PROBLEMAS ELÍPTICOS 1D

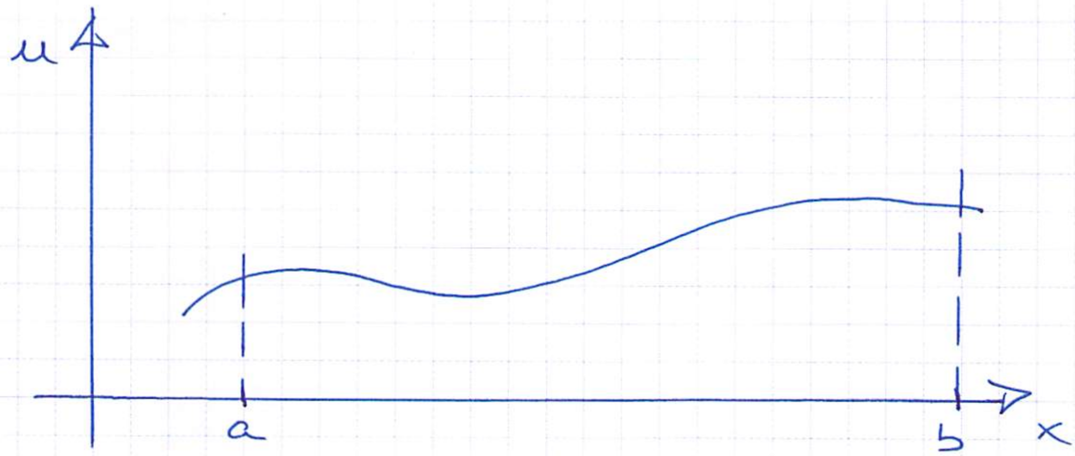
$u_{xx} = \varphi(x) \quad ; x \in (a,b)$ ← EDP

1) Condiciones de contorno tipo DIRICHLET:

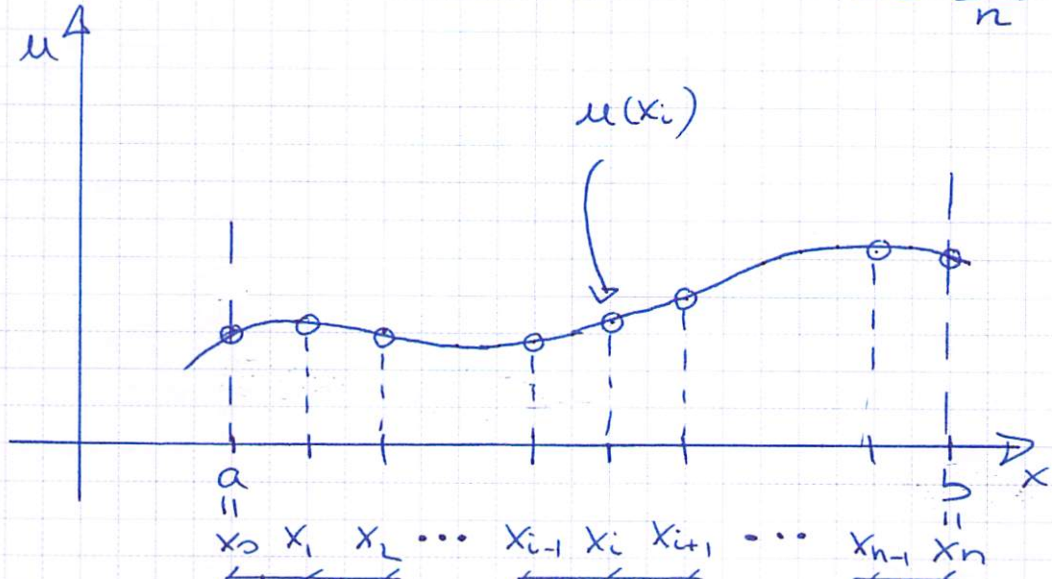
$$\begin{cases} x = a \rightarrow u(x) = u_a \\ x = b \rightarrow u(x) = u_b \end{cases}$$

