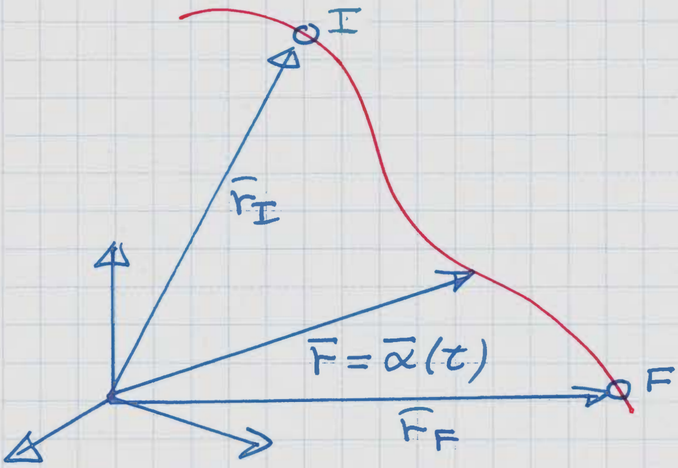
FMC - P4 - E1



Datos: $\bar{r}_I, \bar{r}_F$. // τ_I, τ_F arbitrarios.

$$S = \int_{\tau_I}^{\tau_F} |\bar{\alpha}'(\tau)|\, d\tau$$

$\Downarrow$

$$\boxed{\begin{array}{l} S = \displaystyle\int_{\tau_I}^{\tau_F} \left[L(\tau, \bar{r}, \bar{p}) \Big|_{\substack{\bar{r} = \bar{\alpha}(\tau) \\ \bar{p} = \bar{\alpha}'(\tau)}} \right] d\tau \\ L(\tau, \bar{r}, \bar{p}) = |\bar{p}| = (\bar{p}^T \bar{p})^{1/2} \end{array}}$$

Planteamiento variacional:

$$\boxed{\begin{array}{l} \text{Hallar } \bar{f}(\tau) = \bar{\alpha}(\tau) \text{ que minimice} \\ \mathcal{J}[\bar{f}(\tau)] = \displaystyle\int_{\tau_I}^{\tau_F} \left[L(\tau, \bar{r}, \bar{p}) \Big|_{\substack{\bar{r} = \bar{f}(\tau) \\ \bar{p} = \bar{f}'(\tau)}} \right] d\tau \\ \text{con } L(\tau, \bar{r}, \bar{p}) = |\bar{p}| = (\bar{p}^T \bar{p})^{1/2} \\ \text{verificando: } \bar{f}(\tau_I) = \bar{r}_I, \quad \bar{f}(\tau_F) = \bar{r}_F \end{array}}$$

Ecuaciones de Euler-Lagrange: (Solución $\Rightarrow \bar{f}(\tau) = \bar{\alpha}(\tau)$)

$$\frac{\partial L}{\partial \bar{r}} \Big|_{\substack{\bar{r} = \bar{\alpha}(\tau) \\ \bar{p} = \bar{\alpha}'(\tau)}} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \bar{p}} \Big|_{\substack{\bar{r} = \bar{\alpha}(\tau) \\ \bar{p} = \bar{\alpha}'(\tau)}} \right) = 0$$

donde
$$\begin{cases} \dfrac{\partial L}{\partial \bar{r}} = \bar{0}^T \\ \dfrac{\partial L}{\partial \bar{p}} = \cancel{\dfrac{1}{2}} (\bar{p}^T \bar{p})^{-1/2} \cancel{2} \bar{p}^T = \dfrac{1}{|\bar{p}|} \bar{p}^T \end{cases}$$

luego:
$$\bar{0}^T - \frac{d}{d\tau}\left(\frac{(\bar{\alpha}'(\tau))^T}{|\bar{\alpha}'(\tau)|} \right) = \bar{0}^T$$

$$\Longleftrightarrow \quad \boxed{\frac{d}{d\tau}\left(\frac{\bar{\alpha}'(\tau)}{|\bar{\alpha}'(\tau)|} \right) = \bar{0}}$$

