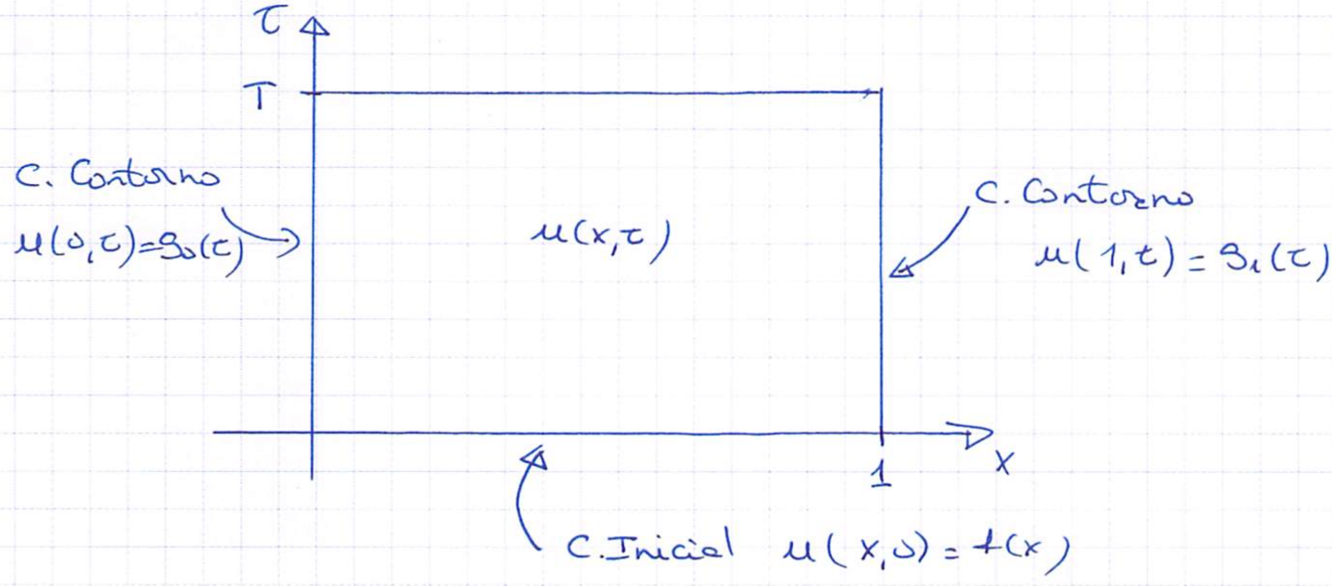
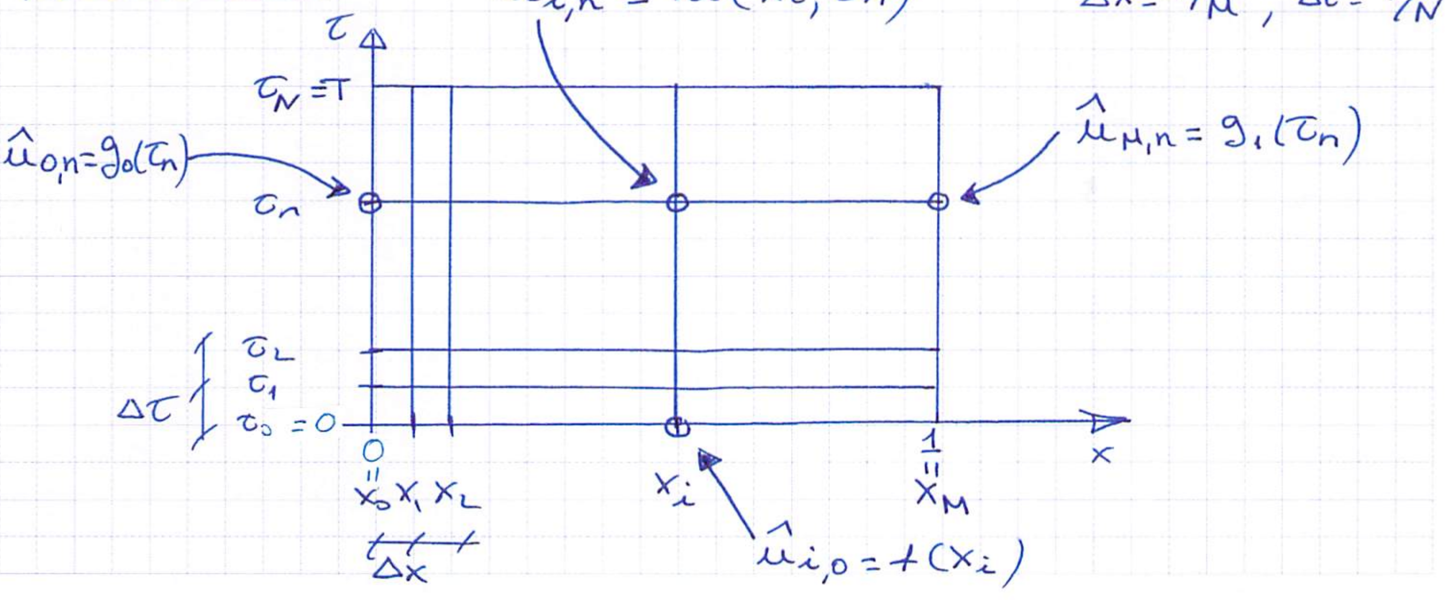
ECUACIONES PARABÓLICAS 1D

