0 & 2 & 6x_2 & 12x_2^2 & 20x_2^3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} z_1 \\ z_2 \\ p_1 \\ p_2 \\ 0 \\ m_2 \end{Bmatrix}$$

c) Para sistematizar el cálculo escribimos el polinomio en la forma: $P_5(x) = \sum_{i=0,n} b_i \xi^i$, con $\xi = \frac{x - x_1}{L}$, $L = x_2 - x_1$

de forma que $\begin{cases} x = x_1 \iff \xi = 0 \\ x = x_2 \iff \xi = 1 \end{cases}$

Luego:

$$\begin{cases} P_5(x) = b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3 + b_4\xi^4 + b_5\xi^5 \\ P'_5(x) = (b_1 + 2b_2\xi + 3b_3\xi^2 + 4b_4\xi^3 + 5b_5\xi^4)\frac{1}{L} \\ P''_5(x) = (2b_2 + 6b_3\xi + 12b_4\xi^2 + 20b_5\xi^3)\frac{1}{L^2} \end{cases}$$

El sistema de ecuaciones que hay que resolver ahora es:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1/L & 0 & 0 & 0 & 0 \\ 0 & 1/L & 2/L & 3/L & 4/L & 5/L \\ 0 & 0 & 2/L^2 & 0 & 0 & 0 \\ 0 & 0 & 2/L^2 & 6/L^2 & 12/L^2 & 20/L^2 \end{bmatrix} \begin{Bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{Bmatrix} = \begin{Bmatrix} z_1 \\ z_2 \\ p_1 \\ p_2 \\ 0 \\ m_2 \end{Bmatrix}$$

Reescalando y reordenando se obtiene

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 6 & 12 & 20 \end{bmatrix}}_{\underset{\sim}{A}} \begin{Bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{Bmatrix} = \begin{Bmatrix} z_1 \\ p_1 \cdot L \\ 0 \\ z_2 \\ p_2 L \\ m_2 L^2 \end{Bmatrix}$$

Luego: $b_0 = z_1$, $b_1 = p_1 L$, $b_2 = 0$

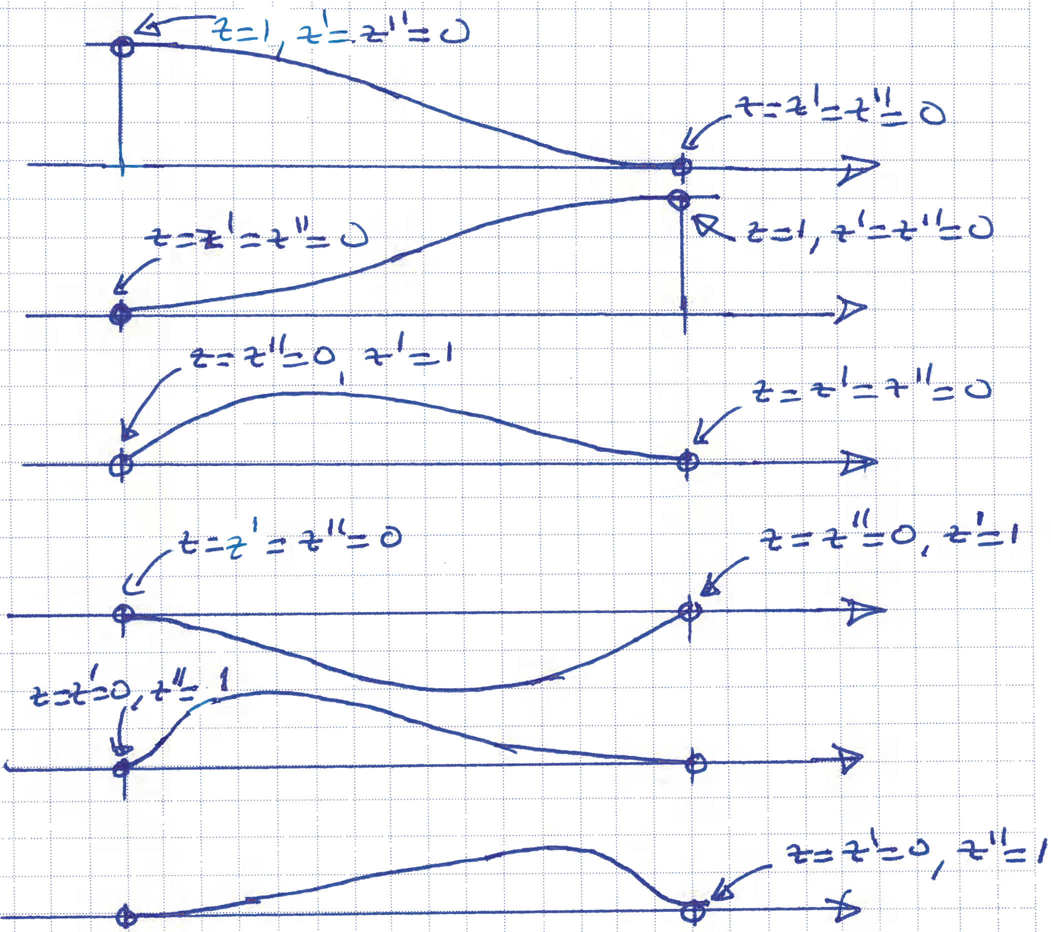
y
$$\boxed{\begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 6 & 12 & 20 \end{bmatrix} \begin{Bmatrix} b_3 \\ b_4 \\ b_5 \end{Bmatrix} = \begin{Bmatrix} z_2 - z_1 - p_1 L \\ (p_2 - p_1) L \\ m_2 L^2 \end{Bmatrix}}$$

Definimos la base de polinomios de 5.º grado

$$\{ B_1(x), B_2(x), D_1(x), D_2(x), S_1(x), S_2(x) \}$$

tal que

$$\left\{ \begin{array}{l} z = B_1(x) \\ z = B_2(x) \\ z = D_1(x) \\ z = D_2(x) \\ z = S_1(x) \\ z = S_2(x) \end{array} \right.$$



lo que permite escribir:

$$\begin{aligned} z = P_5(x) = {} & z_1 \cdot B_1(x) + z_2 B_2(x) \\ & + p_1 D_1(x) + p_2 D_2(x) \\ & + \underbrace{\cancel{m_1}}_{0} \underbrace{S_1(x)}_{\text{innecesario}} + m_2 S_2(x) \end{aligned}$$

Cada uno de ls elements de la base es un $P_5(x)$.

Resolvems les sistems correspondients:

$B_1(x)$: $b_0 = 1,\ b_1 = 0,\ b_2 = 0,\ b_3 = -10,\ b_4 = 15,\ b_5 = -6$

$$B_1(x) = 1 - 10\frac{(x-x_1)^3}{(x_2-x_1)^3} + 15\frac{(x-x_1)^4}{(x_2-x_1)^4} - 6\frac{(x-x_1)^5}{(x_2-x_1)^5}$$

$B_2(x)$: $b_0 = 0,\ b_1 = 0,\ b_2 = 0,\ b_3 = 10,\ b_4 = -15,\ b_5 = 6$

$$B_2(x) = 10\frac{(x-x_1)^3}{(x_2-x_1)^3} - 15\frac{(x-x_1)^4}{(x_2-x_1)^4} + 6\frac{(x-x_1)^5}{(x_2-x_1)^5}$$

$D_1(x)$: $b_0 = 0,\ b_1 = h,\ b_2 = 0,\ b_3 = -6h,\ b_4 = 8h,\ b_5 = -3h$

$$D_1(x) = (x-x_1) - 6\frac{(x-x_1)^3}{(x_2-x_1)^2} + 8\frac{(x-x_1)^4}{(x_2-x_1)^3} - 3\frac{(x-x_1)^5}{(x_2-x_1)^4}$$

$D_2(x)$: $b_0 = 0,\ b_1 = 0,\ b_2 = 0,\ b_3 = -4h,\ b_4 = 7h,\ b_5 = -3h$

$$D_2(x) = -4\frac{(x-x_1)^3}{(x_2-x_1)^2} + 7\frac{(x-x_1)^4}{(x_2-x_1)^3} - 3\frac{(x-x_1)^5}{(x_2-x_1)^4}$$

$S_1(x)$: $b_0 = 0,\ b_1 = 0,\ b_2 = h^2/2,\ b_3 = -\frac{3h^2}{2},\ b_4 = \frac{3h^2}{2},\ b_5 = -h^2/2$

$$S_1(x) = \frac{(x-x_1)^2}{2} - \frac{3(x-x_1)^3}{2(x_2-x_1)} + \frac{3(x-x_1)^4}{2(x_2-x_1)^2} - \frac{(x-x_1)^5}{2(x_2-x_1)^3} \quad (*)$$