Integración :

$$\left\{\begin{array}{l} \dfrac{d}{d\tau}\left(\dfrac{\bar\alpha'(\tau)}{|\bar\alpha'(\tau)|}\right)=\bar 0 \\[2ex] \bar\alpha(\tau_I)=\bar r_I\,,\ \bar\alpha(\tau_F)=\bar r_F \end{array}\right\} \Longrightarrow$$

$$\frac{\bar\alpha'(\tau)}{|\bar\alpha'(\tau)|} = \bar c\,, \quad \text{con } |\bar c| = 1 \text{ pues } \left|\frac{\bar\alpha'(\tau)}{|\bar\alpha'(\tau)|}\right| = 1$$

Sea $|\bar\alpha'(\tau)| = g(\tau) \Longrightarrow \bar\alpha'(\tau) = g(\tau)\,\bar c$

$$\left\{\begin{array}{l} d\bar\alpha(\tau) = g(t)\,\bar c\, d\tau \\[1ex] \bar\alpha(\tau) - \bar\alpha(\tau_I) = \left(\displaystyle\int_{\tau_I}^{\tau} g(\tau)\,d\tau\right)\bar c \end{array}\right.$$

$$\Longrightarrow \bar\alpha(\tau) = \underbrace{\bar\alpha(\tau_I)}_{\bar r_I} + \underbrace{\left(\int_{\tau_I}^{\tau} g(\tau)\,d\tau\right)}_{\eta(\tau)}\bar c\,, \quad \text{con } \eta(\tau_I) = 0$$

Luego $\bar r_F = \bar\alpha(\tau_F) = \bar r_I + \eta(\tau_F)\,\bar c \Longrightarrow \left\{\begin{array}{l} \bar c = \dfrac{\bar r_F - \bar r_I}{|\bar r_F - \bar r_I|} \\[2ex] \eta(\tau_F) = |\bar r_F - \bar r_I| \end{array}\right.$

Tomemos $\eta(\tau) = |\bar r_F - \bar r_I|\,\dfrac{(t-\tau_I)}{(\tau_F-\tau_I)}$ (elección más sencilla y conveniente $\rightarrow$ lineal)

$$\Longrightarrow \boxed{\bar\alpha(t) = \bar r_I + (\bar r_F - \bar r_I)\,\frac{\tau-\tau_I}{\tau_F-\tau_I}} \Longleftrightarrow \text{RECTA QUE UNE } \bar r_I, \bar r_F$$

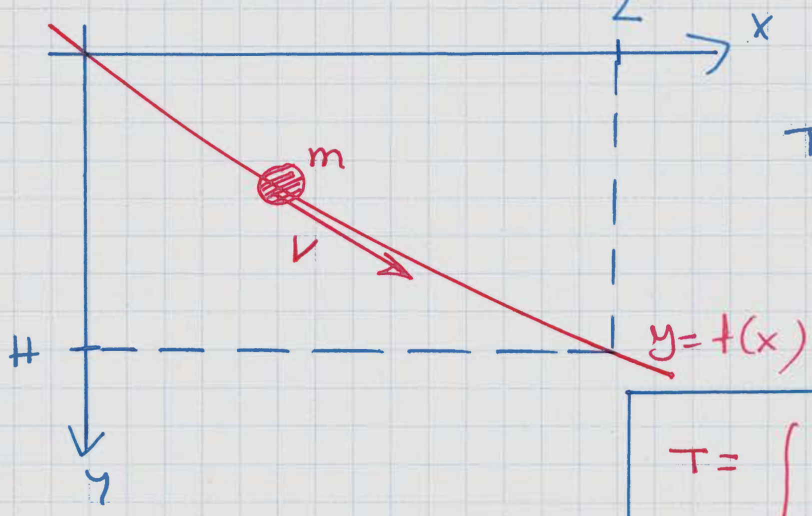
Beltrami (¡OJO!: en el caso de sistemas, es insuficiente)

$$\frac{\partial L}{\partial \tau} = 0 \Rightarrow \left[L - \left(\frac{\partial L}{\partial \bar p}\right)\bar p\right]\Bigg|_{\substack{\bar r = \bar\alpha(\tau)\\ \bar p = \bar\alpha'(\tau)}} = K$$