$$\begin{cases} u_\tau = u_{xx} & ;\ 0 < x < 1 \quad ;\ 0 < \tau < T & \text{(EDP)} \\ u(x,0) = f(x) & 0 \leq x \leq 1 & \text{(C. Inicial)} \\ u(0,\tau) = g_0(\tau) & 0 < \tau \leq T & \text{(C. Contorno)} \\ u(1,\tau) = g_1(\tau) & 0 < \tau \leq T & \text{(C. Contorno)} \end{cases}$$



Discretización:

$\hat{u}_{i,n} \simeq u(x_i, \tau_n)$ $\qquad \Delta x = 1/M \ ; \ \Delta\tau = T/N$



Metodo Explicito

$$\left[u_\tau - u_{xx}\right]\Big|_{\substack{x=x_i\\ \tau=\tau_n}} = 0$$

$$\begin{cases} u_\tau\Big|_{\substack{x=x_i\\ \tau=\tau_n}} = \dfrac{u(x_i,\tau_{n+1})-u(x_i,\tau_n)}{\Delta\tau} - u_{\tau\tau}\dfrac{\Delta\tau}{2} + \Theta(\Delta\tau^2) \\ u_{xx}\Big|_{\substack{x=x_i\\ \tau=\tau_n}} = \dfrac{u(x_{i+1},\tau_n)-2u(x_i,\tau_n)+u(x_{i-1},\tau_n)}{\Delta x^2} - u_{xxxx}\dfrac{\Delta x^2}{12} + \Theta(\Delta x^4) \end{cases}$$

$$\boxed{\begin{gathered} \frac{u(x_i,\tau_{n+1})-u(x_i,\tau_n)}{\Delta\tau} - \frac{u(x_{i+1},\tau_n)-2u(x_i,\tau_n)+u(x_{i-1},\tau_n)}{\Delta x^2} = \tau(x_i,\tau_n) \\ \tau(x_i,\tau_n) = \left[u_{\tau\tau}\frac{\Delta\tau}{2} - u_{xxxx}\frac{\Delta x^2}{12}\right]\Big|_{\substack{x=x_i\\ \tau=\tau_n}} + \Theta(\Delta\tau^2) + \Theta(\Delta x^4) \end{gathered}}$$

$$\begin{cases} \text{Se define : } \lambda = \dfrac{\Delta\tau}{\Delta x^2} \\ u_\tau = u_{xx} \Rightarrow u_{\tau\tau} = u_{xx\tau} = u_{\tau xx} = (u_\tau)_{xx} = u_{xxxx} \end{cases}$$