2) Condiciones de contorno tipo MIXTO (NEUMAN – DIRICHLET):

$$\begin{cases} x = a \rightarrow u_x(x) = u'_a \\ x = b \rightarrow u(x) = u_b \end{cases}$$



Discretización $\quad \hat{u}_i \simeq u(x_i) \qquad h = \frac{b-a}{n}$



1) C.C. tipo DIRICHLET

$$\left[u_{xx} - \varphi(x)\right]\Big|_{x=x_i} = 0 \quad ; i = 1, \dots, n-1$$

$$\to u_{xx}\Big|_{x=x_i} = \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} + \Theta(h^2)$$

$\Longrightarrow$

$$\left\{\begin{array}{ll} \dfrac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} - \varphi(x_i) + \Theta(h^2) = 0 \\ \dfrac{\hat{u}_{i-1} - 2\hat{u}_i + \hat{u}_{i+1}}{h^2} - \varphi(x_i) = 0 \end{array}\right\}$$

para $\underline{i = 1, \dots, n-1}$

$$\text{C.C.}: \left\{\begin{array}{l} u(x_0) = u_a \to \hat{u}_0 = u_a \\ u(x_n) = u_b \to \hat{u}_n = u_b \end{array}\right.$$

$\Longrightarrow$

$$\begin{bmatrix} 1 & 0 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 & \\ & & & 1 & -2 & 1 \\ & & & & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_{n-2} \\ \hat{u}_{n-1} \\ \hat{u}_n \end{Bmatrix} = \begin{Bmatrix} u_a \\ h^2\varphi(x_1) \\ h^2\varphi(x_2) \\ \vdots \\ h^2\varphi(x_{n-2}) \\ h^2\varphi(x_{n-1}) \\ u_b \end{Bmatrix}$$

$\Updownarrow$

$$\boxed{\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_{n-2} \\ \hat{u}_{n-1} \end{Bmatrix} = \begin{Bmatrix} h^2\varphi(x_1) - u_a \\ h^2\varphi(x_2) \\ \vdots \\ h^2\varphi(x_{n-2}) \\ h^2\varphi(x_{n-1}) - u_b \end{Bmatrix}}$$

Además: $\hat{u}_0 = u_a$, $\hat{u}_n = u_b$

2) C.C. tipo MIXTO

2.1.) Sin extensión del dominio (sin "puntos fantasma")

$$\begin{cases} \dfrac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} - \varphi(x_i) + O(h^2) = 0 \\[2ex] \dfrac{\hat{u}_{i-1} - 2\hat{u}_i + \hat{u}_{i+1}}{h^2} - \varphi(x_i) = 0 \end{cases}$$

para $\underline{i = 1, \ldots, n-1}$

C.C:

$$\begin{cases} [u_x - u'_a]\big|_{x=x_0} = 0 \\ \quad \searrow u_x\big|_{x=x_0} = \dfrac{-3u(x_0) + 4u(x_1) - u(x_2)}{2h} + O(h^2) \\ \Rightarrow \begin{cases} \dfrac{-3u(x_0) + 4u(x_1) - u(x_2)}{2h} - u'_a + O(h^2) = 0 \\[2ex] \dfrac{-3\hat{u}_0 + 4\hat{u}_1 - \hat{u}_2}{2h} - u'_a = 0 \end{cases} \\ u(x_n) = u_b \rightarrow \hat{u}_n = u_b \end{cases}$$