$S_2(x)$: $b_0 = 0,\ b_1 = 0,\ b_2 = 0,\ b_3 = h^2/2,\ b_4 = -h^2,\ b_5 = h^2/2$

$$S_2(x) = \frac{(x-x_1)^3}{2(x_2-x_1)} - \frac{(x-x_1)^4}{(x_2-x_1)^2} + \frac{(x-x_1)^5}{2(x_2-x_1)^3}$$

$$\boxed{Z(x) = z_1 B_1(x) + z_2 B_2(x) + p_1 D_1(x) + p_2 D_2(x) + \cancelto{0}{m_1} S_1(x) + m_2 S_2(x)}$$

(*) Innecesario en este caso

CN - 2023/2024 PR1 - 04

a) $P_3(x) = \sum_{i=0,3} f(x_i)\,\ell_i(x)$, $\begin{cases} f(x) = \operatorname{sen}(x) \\ \ell_i(x) = \prod_{\substack{j=0,3 \\ j\neq i}} \dfrac{(x-x_j)}{(x_i-x_j)} \end{cases}$

$x\in[0,\frac{\pi}{2}] \Rightarrow \left\{\begin{array}{l} x_i = \dfrac{0+\pi/2}{2} + z_i\,\dfrac{\pi/2-0}{2} = \pi/4\,(1+z_i) \\ z_i = \cos\left(\dfrac{\pi(1+2i)}{2(n+1)}\right);\ i=0,\dots,n \end{array}\right\}$ (*)

$R_3(x) = f(x) - P_3(x) = \dfrac{f^{(4)}(\mu)}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3)$

$\left.\begin{array}{ll} x = \pi/4\,(1+z) & , z\in[-1,1] \\ x_i = \pi/4\,(1+z_i) & , z_i\in[-1,1] \end{array}\right\} \Rightarrow (x-x_i) = \pi/4\,(z-z_i)$ $\Rightarrow$

$\Rightarrow R_3(x) = \dfrac{f^{(4)}(\mu)}{4!}(\pi/4)^4 \underbrace{(z-z_0)(z-z_1)(z-z_2)(z-z_3)}_{\overline{T}_4(z)=\frac{T_4(z)}{2^3}\ (**)} = \dfrac{f^{(4)}(\mu)}{4!}(\pi/4)^4\dfrac{T_4(z)}{2^3}$

$\left.\begin{array}{l} |T_4(z)|\le 1 \Rightarrow |R_3(x)| \le \max\limits_{\mu\in[0,\pi/2]} |f^{(4)}(\mu)|\,\dfrac{1}{4!}(\pi/4)^4\dfrac{1}{2^3} \\ f(x) = \operatorname{sen}(x) \Rightarrow f^{(4)}(x) = \operatorname{sen}(x) \end{array}\right\} \Rightarrow |R_3(x)| \le \dfrac{(\pi/4)^4}{4!\,2^3}$

Luego : $\boxed{|R_3(x)| \le \dfrac{(\pi/4)^4}{4!\,2^3} \simeq 0{,}19817930\cdot 10^{-2}}$

b) $n=3 \longrightarrow \begin{cases} x_0 = \pi/4\,(1+\cos(\pi/8)) \to f(x_0)\simeq 0{,}99821342 \\ x_1 = \pi/4\,(1+\cos(3\pi/8)) \to f(x_1)\simeq 0{,}88474986 \\ x_2 = \pi/4\,(1+\cos(5\pi/8)) \to f(x_2)\simeq 0{,}46606619 \\ x_3 = \pi/4\,(1+\cos(7\pi/8)) \to f(x_3)\simeq 0{,}59749268\cdot 10^{-1} \end{cases}$

c) <u>NO</u> es Mini-Max

d) Taylor: $\operatorname{sen}(x) = \underbrace{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dots + (-1)^n\dfrac{x^{2n+1}}{(2n+1)!}}_{\text{Aproximación } P_{2n+1}(x)} + \underbrace{(-1)^{n+1}\dfrac{x^{2n+3}}{(2n+3)!}\cos(\xi)}_{\text{Resto } R_{2n+1}(x)}$

$|R_{2n+1}(x)| \le \dfrac{(\pi/2)^{2n+3}}{(2n+3)!} \Rightarrow |R_5(x)| \le 0{,}46817541\cdot 10^{-2}$, $\underline{|R_7(x)| \le 0{,}16044119\cdot 10^{-3}}$

<u>Luego hace falta un polinomio de grado 7</u>

e) $\underline{P_3(x) = \sum_{i=0,3} f(x_i)\,\ell_i(x)}$. $P_7(x) = x - x^3/6 + x^5/120 - x^7/5040$

(*) Puntos de Chebyshev para $n=3$. (**) Polinomio normalizado de Chebyshev.

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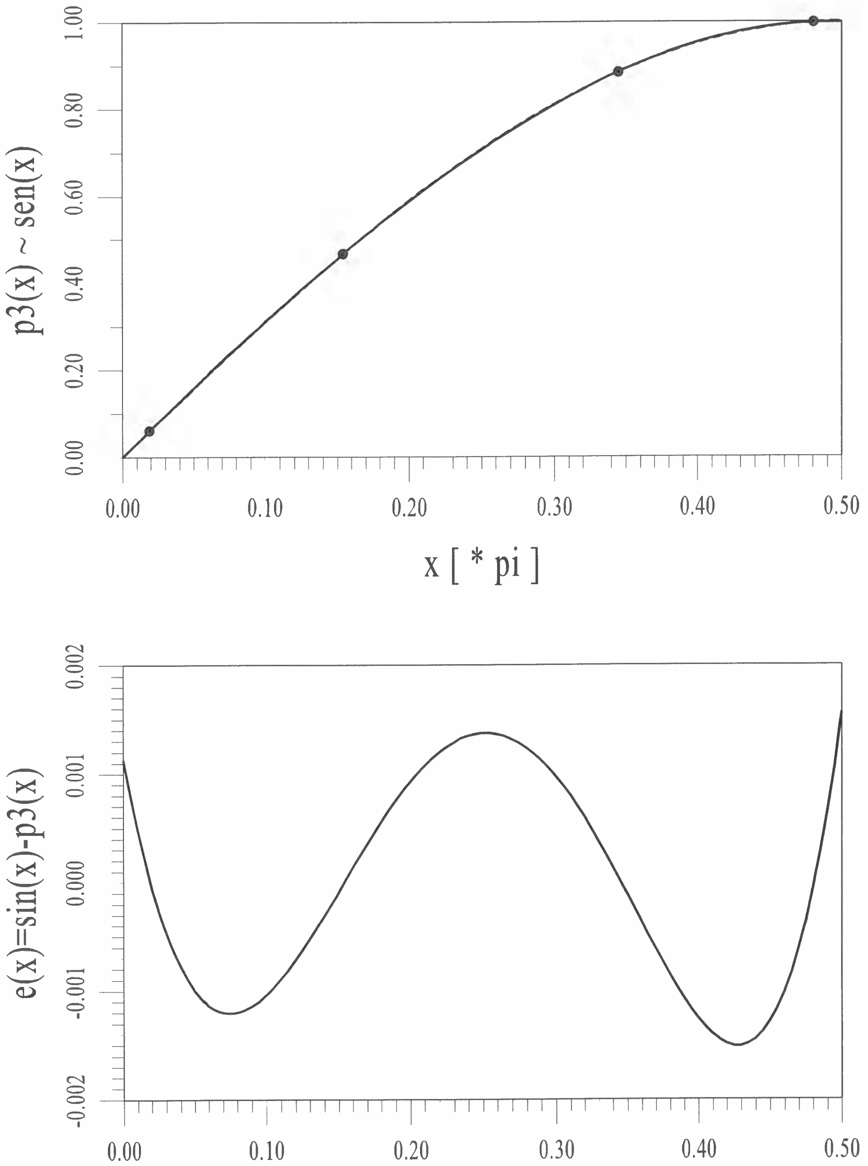
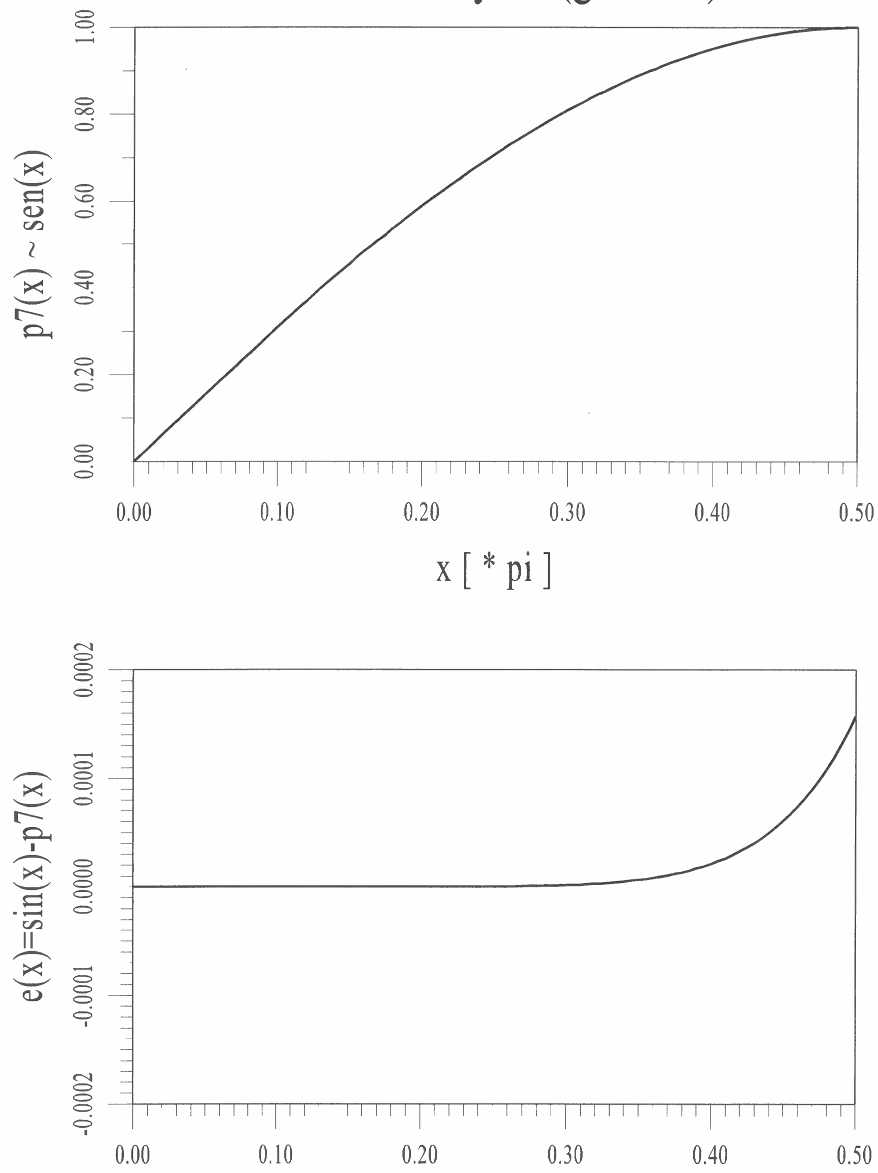
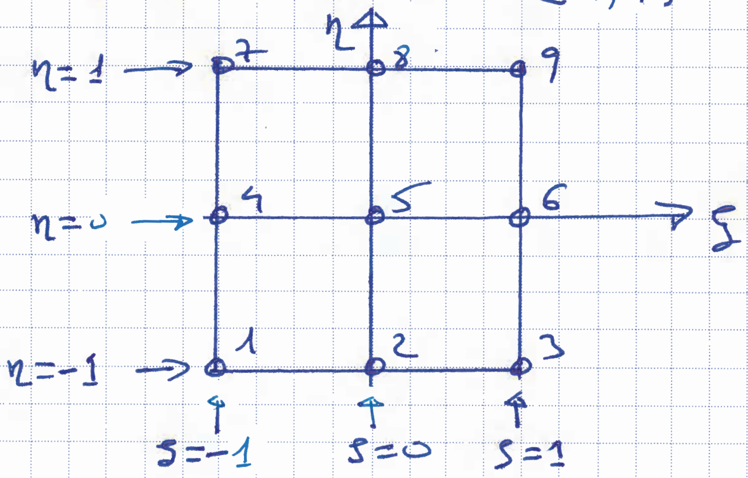
Muestreo Optimo (grado 3)