Luego:

$$\tau(x_i,\tau_n) = \frac{u_{\tau\tau}}{12}\Big|_{\substack{x=x_i\\ \tau=\tau_n}} (6\lambda-1)\,\Delta x^2 + \Theta(\Delta x^4\cdot\lambda^2) + \Theta(\Delta x^4)$$

$$\boxed{\begin{gathered} \begin{cases} u(x_i,\tau_{n+1}) = \lambda\, u(x_{i-1},\tau_n) + (1-2\lambda)\,u(x_i,\tau_n) + \lambda\, u(x_{i+1},\tau_n) + \Delta\tau\cdot\tau(x_i,\tau_n) \\ \hat{u}_{i,n+1} = \lambda\,\hat{u}_{i-1,n} + (1-2\lambda)\,\hat{u}_{i,n} + \lambda\,\hat{u}_{i+1,n} \end{cases} \\ \begin{cases} \hat{u}_{i,0} = u(x_i,\tau_0) = f(x_i) \quad ; \; i = 0,\ldots,M \\ \begin{cases} \hat{u}_{0,n+1} = u(x_0,\tau_{n+1}) = g_0(\tau_{n+1}) \quad ; \; n = 0,\ldots,N-1 \\ \hat{u}_{M,n+1} = u(x_M,\tau_{n+1}) = g_1(\tau_{n+1}) \quad ; \; n = 0,\ldots,N-1 \end{cases} \end{cases} \end{gathered}}$$

Convergencia

Sea: $W_{i,n} = u(x_i, t_n) - \hat{u}_{i,n} \equiv$ error global de truncamiento

$$\boxed{\begin{array}{l} W_{i,n+1} = \lambda W_{i-1,n} + (1-2\lambda) W_{i,n} + \lambda W_{i+1,n} + \Delta\tau\, \tau(x_i, t_n) \\ \left\{\begin{array}{ll} W_{i,0} = 0 & ; i = 0, \dots, M \\ \left\{\begin{array}{ll} W_{0,n+1} = 0 & ; n = 0, \dots, N-1 \\ W_{M,n+1} = 0 & ; n = 0, \dots, N-1 \end{array}\right. \end{array}\right. \end{array}}$$

Si $0 < \lambda \leq 1/2 \Rightarrow 0 \leq (1-2\lambda) < 1$, luego $\left\{\begin{array}{l} \lambda \geq 0 \\ (1-2\lambda) \geq 0 \end{array}\right.$

Por tanto:

$$|W_{i,n+1}| \leq \lambda |W_{i-1,n}| + (1-2\lambda) |W_{i,n}| + \lambda |W_{i+1,n}| + |\Delta\tau|\, |\tau(x_i, \tau_n)|$$

Sea:

$$W_{MAX}(n) = \max_{\substack{1 \leq i \leq M-1 \\ 0 \leq j \leq n}} |W_{i,j}| \quad ; \quad \tau_{MAX}(n) = \max_{\substack{1 \leq i \leq M-1 \\ 0 \leq j \leq n}} |\tau(x_i, \tau_n)|$$

Entonces:

$$|W_{i,n+1}| \leq [\lambda + (1-2\lambda) + \lambda]\, W_{MAX}(n) + |\Delta\tau|\, \tau_{MAX}(n) \quad ; \forall i$$

Luego

$$\boxed{W_{MAX}(n+1) \leq W_{MAX}(n) + |\Delta\tau|\, \tau_{MAX}(n)}$$

Por tanto:

$$\begin{array}{l} W_{MAX}(0) = 0 \\ W_{MAX}(1) \leq W_{MAX}(0) + |\Delta\tau|\, \tau_{MAX}(0) \\ W_{MAX}(2) \leq W_{MAX}(1) + |\Delta\tau|\, \tau_{MAX}(1) \\ \dots \\ W_{MAX}(n+1) \leq W_{MAX}(n) + |\Delta\tau|\, \tau_{MAX}(n) \end{array}$$

$$\Rightarrow \quad W_{MAX}(n+1) \leq |\Delta\tau| \sum_{j=0}^{n} \tau_{MAX}(j) \leq |\Delta\tau| (n+1)\, \tau_{MAX}(n)$$