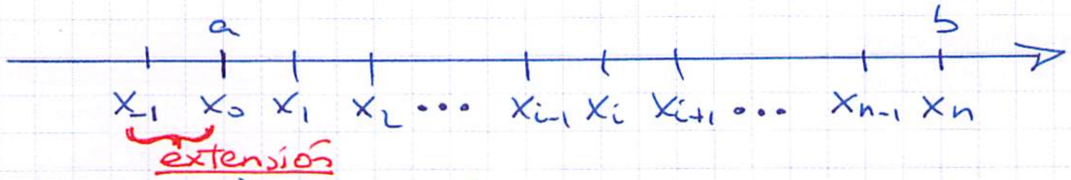
$$\Rightarrow \begin{bmatrix} -3 & 4 & -1 & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_{n-2} \\ \hat{u}_{n-1} \\ \hat{u}_n \end{Bmatrix} = \begin{Bmatrix} 2h\,u'_a \\ h^2\varphi(x_1) \\ h^2\varphi(x_2) \\ \vdots \\ h^2\varphi(x_{n-2}) \\ h^2\varphi(x_{n-1}) \\ u_b \end{Bmatrix}$$

$$\Updownarrow$$

$$\boxed{\begin{bmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix} \begin{Bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_{n-2} \\ \hat{u}_{n-1} \end{Bmatrix} = \begin{Bmatrix} h u'_a + h^2\varphi(x_1)/2 \\ h^2\varphi(x_1) \\ h^2\varphi(x_2) \\ \vdots \\ h^2\varphi(x_{n-2}) \\ h^2\varphi(x_{n-1}) - u_b \end{Bmatrix}}$$

Además: $\hat{u}_n = u_b$

2.2) Con extensión del dominio (con "puntos fantasma")



$$\left\{\begin{array}{l} \dfrac{u(x_{i-1})-2u(x_i)+u(x_{i+1})}{h^2} - \varphi(x_i) + O(h^2) = 0 \\[2ex] \dfrac{\hat{u}_{i-1} - 2\hat{u}_i + \hat{u}_{i+1}}{h^2} - \varphi(x_i) = 0 \end{array}\right\}$$

¡ojo!

para $\underline{i = 0, \ldots, n-1}$

c.c.

$$\left\{\begin{array}{l} [u_x - u'_a]\big|_{x=x_0} = 0 \\[1ex] \qquad \longrightarrow u_x\big|_{x=x_0} = \dfrac{u(x_1)-u(x_{-1})}{2h} + O(h^2) \\[2ex] \Longrightarrow \left\{\begin{array}{l} \dfrac{u(x_1)-u(x_{-1})}{2h} - u'_a + O(h^2) = 0 \\[2ex] \dfrac{\hat{u}_1 - \hat{u}_{-1}}{2h} - u'_a = 0 \end{array}\right. \\[2ex] u(x_n) = u_b \rightarrow \hat{u}_n = u_b \end{array}\right.$$

$\Longrightarrow$

$$\begin{bmatrix} -1 & 0 & 1 & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 & 1 \\ & & & & & & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{u}_{-1} \\ \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_{n-2} \\ \hat{u}_{n-1} \\ \hat{u}_n \end{Bmatrix} = \begin{Bmatrix} 2h u'_a \\ h^2\varphi(x_0) \\ h^2\varphi(x_1) \\ h^2\varphi(x_2) \\ \vdots \\ h^2\varphi(x_{n-2}) \\ h^2\varphi(x_{n-1}) \\ u_b \end{Bmatrix}$$