Pero $\left\{\begin{array}{l} L = |\bar p| \\[1ex] \left(\dfrac{\partial L}{\partial \bar p}\right)\bar p = \left(\dfrac{1}{|\bar p|}\,\bar p^T\right)\bar p = \dfrac{1}{|\bar p|}\,\bar p^T \bar p = \dfrac{|\bar p|^2}{|\bar p|} = |\bar p| \end{array}\right.$

Luego Beltrami $\longrightarrow \left[|\bar p| - |\bar p|\right]\Big|_{\substack{\bar r = \bar\alpha(\tau)\\ \bar p = \bar\alpha'(\tau)}} = 0 = K \Longrightarrow$ No sirve (en este caso)

FMC - P4 - E2



$$T = \int_{x=0}^{x=L} \sqrt{\frac{1+(y')^2}{2gy}}\, dx$$

$$\Downarrow$$

$$\boxed{T = \int_{x=0}^{x=L} \left[L(x,y,p) \Big|_{\substack{y=f(x)\\ p=f'(x)}} \right] dx \qquad L(x,y,p) = \sqrt{\frac{1+p^2}{2gy}}}$$

a) $\boxed{L(x,y,p) = \sqrt{\frac{1+p^2}{2gy}}}$

Problema variacional:

Hallar $f(x) = u(x)$ que minimice

$$J[f(x)] = \int_{x=0}^{x=L} \left[L(x,y,p) \Big|_{\substack{y=f(x)\\ p=f'(x)}} \right] dx$$

verificando: $f(0) = 0$, $f(L) = H$

b) Ecuación de Euler-Lagrange: (Solución => $f(x) = u(x)$)

$$\begin{cases} \dfrac{\partial L}{\partial y} = \dfrac{(1+p^2)^{1/2}}{\sqrt{2g}} \left(-\frac{1}{2} y^{-3/2}\right) \\[2ex] \dfrac{\partial L}{\partial p} = \dfrac{1}{\sqrt{2g}} (y)^{-1/2} \left(\cancel{\tfrac{1}{2}} (1+p^2)^{-1/2} \cancel{2} p \right) \end{cases}$$

$$\frac{\partial L}{\partial y}\Big|_{\substack{y=u(x)\\ p=u'(x)}} - \frac{d}{dx}\left(\frac{\partial L}{\partial p}\Big|_{\substack{y=u(x)\\ p=u'(x)}} \right) = 0$$

$$\Rightarrow \cancel{\frac{1}{\sqrt{2g}}} \left[(1+(u')^2)^{1/2} \left(-\tfrac{1}{2}\right) u^{-3/2} - \frac{d}{dx}\left((1+(u')^2)^{-1/2} u' u^{-1/2} \right) \right] = 0$$

$$-\frac{1}{2}u^{-3/2}\left(1+(u')^2\right)^{1/2}-\frac{d}{dx}\left(u^{-1/2}u'\left(1+(u')^2\right)^{-1/2}\right)=0$$

$$\Rightarrow -\frac{1}{2}u^{-3/2}\left(1+(u')^2\right)^{1/2}-\Big[\left(-\tfrac{1}{2}\right)u^{-3/2}(u')^2\left(1+(u')^2\right)^{-1/2}$$
$$+u^{-1/2}u''\left(1+(u')^2\right)^{-1/2}$$
$$+u^{-1/2}u'\left(-\tfrac{1}{2}\right)\left(1+(u')^2\right)^{-3/2}2u'u''\Big]=$$