Y Finalmente:

$$W_{MAX}(n+1) \leq |\tau_{n+1} - \tau_0| \cdot \max_{\substack{1 \leq i \leq M-1 \\ 0 \leq j \leq n}} \left| \frac{(6\lambda - 1)}{12} \Delta x^2 u_{\tau\tau} \Big|_{\substack{x = x_i \\ \tau = \tau_j}} + \Theta(\Delta x^4 + \lambda^2 \Delta x^4) \right|$$

$$\boxed{\begin{array}{ll} 0 < \lambda \leq 1/2 \Rightarrow & W_{MAX}(n+1) = \Theta(\Delta x^2) \\ \lambda = 1/6 \Rightarrow & W_{MAX}(n+1) = \Theta(\Delta x^4) \end{array}} \Rightarrow \underline{\text{ES CONVERGENTE}}$$

Análisis de Estabilidad de Von Neumann

IDEA: $\begin{cases} \text{BUSCAR SOLUCIONES TIPO} & \hat{u}_{i,n} = \psi(\tau_n) e^{j\beta x_i} \; ; \; j=\sqrt{-1} \\ \text{OBLIGAR A QUE} & \begin{cases} |\xi| \leq 1 \\ \text{con } \xi = \dfrac{\psi(\tau_n+\Delta\tau)}{\psi(\tau_n)} \equiv \text{coeficiente de amplificación} \end{cases} \end{cases}$

$$\boxed{\hat{u}_{i,n+1} = \lambda \hat{u}_{i-1,n} + (1-2\lambda)\hat{u}_{i,n} + \lambda \hat{u}_{i+1,n}}$$

$$\Downarrow$$

$$\psi(\tau_n+\Delta\tau) e^{j\beta x_i} = \lambda \psi(\tau_n) e^{j\beta(x_i-\Delta x)} + (1-2\lambda)\psi(\tau_n) e^{j\beta x_i} + \lambda \psi(\tau_n) e^{j\beta(x_i+\Delta x)}$$

$$\Downarrow$$

$$\xi = \frac{\psi(\tau_n+\Delta\tau)}{\psi(\tau_n)} = \lambda e^{-j\beta\Delta x} + (1-2\lambda) + \lambda e^{j\beta\Delta x}$$

$$= 1 + \lambda \left[e^{-j\beta\Delta x} + e^{j\beta\Delta x} - 2 \right]$$

$$= 1 + \lambda \left[\left(\cos(\beta\Delta x) - \cancel{j\,\mathrm{sen}(\beta\Delta x)}\right) + \left(\cos(\beta\Delta x) + \cancel{j\,\mathrm{sen}(\beta\Delta x)}\right) - 2 \right]$$

$$= 1 + 2\lambda \left[\cos(\beta\Delta x) - 1 \right]$$

$$= 1 - 4\lambda\, \mathrm{sen}^2\left(\frac{\beta\Delta x}{2}\right)$$

$$|\xi| \leq 1 \Rightarrow \left| 1 - 4\lambda\, \mathrm{sen}^2\left(\frac{\beta\Delta x}{2}\right) \right| \leq 1 \quad \forall \beta \Rightarrow |1-4\lambda| \leq 1$$

$$\Rightarrow -1 \leq 1-4\lambda \leq 1 \Rightarrow -2 \leq -4\lambda \leq 0$$

$$\Rightarrow \left. \begin{array}{l} 0 \leq \lambda \leq 1/2 \\ \lambda = 0 \text{ no tiene sentido} \end{array} \right\} \Rightarrow \boxed{0 < \lambda \leq 1/2}$$

¡CONDICIONALMENTE ESTABLE!

