# Derivación Numérica

$$f'_0 = \frac{f_1 - f_0}{h} - \frac{f''_0}{2} h + \mathcal{O}(h^2) \quad \rightarrow \mathcal{O}(h)$$

$$= \frac{f_0 - f_{-1}}{h} + \frac{f''_0}{2} h + \mathcal{O}(h^2) \quad \rightarrow \mathcal{O}(h)$$

$$= \frac{f_1 - f_{-1}}{2h} - \frac{f'''_0}{6} h^2 + \mathcal{O}(h^4) \quad \rightarrow \mathcal{O}(h^2)$$

$$= \frac{-f_2 + 4f_1 - 3f_0}{2h} + \frac{f'''_0 h^2}{3} + \mathcal{O}(h^3) \quad \rightarrow \mathcal{O}(h^2)$$

$$= \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h} + \frac{f_0^{V} h^4}{30} + \mathcal{O}(h^5) \quad \rightarrow \mathcal{O}(h^4)$$

$$f''_0 = \frac{f_1 - 2f_0 + f_{-1}}{h^2} - \frac{f_0^{IV} h^2}{12} + \mathcal{O}(h^4)$$

$$= \frac{f_2 - 2f_1 + f_0}{h^2} - f'''_0 h + \mathcal{O}(h^2)$$

$$= \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2} + \frac{f_0^{VI} h^4}{90} + \mathcal{O}(h^5) \rightarrow \mathcal{O}(h^4)$$

$$f'''_0 = \frac{f_2 - 2f_1 + 2f_{-1} - f_{-2}}{2h^3} - \frac{f_0^{V} h^2}{4} + \mathcal{O}(h^3) \quad \rightarrow \mathcal{O}(h^2)$$

$$f_0^{IV} = \frac{f_2 - 4f_1 + 6f_0 - 4f_{-1} + f_{-2}}{h^4} - \frac{f_0^{VI} h^2}{6} + \mathcal{O}(h^3) \quad \rightarrow \mathcal{O}(h^2)$$

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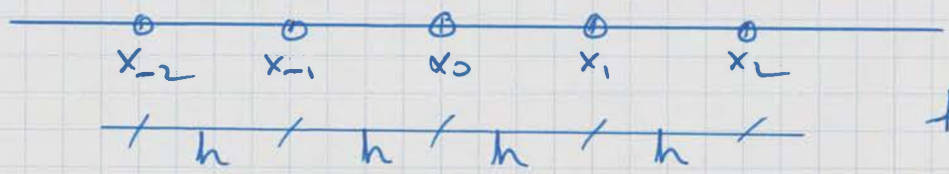
Discretización:




$f_i = f(x_i)$

# Problemas 2D/3D

**2D** : $u(x+h, y+k) = u(x,y) +$

$$+ \sum_{i=1}^{n} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i u(x,y) +$$

$$+ R_n \qquad (*)$$

con $R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} u(x+\xi h, y+\eta k)$

$$\xi, \eta \in [0,1]$$

$$\Rightarrow R_n = O\left((|h|+|k|)^{n+1}\right) \Leftrightarrow \exists M>0 \,/\, |R_n| \leq M(|h|+|k|)^{n+1}$$

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**3D** : $u(x+h, y+k, z+l) = u(x,y,z) +$

$$+ \sum_{i=1}^{n} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^i u(x,y) +$$

$$+ R_n \qquad (*)$$

con $R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^{n+1} u(x+\xi h, y+\eta k, z+l\chi)$

$$\xi, \eta, \chi \in [0,1]$$

$$\Rightarrow R_n = O\left((|h|+|k|+|l|)^{n+1}\right) \Leftrightarrow \exists M>0 \,/\, |R_n| \leq M(|h|+|k|+|l|)^{n+1}$$

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(*) luego:

$$\left.\begin{aligned} &\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)u = h u_x + k u_y \\ &\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 u = h^2 u_{xx} + 2hk u_{xy} + k^2 u_{yy} \\ &\dots \end{aligned}\right\} \Rightarrow$$