$$=-\frac{1}{2}u^{-3/2}\left(1+(u')^2\right)^{-3/2}\Big[\left(1+(u')^2\right)^2-(u')^2\left(1+(u')^2\right)$$
$$+2uu''\left(1+(u')^2\right)-2uu''(u')^2\Big]$$

$$=-\frac{1}{2}u^{-3/2}\left(1+(u')^2\right)^{-3/2}\Big[1+\cancel{(u')^4}+2(u')^2-(u')^2-\cancel{(u')^4}$$
$$+2uu''+\cancel{2uu''(u')^2}-\cancel{2uu''(u')^2}\Big]$$

$$=-\frac{1}{2}u^{-3/2}\left(1+(u')^2\right)^{-3/2}\left[1+(u')^2+2uu''\right]=0$$

Luego: $\boxed{\begin{cases} 1+(u')^2+2uu''=0 \\ \text{con } u(0)=0,\ u(L)=H \end{cases}}$

c) <u>Identidad de Beltrami</u>

$$\frac{\partial L}{\partial x}=0 \Rightarrow \left[L-\left(\frac{\partial L}{\partial p}\right)p\right]\Bigg|_{\substack{y=u(x)\\ p=u'(x)}}=K$$

$$\Rightarrow \frac{1}{\sqrt{2g}}\left\{u^{-1/2}\left(1+(u')^2\right)^{1/2}-\left(u^{-1/2}\left(1+(u')^2\right)^{-1/2}u'\right)u'\right]=K$$

$$\frac{1}{\sqrt{2g}}u^{-1/2}\left(1+(u')^2\right)^{-1/2}\left[\left(1+\cancel{(u')^2}\right)-\cancel{(u')^2}\right]=K$$

$$\left(u\left(1+(u')^2\right)\right)^{-1/2}=K\sqrt{2g}$$

$$\Rightarrow u'=\sqrt{\frac{1}{2gK^2}\,\frac{1}{u}-1} \Rightarrow u=0 \Rightarrow u'=\infty$$

$$\gamma = \frac{1}{2gk^2} \rightarrow$$

$$\boxed{\begin{array}{l} u' = \left(\gamma/u - 1\right)^{1/2} \\ \text{con} \quad u(0) = 0 \rightarrow u'(0) = \infty \\ \qquad u(L) = H \\ \gamma \text{ a determinar.} \end{array}}$$

d) Integración

cambio $u = \gamma \, \mathrm{sen}^2(\theta/2) \Rightarrow \begin{cases} u' = \cancel{2}\gamma \, \mathrm{sen}(\theta/2)\cos(\theta/2)\, \theta'/\cancel{2} \\ (\gamma/u - 1)^{1/2} = \left(\frac{\gamma - u}{u}\right)^{1/2} \end{cases}$

$$= \left(\frac{1 - \mathrm{sen}^2(\theta/2)}{\mathrm{sen}^2(\theta/2)}\right)^{1/2}$$

$$= \frac{\cos(\theta/2)}{\mathrm{sen}(\theta/2)}$$

$$\Rightarrow \gamma \, \mathrm{sen}(\theta/2)\cancel{\cos(\theta/2)}\, \theta' = \frac{\cancel{\cos(\theta/2)}}{\mathrm{sen}(\theta/2)}$$

$$\Rightarrow \theta' = \frac{1}{\gamma} \frac{1}{\mathrm{sen}^2(\theta/2)} \Rightarrow \gamma \, \mathrm{sen}^2(\theta/2)\, d\theta = dx$$

Luego $\displaystyle\int_{x_0}^{x} dx = \gamma \int_{\theta_0}^{\theta} \mathrm{sen}^2(\theta/2)\, d\theta = \gamma \int_{\theta_0}^{\theta} \left(\frac{1 - \cos\theta}{2}\right) d\theta$

$$= \gamma/2 \left(\theta - \mathrm{sen}\,\theta\right)\Big|_{\theta_0}^{\theta}$$

$$x_0 = 0 \Rightarrow u(0) = 0 \Rightarrow \gamma \, \mathrm{sen}^2(\theta_0/2) = 0 \Rightarrow \theta_0 = 0$$

$$\Rightarrow \gamma/2 = R \rightarrow \boxed{x = R\,(\theta - \mathrm{sen}\,\theta)}$$