MÉTODO IMPLÍCITO

$$\left[u_\tau - u_{xx}\right]\Big|_{\substack{x=x_i\\ \tau=\tau_{n+1}}} = 0$$

$$\begin{cases} u_\tau\Big|_{\substack{x=x_i\\ \tau=\tau_{n+1}}} = \dfrac{u(x_i,\tau_{n+1}) - u(x_i,\tau_n)}{\Delta\tau} + u_{\tau\tau}\dfrac{\Delta\tau}{2} + \Theta(\Delta\tau^2) \\ u_{xx}\Big|_{\substack{x=x_i\\ \tau=\tau_{n+1}}} = \dfrac{u(x_{i+1},\tau_{n+1}) - 2u(x_i,\tau_{n+1}) + u(x_{i-1},\tau_{n+1})}{\Delta x^2} - u_{xxxx}\dfrac{\Delta x^2}{12} + \Theta(\Delta x^4) \end{cases}$$

$$\boxed{\begin{gathered} \frac{u(x_i,\tau_{n+1}) - u(x_i,\tau_n)}{\Delta\tau} - \frac{u(x_{i+1},\tau_{n+1}) - 2u(x_i,\tau_{n+1}) + u(x_{i-1},\tau_{n+1})}{\Delta x^2} = \tau(x_i,\tau_{n+1}) \\ \tau(x_i,\tau_{n+1}) = \left[-u_{\tau\tau}\frac{\Delta\tau}{2} - u_{xxxx}\frac{\Delta x^2}{12}\right]\Big|_{\substack{x=x_i\\ \tau=\tau_{n+1}}} - \Theta(\Delta\tau^2) + \Theta(\Delta x^4) \end{gathered}}$$

$$\begin{cases} \text{Se define:} \quad \lambda = \dfrac{\Delta\tau}{\Delta x^2} \\ u_\tau = u_{xx} \Rightarrow u_{\tau\tau} = u_{xx\tau} = u_{\tau xx} = (u_\tau)_{xx} = u_{xxxx} \end{cases}$$

Luego:

$$\tau(x_i,\tau_{n+1}) = -\frac{u_{\tau\tau}}{12}\Big|_{\substack{x=x_i\\ \tau=\tau_{n+1}}} \cdot \Delta x^2 (6\lambda+1) + \Theta(\Delta x^4\lambda^2) + \Theta(\Delta x^4)$$

$$\boxed{\begin{gathered} -\lambda u(x_{i-1},\tau_{n+1}) + (1+2\lambda)u(x_i,\tau_{n+1}) - \lambda u(x_{i+1},\tau_{n+1}) = u(x_i,\tau_n) + \Delta\tau\,\tau(x_i,\tau_{n+1}) \\ -\lambda\hat{u}_{i-1,n+1} + (1+2\lambda)\hat{u}_{i,n+1} - \lambda\hat{u}_{i+1,n+1} = \hat{u}_{i,n} \\ \begin{cases} \hat{u}_{i,0} = u(x_i,\tau_0) = f(x_i) \quad ; i = 0,\dots,M \\ \begin{cases} \hat{u}_{0,n+1} = u(x_0,\tau_{n+1}) = g_0(\tau_{n+1}) \quad ; \quad n = 0,\dots,N-1 \\ \hat{u}_{M,n+1} = u(x_M,\tau_{n+1}) = g_1(\tau_{n+1}) \quad ; \quad n = 0,\dots,N-1 \end{cases} \end{cases} \end{gathered}}$$