$$u(x+h, y+k) = u(x,y) + \left[h u_x(x,y) + k u_y(x,y)\right] +$$

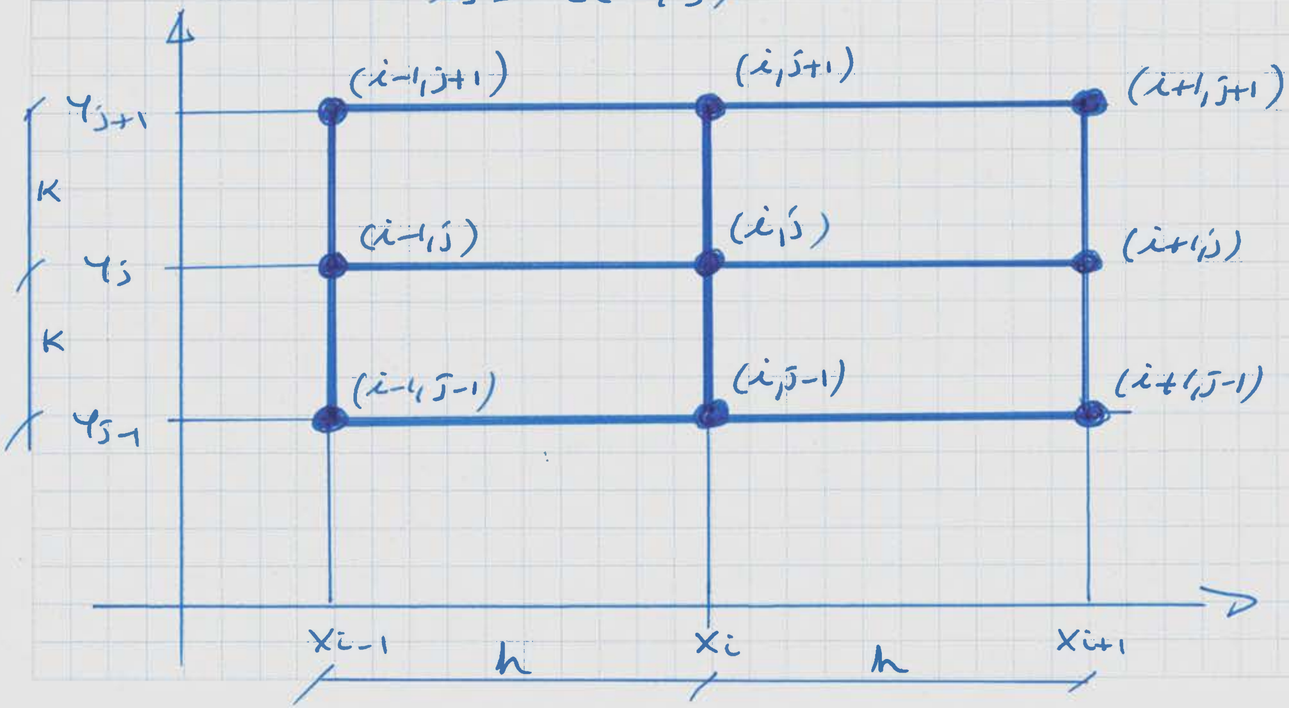
$$+ \left[h^2 u_{xx}(x,y) + 2hk\, u_{xy}(x,y) + k^2 u_{yy}(x,y)\right] +$$

$$+ \dots$$

21)

$$
\begin{Bmatrix} u_{i+1,j} \\ u_{i-1,j} \\ u_{i,j+1} \\ u_{i,j-1} \\ u_{i+1,j+1} \\ u_{i+1,j-1} \\ u_{i-1,j+1} \\ u_{i-1,j-1} \end{Bmatrix}
=
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 2 & 1 & 1 & 3 & 3 & 1 & 1 & 4 & 6 & 4 & 1 \\
1 & 1 & -1 & 1 & -2 & 1 & 1 & -3 & 3 & -1 & 1 & -4 & 6 & -4 & 1 \\
1 & -1 & 1 & 1 & -2 & 1 & -1 & 3 & -3 & 1 & 1 & -4 & 6 & -4 & 1 \\
1 & -1 & -1 & 1 & 2 & 1 & -1 & -3 & -3 & -1 & 1 & 4 & 6 & 4 & 1
\end{bmatrix}
\begin{Bmatrix} u_{ij} \\ u_x h \\ u_y k \\ u_{xx} h^2/2 \\ u_{xy} hk/2 \\ u_{yy} k^2/2 \\ u_{xxx} h^3/6 \\ u_{xxy} h^2k/6 \\ u_{xyy} hk^2/6 \\ u_{yyy} k^3/6 \\ u_{xxxx} h^4/24 \\ u_{xxxy} h^3k/24 \\ u_{xxyy} h^2k^2/24 \\ u_{xyyy} hk^3/24 \\ u_{yyyy} k^4/24 \end{Bmatrix}
+
\begin{Bmatrix} O(h^5) \\ O(h^5) \\ O(k^5) \\ O(k^5) \\ * \\ * \\ * \\ * \end{Bmatrix}
$$

donde $u_{ij} = u(x_i, y_j)$




$$(*) = O\left(h^5 + h^4k + h^3k^2 + h^2k^3 + hk^4 + k^5\right)$$

## 2D con $h=k$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{4h^2}\begin{matrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{matrix} + O(h^2) = \frac{1}{144h^2}\begin{matrix} -1 & 8 & 0 & -8 & 1 \\ 8 & -64 & 0 & 64 & -8 \\ 0 & 0 & 0 & 0 & 0 \\ -8 & 64 & 0 & -64 & 8 \\ 1 & -8 & 0 & 8 & -1 \end{matrix} + O(h^4)$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{h^2}\begin{matrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{matrix} + O(h^2) = \frac{1}{12h^2}\begin{matrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 \\ -1 & 16 & -60 & 16 & -1 \\ 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{matrix} + O(h^4)$$

$$\Delta^2 u = \Delta(\Delta u) = \frac{1}{h^4}\begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -8 & 2 & 0 \\ 1 & -8 & 20 & -8 & 1 \\ 0 & 2 & -8 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{matrix} + O(h^2) = \frac{1}{6h^4}\begin{matrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 14 & -1 & 0 & 0 \\ 0 & -1 & 20 & -77 & 20 & -1 & 0 \\ -1 & 14 & -77 & 184 & -77 & 14 & -1 \\ 0 & -1 & 20 & -77 & 20 & -1 & 0 \\ 0 & 0 & -1 & 14 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{matrix} + O(h^4)$$