CICLOIDE

$$u = \gamma \, \mathrm{sen}^2(\theta/2) = \gamma\left(\frac{1 - \cos\theta}{2}\right) = \gamma/2\,(1 - \cos\theta)$$

$$u(x) = y \rightarrow \boxed{y = u(x) = R\,(1 - \cos\theta)}$$

Efectivamente, es una cicloide

e) Ajuste de parámetros ⟹ Hallar R, θ_1 en función de L, H

$$\begin{cases} x = R(\theta - \operatorname{sen}\theta) \\ y = R(1-\cos\theta) \end{cases}$$

$$\theta = 0 \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \quad \text{O.K.}$$

$$\theta = \theta_1 \Rightarrow \begin{cases} x = R(\theta_1 - \operatorname{sen}\theta_1) = L \\ y = R(1-\cos\theta_1) = H \end{cases}$$

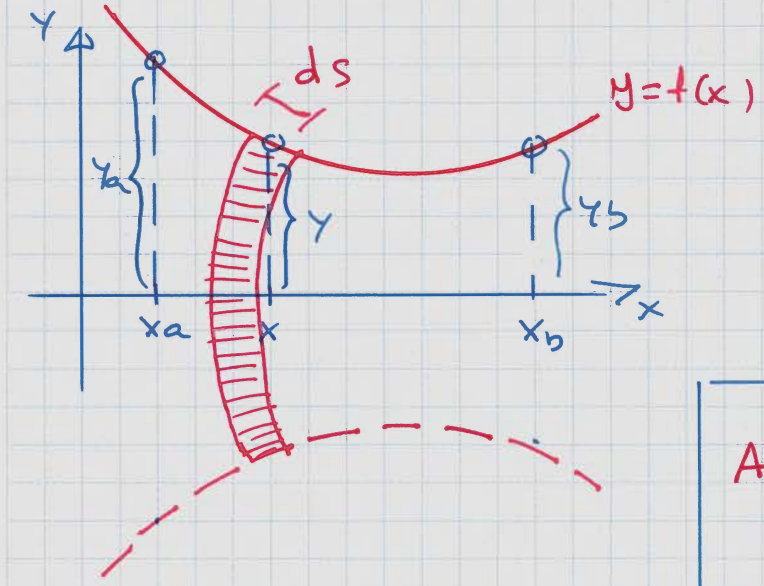
[Solución No trivial]

Casos:

$$\left\{\begin{array}{l} e1) \begin{cases} L = (\pi - 2)\,m \\ H = 2\,m. \end{cases} \rightarrow \begin{cases} R = 2\,m. \\ \theta_1 = \pi/2 \end{cases} \\ e2) \begin{cases} L = \pi\,m \\ H = 2\,m \end{cases} \rightarrow \begin{cases} R = 1\,m. \\ \theta_1 = \pi \end{cases} \\ e3) \begin{cases} L = (3\pi + 2)\,m \\ H = 2\,m \end{cases} \rightarrow \begin{cases} R = 2\,m. \\ \theta_1 = 3\pi/2 \end{cases} \end{array}\right\} (*)$$

(*) En este caso es sencillo obtener R y θ_1 por tanteos.
En general es difícil y hay que emplear métodos numéricos

FMC- P4-E3



$$A = \int_{x_a}^{x_b} (2\pi y)\overbrace{\sqrt{1+(y')^2}\;dx}^{ds}$$

$$\Downarrow$$

$$\boxed{\begin{gathered} A = \int_{x=x_a}^{x=x_b} \left(\mathcal{L}(x,y,p)\Big|_{\substack{y=f(x)\\ p=f'(x)}} \right) dx \\ \mathcal{L}(x,y,p) = 2\pi y\sqrt{1+p^2} \end{gathered}}$$

a) $\boxed{\mathcal{L}(x,y,p) = 2\pi y\sqrt{1+p^2}}$

Problema Variacional:

> Hallar $f(x) = u(x)$ que minimice
>
> $$\mathcal{J}[f(x)] = \int_{x=x_a}^{x=x_b} \left(\mathcal{L}(x,y,p)\Big|_{\substack{y=f(x)\\ p=f'(x)}} \right) dx$$
>
> verificando: $f(x_a) = y_a$, $f(x_b) = y_b$

b) <u>Ecuación de Euler-Lagrange</u>: (Solución $\Rightarrow f(x) = u(x)$)

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial y} = (2\pi)\,(1+p^2)^{1/2} \\[2ex] \dfrac{\partial \mathcal{L}}{\partial p} = (2\pi)\,y\,\cancel{\tfrac{1}{2}}\,(1+p^2)^{-1/2}\,\cancel{2}p \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial y}\Big|_{\substack{y=u(x)\\ p=u'(x)}} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial p}\Big|_{\substack{y=u(x)\\ p=u'(x)}} \right) = 0$$

$$\Rightarrow \cancel{2\pi}\left[(1+(u')^2)^{1/2} - \frac{d}{dx}\left((1+(u')^2)^{-1/2}\,u\,u' \right) \right] = 0$$

$$(1+(u')^2)^{1/2} + \cancel{\tfrac{1}{2}}(1+(u')^2)^{-3/2}\,\cancel{2}u'u''\,u\,u' - (1+(u')^2)^{-1/2}\left((u')^2 + u\,u''\right) = 0$$

$$\left(1+(u')^2\right)^{-3/2}\left[\left(1+(u')^2\right)^2+u(u')^2u''-\left(1+(u')^2\right)\left((u')^2+uu''\right)\right]=0$$

$$\Rightarrow\left(1+(u')^2\right)^{-3/2}\left[1+2(u')^2+\cancel{(u')^4}+\cancel{u(u')^2u''}-(u')^2-uu''-\cancel{(u')^4}-\cancel{u(u')^2u''}\right]$$

$$=\left(1+(u')^2\right)^{-3/2}\left[1+(u')^2-uu''\right]=0$$

Luego:

$$\boxed{\begin{array}{l} 1+(u')^2-uu''=0 \\ \text{con } u(x_a)=y_a,\ u(x_s)=y_s \end{array}}$$

c) Identidad de Beltrami

$$\frac{\partial L}{\partial x}=0 \Rightarrow \left[L-\left(\frac{\partial L}{\partial p}\right)p\right]\Bigg|_{\substack{y=u(x)\\ p=u'(x)}}=K$$

$$\Rightarrow (2\pi)\left[u\left(1+(u')^2\right)^{1/2}-\left(\left(1+(u')^2\right)^{-1/2}uu'\right)u'\right]=K$$

$$(2\pi)\,u\left(1+(u')^2\right)^{-1/2}\left[\left(1+\cancel{(u')^2}\right)-\cancel{(u')^2}\right]=K$$

$$(2\pi)\,u\left(1+(u')^2\right)^{-1/2}=K$$

$$\Rightarrow u'=\sqrt{\left(\frac{2\pi u}{K}\right)^2-1}$$

$$a=\frac{K}{2\pi} \Rightarrow \boxed{\begin{array}{l} u'=\left(\left(u/a\right)^2-1\right)^{1/2} \\ \text{con } u(x_a)=y_a,\ u(x_s)=y_s \\ a \equiv \text{parámetro a determinar} \end{array}}$$

d) Integración

$$u/a=\mathrm{ch}(\eta) \Rightarrow u'=a\,\mathrm{sh}(\eta)\,\eta'=\left(\mathrm{ch}^2(\eta)-1\right)^{1/2}=\mathrm{sh}(\eta)$$