Luego:

$$\begin{bmatrix} 1 & & & & & \\ -\lambda & (1+2\lambda) & -\lambda & & & \\ & -\lambda & (1+2\lambda) & -\lambda & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\lambda & (1+2\lambda) & -\lambda \\ & & & & & 1 \end{bmatrix} \begin{Bmatrix} \hat{u}_{0,n+1} \\ \hat{u}_{1,n+1} \\ \hat{u}_{2,n+1} \\ \vdots \\ \hat{u}_{M-1,n+1} \\ \hat{u}_{M,n+1} \end{Bmatrix} = \begin{Bmatrix} g_0(\tau_{n+1}) \\ \hat{u}_{1,n} \\ \hat{u}_{2,n} \\ \vdots \\ \hat{u}_{M-1,n} \\ g_1(\tau_{n+1}) \end{Bmatrix}$$

y $\hat{u}_{i,0} = f(x_i)$; $i = 0, -, M$

Análisis de Estabilidad de Von Neumann

$$\hat{u}_{i,n} = \psi(\tau_n)\, e^{j\beta x_i}$$

$$\boxed{-\lambda \hat{u}_{i-1,n+1} + (1+2\lambda)\,\hat{u}_{i,n+1} - \lambda\, \hat{u}_{i+1,n+1} = \hat{u}_{i,n}}$$

$$\Downarrow$$

$$-\lambda \psi(\tau_{n+1}) e^{j\beta(x_i-\Delta x)} + (1+2\lambda)\psi(\tau_{n+1}) e^{j\beta x_i} - \lambda \psi(\tau_{n+1}) e^{j\beta(x_i+\Delta x)} = \psi(\tau_n) e^{j\beta x_i}$$

$$\Downarrow$$

$$\xi = \frac{\psi(\tau_{n+1})}{\psi(\tau_n)} = \frac{1}{-\lambda e^{-j\beta\Delta x} + (1+2\lambda) - \lambda e^{j\beta\Delta x}} =$$

$$= \frac{1}{1+\lambda\left[2 - e^{-j\beta\Delta x} - e^{j\beta\Delta x}\right]} = \frac{1}{1+\lambda\left[2 - 2\cos\beta\Delta x\right]}$$

$$= \frac{1}{1+2\lambda(1-\cos\beta\Delta x)} = \frac{1}{1+4\lambda \operatorname{sen}^2\left(\frac{\beta\Delta x}{2}\right)}$$

Se observa que $\boxed{\lambda > 0} \Rightarrow \xi \leq 1 \quad \forall \beta$

¡INCONDICIONALMENTE ESTABLE!

Metodo de Crank-Nicolson

$$\left[u_\tau - u_{xx}\right]\Big|_{\substack{x=x_i \\ \tau=\tau_n+\frac{\Delta\tau}{2}}} = 0$$

$$\begin{cases} u_\tau\Big|_{\substack{x=x_i \\ \tau=\tau_n+\frac{\Delta\tau}{2}}} = \dfrac{u(x_i,\tau_{n+1})-u(x_i,\tau_n)}{\Delta\tau} - u_{\tau\tau\tau}\dfrac{\Delta\tau^2}{24} + \Theta(\Delta\tau^4) \\ u_{xx}\Big|_{\substack{x=x_i \\ \tau=\tau_n+\frac{\Delta\tau}{2}}} = \varphi\left[\dfrac{u(x_{i+1},\tau_{n+1})-2u(x_i,\tau_{n+1})+u(x_{i-1},\tau_{n+1})}{\Delta x^2}\right] + \end{cases}$$

$$(1-\varphi)\left[\frac{u(x_{i+1},\tau_n)-2u(x_i,\tau_n)+u(x_{i-1},\tau_n)}{\Delta x^2}\right]$$

$$-u_{xx\tau}\left(\frac{2\varphi-1}{2}\right)\Delta\tau - u_{xxxx}\frac{\Delta x^2}{12} + \Theta(\Delta\tau^2+\dots)$$