Serie de Taylor (grado 7)



Disponemos el origen de coordenadas en el centro del cuadrado y normalizamos de forma que las coordenadas ξ, η se mueven en el intervalo $[-1, 1]$



$$z(\xi,\eta) = \sum_{i=1,9} z_i B_i(\xi,\eta) \quad \text{con} \quad B_i(\xi_j,\eta_j) = \delta_{ij}$$

$$\Rightarrow \left.\begin{aligned}
B_1(\xi,\eta) &= \frac{\xi(\xi-1)}{2} \, \frac{\eta(\eta-1)}{2} \\
B_2(\xi,\eta) &= (1-\xi)(1+\xi) \, \frac{\eta(\eta-1)}{2} \\
B_3(\xi,\eta) &= \frac{\xi(\xi+1)}{2} \, \frac{\eta(\eta-1)}{2} \\
B_4(\xi,\eta) &= \frac{\xi(\xi-1)}{2} \, (1-\eta)(1+\eta) \\
B_5(\xi,\eta) &= (1-\xi)(1+\xi)\,(1-\eta)(1+\eta) \\
B_6(\xi,\eta) &= \frac{\xi(\xi+1)}{2} \, (1-\eta)(1+\eta) \\
B_7(\xi,\eta) &= \frac{\xi(\xi-1)}{2} \, \frac{\eta(\eta+1)}{2} \\
B_8(\xi,\eta) &= (1-\xi)(1+\xi) \, \frac{\eta(\eta+1)}{2} \\
B_9(\xi,\eta) &= \frac{\xi(\xi+1)}{2} \, \frac{\eta(\eta+1)}{2}
\end{aligned}\right\} \rightsquigarrow$$

$$\rightsquigarrow z(\xi,\eta) = a_{00} + a_{10}\xi + a_{01}\eta + a_{20}\xi^2 + a_{11}\xi\eta + a_{02}\eta^2 + a_{21}\xi^2\eta + a_{22}\xi^2\eta^2 + a_{12}\xi\eta^2$$

$\rightarrow$ {Polinomio completo de grado 2 en ξ y η

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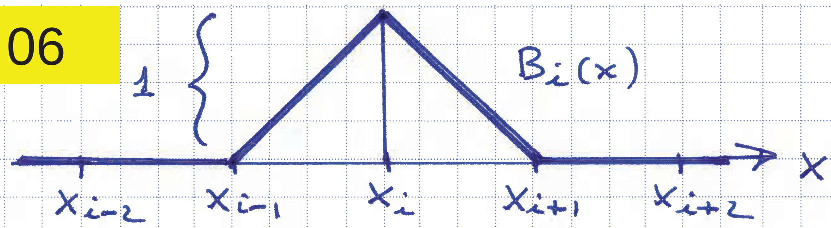


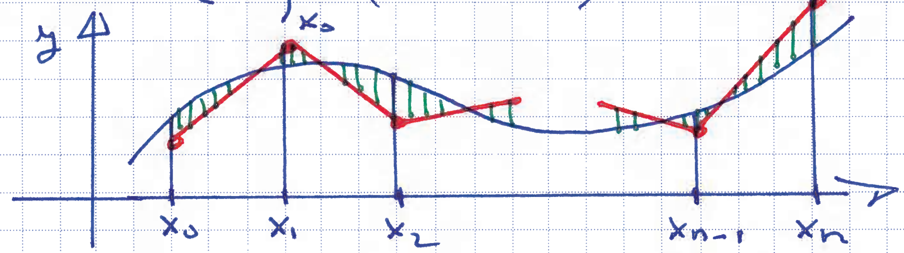
$$S(x) = \sum_{i=0,n} y_i B_i(x)$$

$$Q(y_0, \text{——}, y_n) = \langle f - S, f - S \rangle \; ; \; \langle g, h \rangle = \int_{x_0}^{x_n} g(x) h(x)\, dx$$

$$\boxed{\frac{\partial Q}{\partial y_j} = 0 \quad ; \; j = 0, \text{—}, n} \Leftarrow Q \text{ mínimo}$$

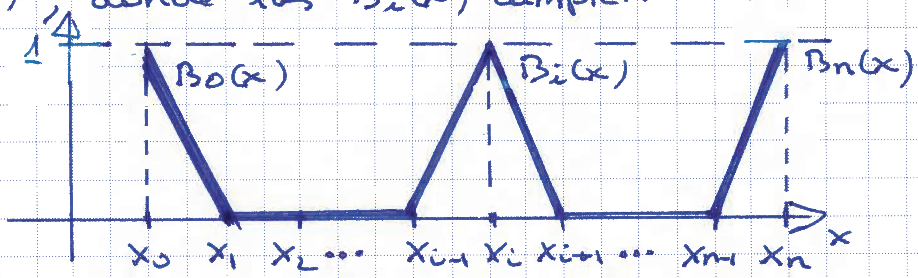
a) Se está utilizando el criterio de MÍNIMOS CUADRADOS, pues se minimiza

$$Q = \int_{x_0}^{x_n} (f(x) - S(x))^2\, dx$$



b) $S(x) = \sum_{i=0,n} y_i B_i(x)$, donde los $B_i(x)$ cumplen

$$B_i(x_j) = \delta_{ij} \iff$$



c) $$Q(y_0, \text{——}, y_n) = \int_{x_0}^{x_n} \Big(f(x) - \sum_{i=0,n} y_i B_i(x)\Big)^2 dx$$

$$\frac{\partial Q}{\partial y_j} = 2 \int_{x_0}^{x_n} \Big(f(x) - \sum_{i=0,n} y_i B_i(x)\Big) B_j(x)\, dx$$

$$\frac{\partial Q}{\partial y_j} = 0 \iff \sum_{i=0,n} \left[\int_{x_0}^{x_n} B_j(x) B_i(x)\, dx \right] y_i = \int_{x_0}^{x_n} B_j(x) f(x)\, dx \, ; \quad j = 0, \text{—}, n$$