$\Updownarrow$

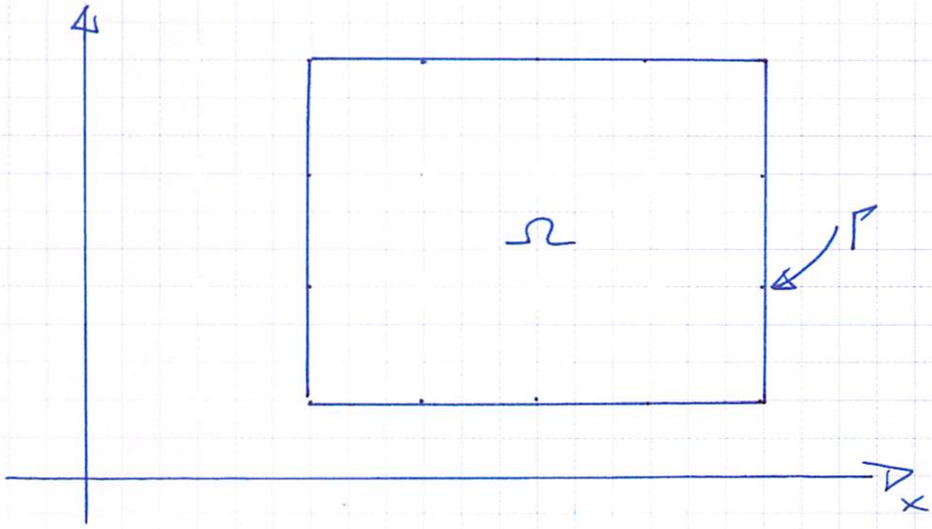
¡ojo!

$$\boxed{\begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{Bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_{n-2} \\ \hat{u}_{n-1} \end{Bmatrix} = \begin{Bmatrix} h u'_a + h^2\varphi(x_0)/2 \\ h^2\varphi(x_1) \\ h^2\varphi(x_2) \\ \vdots \\ h^2\varphi(x_{n-2}) \\ h^2\varphi(x_{n-1}) - u_b \end{Bmatrix}}$$

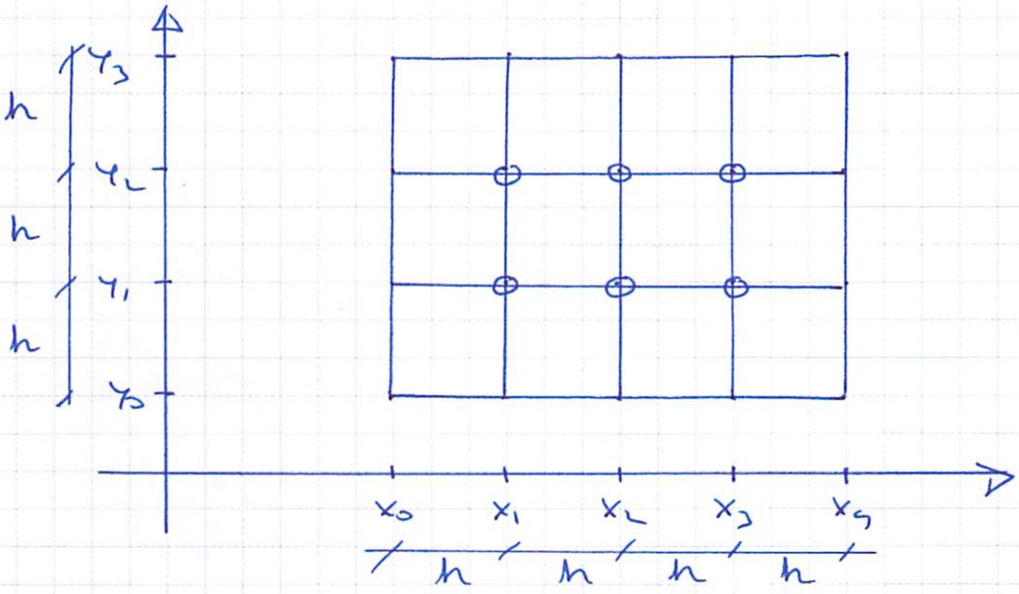
Además $\hat{u}_n = u_b$

Problemas Elípticos 2D

$$\xi: \begin{cases} u_{xx} + u_{yy} = f(x,y) & (x,y) \in \Omega \\ u(x,y) = g(x,y) & (x,y) \in \Gamma \end{cases}$$