$$\boxed{\begin{aligned} &\frac{u(x_i,\tau_{n+1})-u(x_i,\tau_n)}{\Delta\tau} - \varphi\left[\frac{u(x_{i+1},\tau_{n+1})-2u(x_i,\tau_{n+1})+u(x_{i-1},\tau_{n+1})}{\Delta x^2}\right] \\ &\quad -(1-\varphi)\left[\frac{u(x_{i+1},\tau_n)-2u(x_i,\tau_n)+u(x_{i-1},\tau_n)}{\Delta x^2}\right] = \tau\left(x_i,\tau_n+\frac{\Delta\tau}{2}\right) \\ &\tau\left(x_i,t_n+\frac{\Delta\tau}{2}\right) = \left[-u_{xx\tau}\frac{2\varphi-1}{2}\Delta\tau - u_{xxxx}\frac{\Delta x^2}{12} + u_{\tau\tau\tau}\frac{\Delta\tau^2}{24}\right]\Bigg|_{\substack{x=x_i \\ \tau=\tau_n+\frac{\Delta\tau}{2}}} + \Theta(\Delta\tau^2+\dots) \end{aligned}}$$

Se define $\lambda = \dfrac{\Delta\tau}{\Delta x^2}$

$$u_\tau = u_{xx} \Rightarrow \begin{cases} u_{xx\tau} = (u_\tau)_{xx} = u_{xxxx} \\ u_{\tau\tau\tau} = u_{xxxxxx} \end{cases}$$

Luego:

$$\tau\left(x_i, t_n+\frac{\Delta\tau}{2}\right) = -\left[u_{xxxx}\frac{\Delta x^2}{12}\left((2\varphi-1)6\lambda+1\right)\right]\Bigg|_{\substack{x=x_i \\ \tau=\tau_n+\Delta\tau/2}} + \Theta(\Delta t^2+\dots)$$

$$\begin{cases} \text{CRANK-NICOLSON:} \quad \varphi = 1/2 \Rightarrow \tau \equiv \Theta(\Delta x^2+\Delta\tau^2) \\ \text{FORSYTHE-WASOW:} \begin{cases} \varphi = \dfrac{6\lambda-1}{12\lambda} \Rightarrow \tau \equiv \Theta(\Delta x^4) \\ \lambda = \dfrac{1}{\sqrt{20}} \Rightarrow \tau \equiv \Theta(\Delta x^6) \end{cases} \end{cases}$$

$\varphi = 1/2 \Rightarrow$

$$-\lambda u(x_{i-1}, \tau_{n+1}) + 2(1+\lambda) u(x_i, \tau_{n+1}) - \lambda u(x_{i+1}, \tau_{n+1}) =$$
$$= \lambda u(x_{i-1}, \tau_n) + 2(1-\lambda) u(x_i, \tau_n) + \lambda u(x_{i+1}, \tau_n) +$$
$$+ \Delta\tau\, \sigma\left(x_i, \tau_n + \frac{\Delta\tau}{2}\right)$$

$$-\lambda \hat{u}_{i-1,n+1} + 2(1+\lambda)\hat{u}_{i,n+1} - \lambda \hat{u}_{i+1,n+1} =$$
$$= \lambda \hat{u}_{i-1,n} + 2(1-\lambda)\hat{u}_{i,n} + \lambda \hat{u}_{i+1,n}$$

$$\begin{cases} \hat{u}_{i,0} = u(x_i, t_0) = f(x_i) & ; i = 0, \ldots, M \\ \begin{cases} \hat{u}_{0,n+1} = u(x_0, \tau_{n+1}) = g_0(\tau_{n+1}) & ; n = 0, \ldots, N-1 \\ \hat{u}_{M,n+1} = u(x_M, \tau_{n+1}) = g_1(\tau_{n+1}) & ; n = 0, \ldots, N-1 \end{cases} \end{cases}$$

Luego:

$$\begin{bmatrix} 1 & & & & & \\ -\lambda & 2(1+\lambda) & -\lambda & & & \\ & -\lambda & 2(1+\lambda) & -\lambda & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\lambda & 2(1+\lambda) & -\lambda \\ & & & & & 1 \end{bmatrix} \begin{Bmatrix} \hat{u}_{0,n+1} \\ \hat{u}_{1,n+1} \\ \hat{u}_{2,n+1} \\ \vdots \\ \hat{u}_{M-1,n+1} \\ \hat{u}_{M,n+1} \end{Bmatrix} = \begin{bmatrix} 0 & & & & & \\ \lambda & 2(1-\lambda) & \lambda & & & \\ & \lambda & 2(1-\lambda) & \lambda & & \\ & & \ddots & \ddots & \ddots & \\ & & & \lambda & 2(1-\lambda) & \lambda \\ & & & & & 0 \end{bmatrix} \begin{Bmatrix} \hat{u}_{0,n} \\ \hat{u}_{1,n} \\ \hat{u}_{2,n} \\ \vdots \\ \hat{u}_{M-1,n} \\ \hat{u}_{M,n} \end{Bmatrix} +$$