Luego:

$$\boxed{\begin{array}{l} \underset{\sim}{K}\, \overline{y} = \overline{f} \longrightarrow \text{SISTEMA DE ECUACIONES LINEALES} \\ \left.\begin{array}{l} \underset{\sim}{K} = [K_{ij}] \, , \; K_{ij} = \displaystyle\int_{x_0}^{x_n} B_j(x) B_i(x)\, dx \\ \overline{f} = \{f_j\} \, , \; f_j = \displaystyle\int_{x_0}^{x_n} B_j(x) f(x)\, dx \end{array}\right\} \left\{\begin{array}{l} i = 0, \text{—}, n \\ j = 0, \text{—}, n \end{array}\right. \\ \overline{y} = \{y_i\} \end{array}}$$

d) $\underset{\sim}{K}$ es:

1) SIMÉTRICA, pues $K_{ij} = K_{ji}$

2) DEFINIDA POSITIVA, pues

$$\bar{v}^T \underset{\sim}{K} \bar{v} = \sum_{i=0,n} \sum_{j=0,n} v_j K_{ji} v_i = \sum_{i=0,n} \sum_{j=0,n} v_j \left[\int_{x_0}^{x_n} B_j(x) B_i(x) dx \right] v_i$$

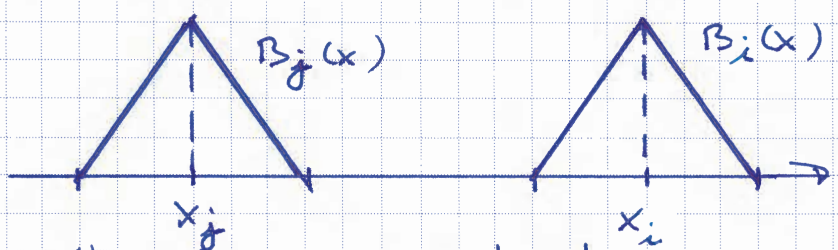
$$= \int_{x_0}^{x_n} \underbrace{\left(\sum_{i=0,n} v_i B_i(x) \right)}_{v(x)} \underbrace{\left(\sum_{j=0,n} v_j B_j(x) \right)}_{v(x)} dx = \int_{x_0}^{x_n} (v(x))^2 dx$$

luego, $\begin{cases} \bar{v}^T \underset{\sim}{K} \bar{v} \geq 0 \quad \forall \bar{v} = \{v_i\} \\ \bar{v}^T \underset{\sim}{K} \bar{v} = 0 \iff v(x) = 0 \iff \bar{v} = \bar{0} \end{cases}$

3) TRIDIAGONAL, pues

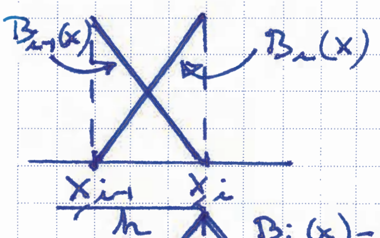
$$K_{ji} = \int_{x_0}^{x_n} B_j(x) B_i(x) dx = 0 \quad \forall i,j \,/\, |i-j| > 1$$

puesto que los rangos de $B_j(x)$ y $B_i(x)$ no se solapan en tal caso

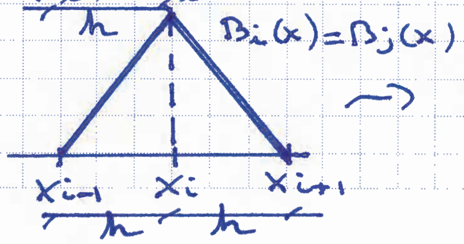


Por tanto, sólo podrán ser no nulas las componentes $\begin{cases} K_{i-1,i} \\ K_{i,i} \\ K_{i+1,i} \end{cases}$

Si las abscisas $\{x_i\}$ son equiespaciadas sólo habrá dos tipos de integrales:



$$\rightsquigarrow \quad K_{i-1,i} = \int_{x_{i-1}}^{x_i} \left(1 - \frac{x - x_{i-1}}{h}\right) \left(\frac{x - x_{i-1}}{h}\right) dx \;;\; i = 1, \ldots, n$$



$$\rightsquigarrow \quad K_{i,i} = \underbrace{\int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{h}\right)^2 dx}_{} + \underbrace{\int_{x_i}^{x_{i+1}} \left(1 - \frac{x - x_i}{h}\right)^2 dx}_{} \;;\; i = 0, \ldots, n$$

igual valor

$k_{0,0}$ y $k_{n,n}$ valen la mitad de las restantes $k_{i,i}$

Integrando se obtiene:

$$\begin{cases} k_{i-1,i} = h/6 \quad ; i = 1, \dots, n \\ k_{0,0} = h/3 \ , \ k_{n,n} = h/3 \ , \ k_{i,i} = 2h/3 \quad ; i = 1, \dots, n-1 \end{cases}$$

luego:

$$\underset{\sim}{K} = h/6 \begin{bmatrix} 2 & 1 & & & 0 \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & 4 & 1 \\ 0 & & & 1 & 2 \end{bmatrix}$$

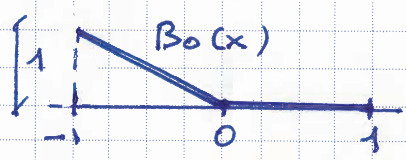
e) Lo más correcto es utilizar el método de $\underline{\text{CHOLESKY}}$ para matrices tridiagonales, pues la matriz es SIMÉTRICA y DEFINIDA POSITIVA, lo que asegura que el método funcionará.

f) Tenemos que resolver el sistema $\boxed{\underset{\sim}{K}\ \bar{q} = \bar{f}}$, con

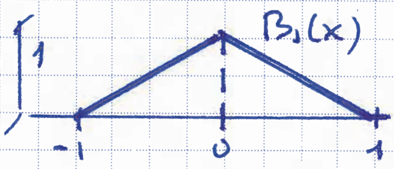
$$\boxed{\underset{\sim}{K} = h/6 \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad ; \quad h = 1}$$

$x_0 = -1 \quad x_1 = 0 \quad x_2 = 1 \quad x$

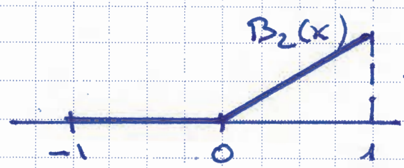
$$\bar{f} = \begin{Bmatrix} f_0 \\ f_1 \\ f_2 \end{Bmatrix} \ , \quad f_j = \int_{x_0}^{x_2} B_j(x)\, x^2\, dx$$



$$\rightarrow f_0 = \int_{-1}^{0} \left(1 - \frac{x+1}{1}\right) x^2\, dx = -\int_{-1}^{0} x^3\, dx = x^4/4 \Big|_{-1}^{0} = 1/4$$