Discretización: $\hat{u}_{ij} \simeq u(x_i, y_j)$



Nota:

- En este caso $\Delta x = \Delta y = h$ (no tiene por qué ser así)
- Incógnitas $\begin{cases} \hat{u}_{12}, \hat{u}_{22}, \hat{u}_{32} \\ \hat{u}_{11}, \hat{u}_{21}, \hat{u}_{31} \end{cases}$

$$\left[\underbrace{(u_{xx}+u_{yy})}-\varphi(x,y)\right]\Bigg|_{\substack{x=x_i\\ y=y_j}} = 0 \qquad ;\ i=1,\dots,3 \quad ;\ j=1,2$$

$$(u_{xx}+u_{yy})\Big|_{\substack{x=x_i\\ y=y_j}} = \frac{u(x_{i-1},y_j)-2u(x_i,y_j)+u(x_{i+1},y_j)}{h^2}+O(h^2)$$

$$+\ \frac{u(x_i,y_{j-1})-2u(x_i,y_j)+u(x_i,y_{j+1})}{h^2}+O(h^2)$$

$$\Longrightarrow \left\{\begin{array}{l} \dfrac{u(x_{i-1},y_j)+u(x_i,y_{j-1})-4u(x_i,y_j)+u(x_i,y_{j+1})+u(x_{i+1},y_j)}{h^2}-\varphi(x_i,y_j)+O(h^2)=0 \\[2ex] \dfrac{\hat{u}_{i-1,j}+\hat{u}_{i,j-1}-4\hat{u}_{ij}+\hat{u}_{i,j+1}+\hat{u}_{i+1,j}}{h^2}-\varphi(x_i,y_j)=0 \end{array}\right.$$

para $\begin{cases} i=1,\dots,3 \\ j=1,2 \end{cases}$

C.C: $u(x_i,y_j)=g(x_i,y_j) \longrightarrow \hat{u}_{ij}=g(x_i,y_j)$ en el contorno Γ

$$\Longrightarrow \boxed{\left[\begin{array}{ccc|ccc} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 & 1 & -4 \end{array}\right] \begin{pmatrix} \hat{u}_{12} \\ \hat{u}_{22} \\ \hat{u}_{32} \\ \hat{u}_{11} \\ \hat{u}_{21} \\ \hat{u}_{31} \end{pmatrix} = \begin{pmatrix} h^2\varphi_{12}-g_{02}-g_{13} \\ h^2\varphi_{22}-g_{23} \\ h^2\varphi_{32}-g_{42}-g_{33} \\ h^2\varphi_{11}-g_{01}-g_{10} \\ h^2\varphi_{21}-g_{20} \\ h^2\varphi_{31}-g_{41}-g_{30} \end{pmatrix}}$$

Matriz $\begin{cases} \text{– Simétrica} \\ \text{– Diagonalmente dominante} \end{cases}$

Nota:

Al discretizar el problema 1D con c.c. tipo Dirichlet, se desprecia el error local de truncamiento $\tau_i(h^2)$ tal que:

$$u_{xx}\Big|_{x=x_i} = \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} + \tau_i(h^2),$$

$$\text{con } \tau_i(h^2) = \underbrace{\frac{-u_{xxxx}\big|_{x=x_i} \; h^2}{12}}_{\Theta(h^2)} + \Theta(h^4)$$

Los errores $\Theta(h^4)$ dependen de la derivada sexta y sucesivas de $u(x)$, luego

$$u_{xxxx} = 0 \;\; \forall x \implies \tau_i = 0 \qquad (*)$$

Por tanto, cuando la solución sea un polinomio de orden 3 o inferior el método numérico será <u>teóricamente exacto</u> (salvo errores de redondeo)

(*) Por ser $u_{xx} = \varphi(x)$, cuando $\varphi'' = 0 \iff \varphi(x) = P_1(x)$ el método numérico proporcionará la solución exacta (salvo errores de redondeo).