$$+ \begin{Bmatrix} g_0(\tau_{n+1}) \\ 0 \\ \vdots \\ 0 \\ g_1(\tau_{n+1}) \end{Bmatrix}$$

y $\hat{u}_{i,0} = f(x_i) \quad ; i = 0, \ldots, M$

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SE PUEDE DEMOSTRAR QUE ES ¡INCONDICIONALMENTE ESTABLE!

ATENCIÓN

1) HAY MÉTODOS INCONDICIONALMENTE INESTABLES

Ej: Método de Richardson

$$\left[u_\tau - u_{xx}\right]\Big|_{\substack{x=x_i\\ \tau=\tau_n}} = 0$$

$$\begin{cases} u_\tau\Big|_{\substack{x=x_i\\ \tau=\tau_n}} = \dfrac{u(x_i,\tau_{n+1}) - u(x_i,\tau_{n-1})}{2\Delta\tau} + \mathcal{O}(\Delta\tau^2) \\ u_{xx}\Big|_{\substack{x=x_i\\ \tau=\tau_n}} = \dfrac{u(x_{i+1},\tau_n) - 2u(x_i,\tau_n) + u(x_{i-1},\tau_n)}{\Delta x^2} + \mathcal{O}(\Delta x^2) \end{cases}$$

$$\boxed{\hat{u}_{i,n+1} = \hat{u}_{i,n-1} + 2\lambda\,\hat{u}_{i-1,n} - 4\lambda\,\hat{u}_{i,n} + 2\lambda\,\hat{u}_{i+1,n}}$$

$$\hat{u}_{i,n} = \psi(\tau_n)\,e^{j\rho x_i} \Rightarrow \quad g - \frac{1}{g} = -8\lambda \,\mathrm{sen}^2\frac{\rho\Delta x}{2}$$

Es imposible conseguir que $|g| \leq 1 \quad \forall \rho \Rightarrow$ Incondicionalmente Inestable

2) HAY MÉTODOS CONDICIONALMENTE CONSISTENTES

Ej: Método de DuFort - Frankel

$$\left[u_\tau - u_{xx}\right]\Big|_{\substack{x=x_i\\ \tau=\tau_n}} = 0$$

$$\begin{cases} u_\tau\Big|_{\substack{x=x_i\\ \tau=\tau_n}} = \dfrac{u(x_i,\tau_{n+1}) - u(x_i,\tau_{n-1})}{2\Delta\tau} + \mathcal{O}(\Delta\tau^2) \\ u_{xx}\Big|_{\substack{x=x_i\\ \tau=\tau_n}} = \dfrac{u(x_{i+1},\tau_n) - u(x_i,\tau_{n+1}) - u(x_i,\tau_{n-1}) + u(x_{i-1},\tau_n)}{\Delta x^2} \\ \qquad + u_{\tau\tau}\left(\dfrac{\Delta\tau}{\Delta x}\right)^2 + \mathcal{O}(\Delta x^2 + \ldots) \end{cases}$$

Si $\Delta\tau = \lambda\Delta x^2$; $\lambda = \text{cte}$ y $\Delta x \to 0 \Rightarrow$ Consistente

Si $\Delta\tau = c\,\Delta x$; $c = \text{cte.}$ y $\Delta x \to 0 \Rightarrow$ NO CONSISTENTE (*)

(*) ES CONSISTENTE CON $u_\tau - u_{xx} + c^2 u_{\tau\tau} = 0$