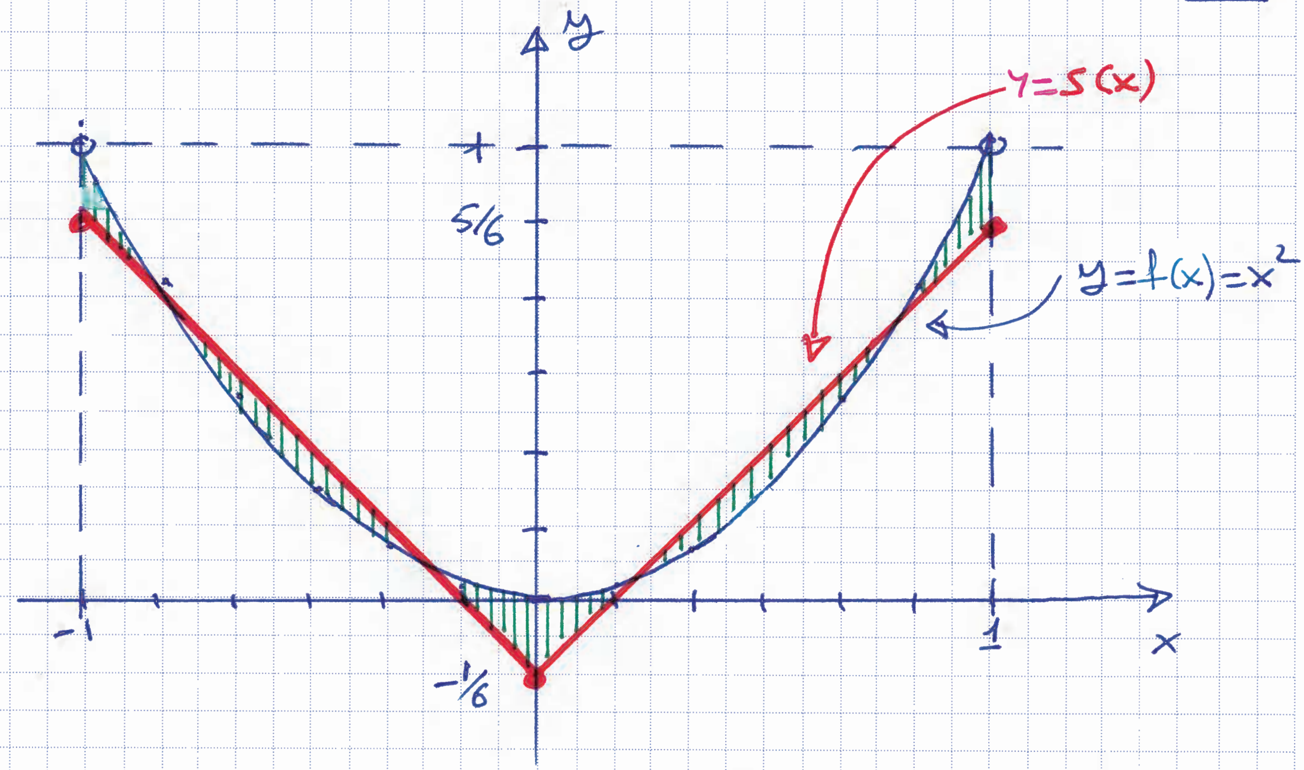
$$\rightarrow f_1 = \int_{-1}^{0} (1+x)\, x^2\, dx + \int_{0}^{1} (1-x)\, x^2\, dx = \dots = 1/6$$



$$\rightarrow f_2 = \int_{0}^{1} (x)\, x^2\, dx = \int_0^1 x^3\, dx = x^4/4 \Big|_0^1 = 1/4$$

Por tanto:

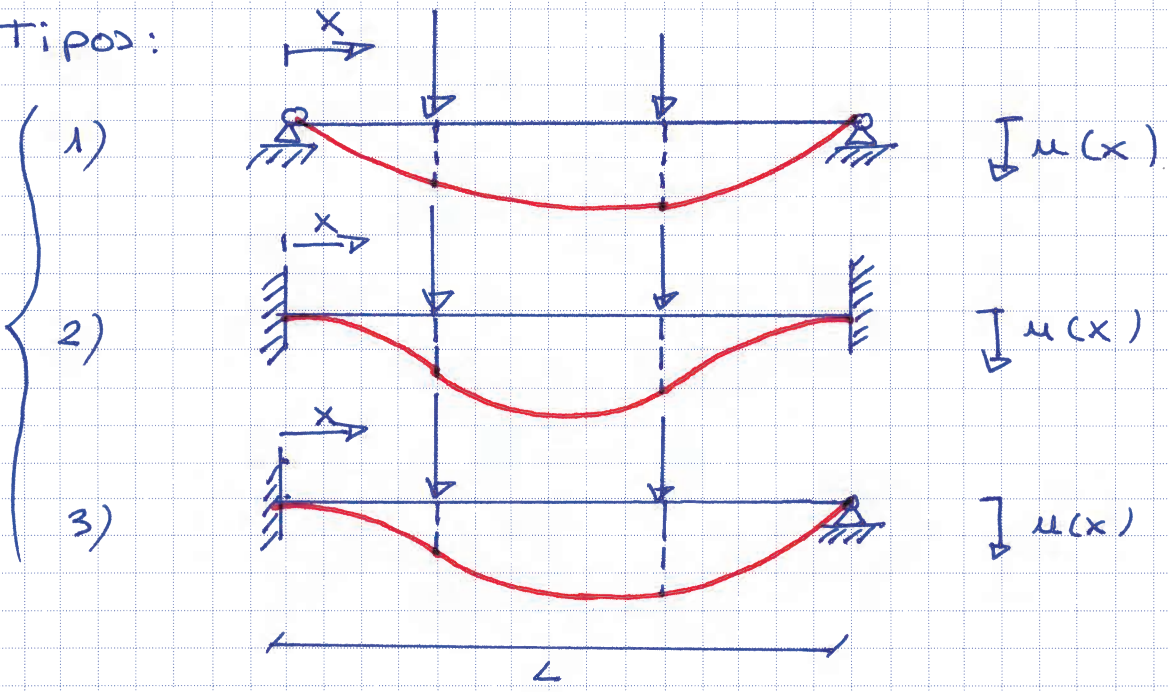
$$\boxed{\bar{f} = \frac{1}{12} \begin{Bmatrix} 3 \\ 2 \\ 3 \end{Bmatrix}} \Rightarrow \boxed{\bar{q} = 1/6 \begin{Bmatrix} 5 \\ -1 \\ 5 \end{Bmatrix}}$$



Observamos que $s(x)$ se ajusta a $f(x) = x^2$ de forma que se minimice el error cuadrático

$$Q = \int_{-1}^{1} \left(f(x) - s(x) \right)^2 dx$$

Tipos:



En cada tramo (apoyo - carga 1, carga 1 - carga 2, carga 2 - apoyo) no hay cargas repartidas $\Rightarrow$ la ecuación de la elástica es $\underline{u^{IV}=0}$

Por tanto $u(x)$ es un polinomio de 3^{er} grado en x en cada tramo. Bajo las cargas puntuales la elástica es continua, su pendiente también (giros) y su derivada segunda también (momentos).

Luego $u(x)$ es un <u>SPLINE CÚBICO CON 2ª DERIVADA CONTINUA</u>, con las condiciones de contorno:

$$\begin{cases} \text{caso 1)} \rightarrow u''(0) = u''(L) = 0 \\ \text{caso 2)} \rightarrow u'(0) = u'(L) = 0 \\ \text{caso 3)} \rightarrow \begin{cases} u'(0) = u''(L) = 0 & \text{(empotramiento a la izquierda)} \\ u''(0) = u'(L) = 0 & \text{(empotramiento a la derecha)} \end{cases} \end{cases}$$

Por tanto, si el programa de cálculo de estructuras facilita las flechas en los puntos de aplicación de las cargas y sabiendo que la flecha en los extremos es nula, interpolaremos:

> 1) un SPLINE NATURAL,
> 2) un SPLINE FORZADO con pendientes nulas en los extremos,
> 3) un SPLINE con condiciones mixtas.

Dados $\{(\tau_i, y_i)\}_{i=0,-,n} \Rightarrow$ obtener $\{(\tau_i, \eta_i)\}_{i=0,-,n}$

que minimicen:

$$Q(\eta_0, -, \eta_n) = \sum_{i=0,n} w_i (y_i - \eta_i)^2 + \sum_{i=0,n-3} \hat{w}_i \left(\eta[\tau_{i+3}, \tau_{i+2}, \tau_{i+1}, \tau_i]\right)^2$$

a) $\boxed{\dfrac{\partial Q}{\partial \eta_j} = 0 \quad ; \quad j = 0, -, n}$

$$\eta[\tau_{i+3}, \tau_{i+2}, \tau_{i+1}, \tau_i] = \frac{\eta[\tau_{i+3}, \tau_{i+2}, \tau_{i+1}] - \eta[\tau_{i+2}, \tau_{i+1}, \tau_i]}{\tau_{i+3} - \tau_i}$$

$$\begin{cases} \eta[\tau_{i+3}, \tau_{i+2}, \tau_{i+1}] = \dfrac{\eta[\tau_{i+3}, \tau_{i+2}] - \eta[\tau_{i+2}, \tau_{i+1}]}{\tau_{i+3} - \tau_{i+1}} \\ \\ \eta[\tau_{i+2}, \tau_{i+1}, \tau_i] = \dfrac{\eta[\tau_{i+2}, \tau_{i+1}] - \eta[\tau_{i+1}, \tau_i]}{\tau_{i+2} - \tau_i} \end{cases}$$

$$\begin{cases} \eta[\tau_{i+3}, \tau_{i+2}] = \dfrac{\eta_{i+3} - \eta_{i+2}}{\tau_{i+3} - \tau_{i+2}} \\ \\ \eta[\tau_{i+2}, \tau_{i+1}] = \dfrac{\eta_{i+2} - \eta_{i+1}}{\tau_{i+2} - \tau_{i+1}} \\ \\ \eta[\tau_{i+1}, \tau_i] = \dfrac{\eta_{i+1} - \eta_i}{\tau_{i+1} - \tau_i} \end{cases}$$

$$\Rightarrow \boxed{\frac{\partial Q}{\partial \eta_j} = -2 w_j (y_j - \eta_j) + 2 \sum_{i=\max(0,j-3)}^{\min(j,n-3)} \hat{w}_i \, \eta[\tau_{i+3}, \tau_{i+2}, \tau_{i+1}, \tau_i] \frac{\partial}{\partial \eta_j} \eta[\tau_{i+3}, \tau_{i+2}, \tau_{i+1}, \tau_i]}$$

Se obtiene un sistema en banda con semiancho = 3 (*)

(*) Pues $\dfrac{\partial}{\partial \eta_j} \eta[\tau_{i+3}, \tau_{i+2}, \tau_{i+1}, \tau_i] = 0$ si $\begin{cases} j < i \\ \text{ó} \\ j > i+3 \end{cases}$

TABLA DE DIFERENCIAS

$\tau_i,\ \eta_i$

$$\left(\frac{\eta_{i+1}-\eta_i}{\tau_{i+1}-\tau_i}\right)$$

$\tau_{i+1},\ \eta_{i+1}$

$$\frac{\left(\dfrac{\eta_{i+2}-\eta_{i+1}}{\tau_{i+2}-\tau_{i+1}}\right)-\left(\dfrac{\eta_{i+1}-\eta_i}{\tau_{i+1}-\tau_i}\right)}{\tau_{i+2}-\tau_i}$$

$$\left(\frac{\eta_{i+2}-\eta_{i+1}}{\tau_{i+2}-\tau_{i+1}}\right)$$

$$\frac{\left(\dfrac{\dfrac{\eta_{i+3}-\eta_{i+2}}{\tau_{i+3}-\tau_{i+2}}-\dfrac{\eta_{i+2}-\eta_{i+1}}{\tau_{i+2}-\tau_{i+1}}}{\tau_{i+3}-\tau_{i+1}}\right)-\left(\dfrac{\dfrac{\eta_{i+2}-\eta_{i+1}}{\tau_{i+2}-\tau_{i+1}}-\dfrac{\eta_{i+1}-\eta_i}{\tau_{i+1}-\tau_i}}{\tau_{i+2}-\tau_i}\right)}{\tau_{i+3}-\tau_i}$$

$\tau_{i+2},\ \eta_{i+2}$

$$\frac{\left(\dfrac{\eta_{i+3}-\eta_{i+2}}{\tau_{i+3}-\tau_{i+2}}\right)-\left(\dfrac{\eta_{i+2}-\eta_{i+1}}{\tau_{i+2}-\tau_{i+1}}\right)}{\tau_{i+3}-\tau_{i+1}}$$

$$\left(\frac{\eta_{i+3}-\eta_{i+2}}{\tau_{i+3}-\tau_{i+2}}\right)$$

$\tau_{i+3},\ \eta_{i+3}$

Luego,

$$\eta[\tau_{i+3},\tau_{i+2},\tau_{i+1},\tau_i] = \eta_{i+3}\ B_{i,3} + $$
$$+ \eta_{i+2}\ B_{i,2} +$$
$$+ \eta_{i+1}\ B_{i,1} +$$
$$+ \eta_i\ B_{i,0}$$

con

$$\left\{\begin{array}{l}
B_{i,3} = \dfrac{1}{\tau_{i+3}-\tau_i}\cdot\dfrac{1}{\tau_{i+3}-\tau_{i+1}}\cdot\dfrac{1}{\tau_{i+3}-\tau_{i+2}} \\[2ex]
B_{i,2} = \dfrac{1}{\tau_{i+3}-\tau_i}\left[\dfrac{1}{\tau_{i+3}-\tau_{i+1}}\left(\dfrac{-1}{\tau_{i+3}-\tau_{i+2}}-\dfrac{1}{\tau_{i+2}-\tau_{i+1}}\right)\right. \\[2ex]
\qquad\qquad \left. -\dfrac{1}{\tau_{i+2}-\tau_i}\left(\dfrac{1}{\tau_{i+2}-\tau_{i+1}}\right)\right] \\[2ex]
B_{i,1} = \dfrac{1}{\tau_{i+3}-\tau_i}\left[\dfrac{1}{\tau_{i+3}-\tau_{i+1}}\left(\dfrac{1}{\tau_{i+2}-\tau_{i+1}}\right)\right. \\[2ex]
\qquad\qquad \left. -\dfrac{1}{\tau_{i+2}-\tau_i}\left(\dfrac{-1}{\tau_{i+2}-\tau_{i+1}}-\dfrac{1}{\tau_{i+1}-\tau_i}\right)\right] \\[2ex]
B_{i,0} = \dfrac{1}{\tau_{i+3}-\tau_i}\cdot\dfrac{1}{\tau_{i+2}-\tau_i}\cdot\dfrac{-1}{\tau_{i+1}-\tau_i}
\end{array}\right.$$

$$\Rightarrow \boxed{\dfrac{\partial\, \eta[\tau_{i+3},\tau_{i+2},\tau_{i+1},\tau_i]}{\partial \eta_j} = \begin{cases} B_{i,j-i} & \text{si } i \le j \le i+3 \\ 0 & \text{en caso contrario} \end{cases}}$$

La ecuación típica $(3 \leq j \leq n-3)$ será de la forma:

$$
\begin{aligned}
& -2 w_j (y_j - \eta_j) + \\
& + 2 \Big[\hat{w}_{j-3} (\eta_{j-3} B_{j-3,0} + \eta_{j-2} B_{j-3,1} + \eta_{j-1} B_{j-3,2} + \eta_j B_{j-3,3}) B_{j-3,3} \\
& \quad + \hat{w}_{j-2} (\eta_{j-2} B_{j-2,0} + \eta_{j-1} B_{j-2,1} + \eta_j B_{j-2,2} + \eta_{j-1} B_{j-2,3}) B_{j-2,2} \\
& \quad + \hat{w}_{j-1} (\eta_{j-1} B_{j-1,0} + \eta_j B_{j-1,1} + \eta_{j+1} B_{j-1,2} + \eta_{j+2} B_{j-1,3}) B_{j-1,1} \\
& \quad + \hat{w}_j (\eta_j B_{j,0} + \eta_{j+1} B_{j,1} + \eta_{j+2} B_{j,2} + \eta_{j+3} B_{j,3}) B_{j,0} \\
& = 0 \quad ; \quad j = 3, \ldots, n-3
\end{aligned}
$$

$$\Updownarrow$$

$$
\boxed{
\begin{aligned}
& [\hat{w}_{j-3} B_{j-3,0} B_{j-3,3}] \eta_{j-3} + \\
& [\hat{w}_{j-3} B_{j-3,1} B_{j-3,3} + \hat{w}_{j-2} B_{j-2,0} B_{j-2,2}] \eta_{j-2} + \\
& [\hat{w}_{j-3} B_{j-3,2} B_{j-3,3} + \hat{w}_{j-2} B_{j-2,1} B_{j-2,2} + \hat{w}_{j-1} B_{j-1,0} B_{j-1,1}] \eta_{j-1} + \\
& [\hat{w}_{j-3} B_{j-3,3} B_{j-3,3} + \hat{w}_{j-2} B_{j-2,2} B_{j-2,2} + \\
& \qquad + \hat{w}_{j-1} B_{j-1,1} B_{j-1,1} + \hat{w}_j B_{j,0} B_{j,0} + w_j] \eta_j + \\
& [\hat{w}_{j-2} B_{j-2,3} B_{j-2,2} + \hat{w}_{j-1} B_{j-1,2} B_{j-1,1} + \hat{w}_j B_{j,1} B_{j,0}] \eta_{j+1} + \\
& [\hat{w}_{j-1} B_{j-1,3} B_{j-1,1} + \hat{w}_j B_{j,2} B_{j,0}] \eta_{j+2} + \\
& [\hat{w}_j B_{j,3} B_{j,0}] \eta_{j+3} \\
& = w_j y_j \quad ; \quad j = 3, \ldots, n-3
\end{aligned}
}
$$

Luego el sistema será HEPTADIAGONAL.

Las ecuaciones $j<3$, $j>n-3$ serán incompletas (le faltarán términos)

08e

b) Puntos equiespaciados $\Rightarrow$ $x_{i+u} = x_i + u \cdot h$

Mismo ponderación $\Rightarrow$ $\left.\begin{cases} w_i = \lambda \\ \hat{w}_i = \mu \end{cases}\right\} \forall i$

$$\eta[t_{i+3}, t_{i+2}, t_{i+1}, t_i] = \frac{\eta[t_{i+3}, t_{i+2}, t_{i+1}] - \eta[t_{i+2}, t_{i+1}, t_i]}{3h}$$

$$\begin{cases} \eta[t_{i+3}, t_{i+2}, t_{i+1}] = \dfrac{\eta[t_{i+3}, t_{i+2}] - \eta[t_{i+2}, t_{i+1}]}{2h} \\ \eta[t_{i+2}, t_{i+1}, t_i] = \dfrac{\eta[t_{i+2}, t_{i+1}] - \eta[t_{i+1}, t_i]}{2h} \end{cases}$$

$$\begin{cases} \eta[t_{i+3}, t_{i+2}] = \dfrac{\eta_{i+3} - \eta_{i+2}}{h} \\ \eta[t_{i+2}, t_{i+1}] = \dfrac{\eta_{i+2} - \eta_{i+1}}{h} \\ \eta[t_{i+1}, t_i] = \dfrac{\eta_{i+1} - \eta_i}{h} \end{cases}$$

Sustituyendo:

$$\eta[t_{i+3}, t_{i+2}, t_{i+1}, t_i] = \frac{1}{3!\,h^3}\left(\eta_{i+3} - 3\eta_{i+2} + 3\eta_{i+1} - \eta_i\right)$$

Luego,

$$Q(\eta_0, \dots, \eta_n) = \sum_{i=0,n} \lambda (y_i - \eta_i)^2 + \sum_{i=0,n-3} \mu\left(\frac{1}{6h^3}\left(\eta_{i+3} - 3\eta_{i+2} + \eta_{i+1} - \eta_i\right)\right)^2$$

$$= \lambda \sum_{i=0,n} (y_i - \eta_i)^2 + \frac{\mu}{36h^6} \sum_{i=0,n-3} \left(\eta_{i+3} - 3\eta_{i+2} + 3\eta_{i+1} - \eta_i\right)^2$$

$$\frac{\partial Q}{\partial \eta_j} = 2\lambda \sum_{i=0,n} (y_i-\eta_i)\frac{\partial (y_i-\eta_i)}{\partial \eta_j} +$$

$$+ \frac{2\mu}{36h^6}\sum_{i=0,n-3} (\eta_{i+3}-3\eta_{i+2}+3\eta_{i+1}-\eta_i)\frac{\partial(\eta_{i+3}-3\eta_{i+2}+3\eta_{i+1}-\eta_i)}{\partial \eta_j}$$

donde,

$$\left\{\begin{array}{l} \dfrac{\partial (y_i-\eta_i)}{\partial \eta_j} = \begin{cases} -1 & \text{si } i=j \\ 0 & \text{si } i\neq j \end{cases} \\[2ex] \dfrac{\partial(\eta_{i+3}-3\eta_{i+2}+3\eta_{i+1}-\eta_i)}{\partial \eta_j} = \begin{cases} 1 & \text{si } j=i+3 \\ -3 & \text{si } j=i+2 \\ 3 & \text{si } j=i+1 \\ -1 & \text{si } j=i \\ 0 & \text{en los otros casos} \end{cases} \end{array}\right.$$

Luego,

$$\frac{\partial Q}{\partial \eta_j} = -2\lambda(y_j-\eta_j) + $$

$$+\frac{2\mu}{36h^6}\sum_{i=\max(0,j-3)}^{\min(n-3,j)} (\eta_{i+3}-3\eta_{i+2}+3\eta_{i+1}-\eta_i)\frac{\partial(\eta_{i+3}-3\eta_{i+2}+3\eta_{i+1}-\eta_i)}{\partial \eta_j}$$

Para $\left\{\begin{array}{l} j<3 \\ j>n-3 \end{array}\right\}$ las ecuaciones son incompletas (faltan términos)

Particularizamos:

$$j=0 \Rightarrow \; -2\cancel{\lambda}(y_0-\eta_0) + \frac{\cancel{2}\mu}{36h^6}\left[(\eta_3-3\eta_2+3\eta_1-\eta_0)(-1)\right] = 0$$

$$j=1 \Rightarrow \; -2\cancel{\lambda}(y_1-\eta_1) + \frac{\cancel{2}\mu}{36h^6}\Big[(\eta_3-3\eta_2+3\eta_1-\eta_0)(3) + (\eta_4-3\eta_3+3\eta_2-\eta_1)(1)\Big] = 0$$

$$j=2 \Rightarrow \; -2\cancel{\lambda}(y_2-\eta_2) + \frac{\cancel{2}\mu}{36h^6}\Big[(\eta_3-3\eta_2+3\eta_1-\eta_0)(-3) + (\eta_4-3\eta_3+3\eta_2-\eta_1)(3) + (\eta_5-3\eta_4+3\eta_3-\eta_2)(-1)\Big] = 0$$

$$3 \le j \le n-3 \Rightarrow -\cancel{2}\lambda(y_j-\eta_j)+\frac{\cancel{2}\mu}{36h^6}\Big[(\eta_j-3\eta_{j-1}+3\eta_{j-2}-\eta_{j-3})(1)+$$
$$+(\eta_{j+1}-3\eta_j+3\eta_{j-1}-\eta_{j-2})(-3)+$$
$$+(\eta_{j+2}-3\eta_{j+1}+3\eta_j-\eta_{j-1})(3)+$$
$$+(\eta_{j+3}-3\eta_{j+2}+3\eta_{j+1}-\eta_j)(-1)\Big]=0$$

$$j=n-2 \Rightarrow -\cancel{2}\lambda(y_{n-2}-\eta_{n-2})+\frac{\cancel{2}\mu}{36h^6}\Big[(\eta_{n-2}-3\eta_{n-3}+3\eta_{n-4}-\eta_{n-5})(1)+$$
$$+(\eta_{n-1}-3\eta_{n-2}+3\eta_{n-3}-\eta_{n-4})(-3)+$$
$$+(\eta_n-3\eta_{n-1}+3\eta_{n-2}-\eta_{n-3})(3)\Big]=0$$

$$j=n-1 \Rightarrow -\cancel{2}\lambda(y_{n-1}-\eta_{n-1})+\frac{\cancel{2}\mu}{36h^6}\Big[(\eta_{n-1}-3\eta_{n-2}+3\eta_{n-3}-\eta_{n-4})(1)+$$
$$+(\eta_n-3\eta_{n-1}+3\eta_{n-2}-\eta_{n-3})(-3)\Big]=0$$

$$j=n \Rightarrow -\cancel{2}\lambda(y_n-\eta_n)+\frac{\cancel{2}\mu}{36h^6}\Big[(\eta_n-3\eta_{n-1}+3\eta_{n-2}-\eta_{n-3})(1)\Big]=0$$

c) $\varphi = 36h^6\dfrac{\lambda}{\mu}$

$$\begin{bmatrix}
(1+\varphi) & -3 & 3 & -1 & & & & & & \\
-3 & (10+\varphi) & -12 & 6 & -1 & & & & & \\
3 & -12 & (19+\varphi) & -15 & 6 & -1 & & & & \\
-1 & 6 & -15 & (20+\varphi) & -15 & 6 & -1 & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & -1 & 6 & -15 & (20+\varphi) & -15 & 6 & -1 & \\
 & & & -1 & 6 & -15 & (19+\varphi) & -12 & 3 & \\
 & & & & -1 & 6 & -12 & (10+\varphi) & -3 & \\
 & & & & & -1 & 3 & -3 & (1+\varphi) &
\end{bmatrix}
\begin{Bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \vdots \\ \eta_{n-3} \\ \eta_{n-2} \\ \eta_{n-1} \\ \eta_n \end{Bmatrix}
= \varphi \begin{Bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{Bmatrix}$$

La matriz es <u>simétrica</u> y será <u>definida positiva</u> (por provenir de un planteamiento de mínimos cuadrados) $\Rightarrow$ <u>CHOLESKY</u> (semiancho = 3).

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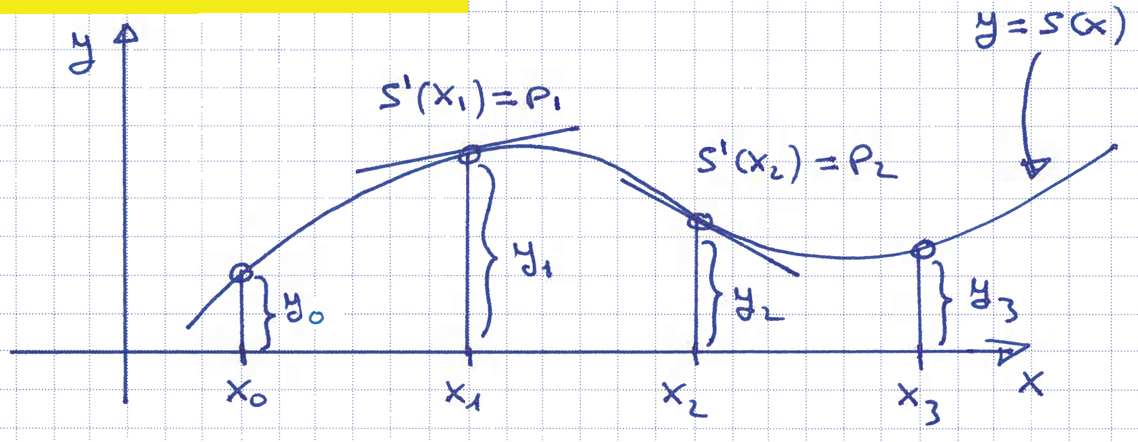


a) Sea $y = s(x)$ la función interpoladora.

Exigimos: $\begin{cases} s(x_0) = y_0;\; s(x_1) = y_1;\; s(x_2) = y_2;\; s(x_3) = y_3; \\ s'(x_1) = p_1;\; s'(x_2) = p_2; \end{cases}$

I) <u>Polinomio de grado suficientemente alto</u>

Hay 6 condiciones $\Longrightarrow$ $s(x) = P_5(x)$

Luego: $\begin{cases} s(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 \\ s'(x) = \quad a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 \end{cases}$

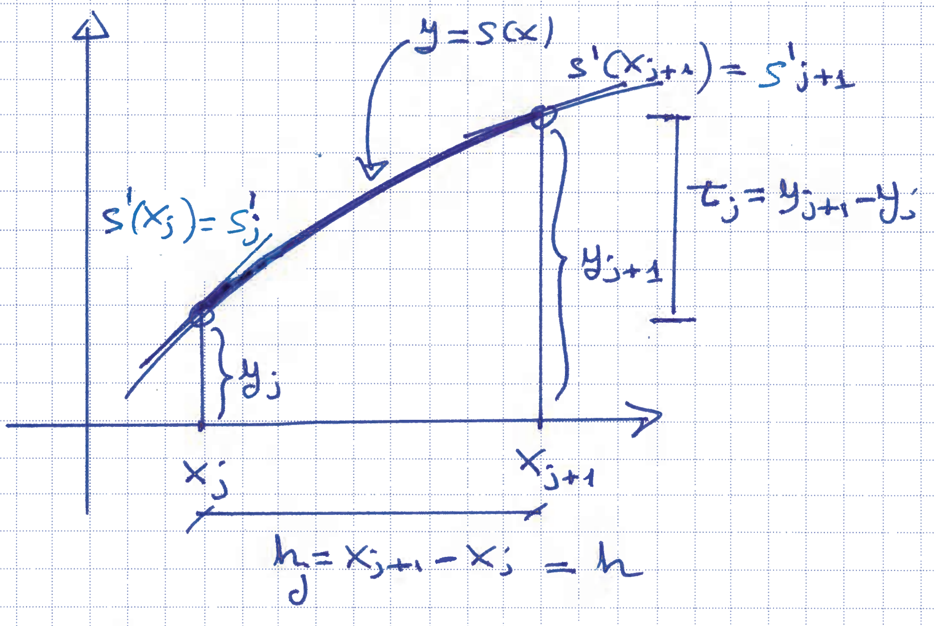
Para hallar los coeficientes $\{a_i\}_{i=0,\dots,5}$ hay que resolver el sistema:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & x_0^4 & x_0^5 \\ 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 & x_2^5 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 & x_3^5 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 \\ 0 & 1 & 2x_2 & 3x_2^2 & 4x_2^3 & 5x_2^4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ p_1 \\ p_2 \end{Bmatrix}$$

II) <u>Spline cúbico con 2ª derivada continua</u>

Normalmente conocemos las pendientes (o las derivadas 2ªs) en los extremos. En este caso las conocemos en los puntos intermedios. En cada subintervalo $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, $s(x)$ es un polinomio de 3er grado.

En cada subintervalo:



$$\begin{cases} s(x_j)=y_j \\ s(x_{j+1})=y_{j+1} \\ s'(x_j)=s'_j \\ s'(x_{j+1})=s'_{j+1} \end{cases}$$

$$s(x)= y_j+s'_j(x-x_j)+ \left[3t_j-h_j(s'_{j+1}+2s'_j)\right]\frac{(x-x_j)^2}{h_j^2}+ \left[h_j(s'_{j+1}+s'_j)-2t_j\right]\frac{(x-x_j)^3}{h_j^3} \quad ; \quad x\in[x_j,x_{j+1}]$$

$j=0,\dots,2$

Compatibilidad de 2ª derivada:

$$s''(x_1^-)=s''(x_1^+) \iff \lambda_1 s'_0+2s'_1+\mu_1 s'_2=e_1$$

con $\lambda_1=\dfrac{h_1}{h_0+h_1}=1/2$; $\mu_1=\dfrac{h_0}{h_0+h_1}=1/2$

$$e_1=3\mu_1\frac{t_1}{h_1}+3\lambda_1\frac{t_0}{h_0}=\frac{3}{2}\,\frac{y_2-y_0}{h}$$

$$s''(x_2^-)=s''(x_2^+) \iff \lambda_2 s'_1+2s'_2+\mu_2 s'_3=e_2$$

con $\lambda_2=\dfrac{h_2}{h_1+h_2}=1/2$; $\mu_2=\dfrac{h_1}{h_1+h_2}=1/2$

$$e_2=3\mu_2\frac{t_2}{h_2}+3\lambda_2\frac{t_1}{h_1}=\frac{3}{2}\,\frac{y_3-y_1}{h}$$

b) Luego,

$$\begin{bmatrix} 1/2 & 2 & 1/2 & 0 \\ 0 & 1/2 & 2 & 1/2 \end{bmatrix} \begin{Bmatrix} s'_0 \\ s'_1 \\ s'_2 \\ s'_3 \end{Bmatrix} = \frac{3}{2h} \begin{Bmatrix} y_2 - y_0 \\ y_3 - y_1 \end{Bmatrix}$$

$$\left.\begin{array}{l} s'_1 = p_1 \\ s'_2 = p_2 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} s'_0 = \frac{3}{h}(y_2 - y_0) - 4p_1 - p_2 \\ s'_3 = \frac{3}{h}(y_3 - y_1) - p_1 - 4p_2 \end{array}\right.$$

Portanto,

i	x_i	$s(x_i)$	$s'(x_i)$
0	x_0	y_0	$s'_0 = 3/h\,(y_2-y_0) - 4p_1 - p_2$
1	$x_1 = x_0 + h$	y_1	$s'_1 = p_1$
2	$x_2 = x_0 + 2h$	y_2	$s'_2 = p_2$
3	$x_3 = x_0 + 3h$	y_3	$s'_3 = 3/h\,(y_3-y_1) - p_1 - 4p_2$

$$x \in [x_0, x_1] \Rightarrow s(x) = y_0 + [3(y_2 - y_0) - h(4p_1 + p_2)]\,(x-x_0)/h + \\ + [3(-2y_2 + y_1 + y_0) + h(7p_1 + 2p_2)]\,(x-x_0)^2/h^2 + \\ + [(3y_2 - 2y_1 - y_0) - h(3p_1 + p_2)]\,(x-x_0)^3/h^3$$

$$x \in [x_1, x_2] \Rightarrow s(x) = y_1 + [h p_1]\,(x-x_1)/h + \\ + [3(y_2 - y_1) - h(p_2 + 2p_1)]\,(x-x_1)^2/h^2 + \\ + [-2(y_2 - y_1) + h(p_2 + p_1)]\,(x-x_1)^3/h^3$$

$$x \in [x_2, x_3] \Rightarrow s(x) = y_2 + [h p_2]\,(x-x_2)/h + \\ + [3(-y_2 + y_1) + h(2p_2 + p_1)]\,(x-x_2)^2/h^2 + \\ + [(y_3 + 2y_2 - 3y_1) - h(3p_2 + p_1)]\,(x-x_2)^3/h^3$$

c) El spline natural es el más suave (y la curva más suave que pasa por los puntos).

Luego p_1 y p_2 han de ser tales que se verifique:

$$s''(x_0) = 0 \,, \qquad s''(x_3) = 0.$$

$$x \in [x_0, x_1] \Rightarrow s''(x) = 2\left[3(-2y_2+y_1+y_0) + h\,(7p_1+2p_2)\right]\frac{1}{h^2} +$$

$$+\, 6\left[(3y_2-2y_1-y_0) - h(3p_1+p_2)\right]\frac{(x-x_0)}{h^3}$$

$$s''(x_0) = \left[6\,(-2y_2+y_1+y_0) + 2h\,(7p_1+2p_2)\right]\frac{1}{h^2}$$

$$x \in [x_2, x_3] \Rightarrow s''(x) = 2\left[3\,(-y_2+y_1) + h\,(2p_2+p_1)\right]\frac{1}{h^2} +$$

$$+\, 6\left[(y_3+2y_2-3y_1) - h\,(3p_2+p_1)\right]\frac{(x-x_2)}{h^3}$$

$$s''(x_3) = \left[6\,(y_3+y_2-2y_1) - 2h\,(7p_2+2p_1)\right]\frac{1}{h^2}$$

Por tanto:

$$\begin{bmatrix} 7 & 2 \\ 2 & 7 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \frac{3}{h} \begin{Bmatrix} 2y_2 - y_1 - y_0 \\ y_3 + y_2 - 2y_1 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} p_1 = \dfrac{1}{15h}\left(-7y_0 - 3y_1 + 12y_2 - 2y_3\right) \\[2ex] p_2 = \dfrac{1}{15h}\left(2y_0 - 12y_1 + 3y_2 + 7y_3\right) \end{cases}$$