$$\Rightarrow \eta'=1/a \Rightarrow \eta=x/a-c$$

$$\Rightarrow \boxed{y=u(x)\to y/a=\mathrm{ch}\left(x/a-c\right)}$$

CATENARIA

La superficie de revolución se llama CATENOIDE

e) <u>Ajuste de parámetros</u> $\Rightarrow$ Hallar a, c en función de (x_a, y_a), (x_b, y_b)

$$\begin{cases} x_a = -L/2 \;, & y_a = H \\ x_b = +L/2 \;, & y_b = H \end{cases}$$

$$(y/a) = ch(x/a - c)$$

$$\Rightarrow \left.\begin{cases} x = x_a \longrightarrow (H/a) = ch\left((-L/2)/a - c\right) \\ x = x_b \longrightarrow (H/a) = ch\left((+L/2)/a - c\right) \end{cases}\right\} \Rightarrow \boxed{c=0}$$

ch(x) es simétrica respecto a $x=0$

$$\Rightarrow (H/a) = ch\left(\frac{L/2}{a}\right) \Bigg\} \Rightarrow (H/a) = ch\left((H/a)\,\mu/2\right) \Bigg\} \Rightarrow$$

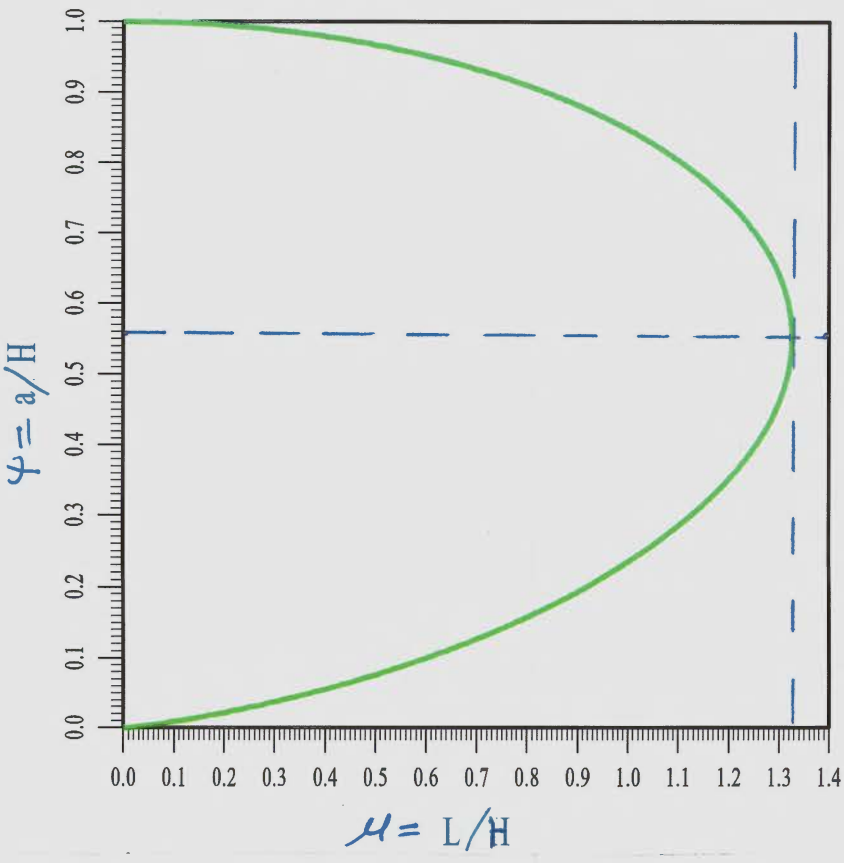
Sea $\boxed{\mu = L/H}$

Definimos $\boxed{\psi = a/H}$

$$\Rightarrow 1/\psi = ch\left(1/\psi \; \mu/2\right) \Rightarrow \mu = 2\,\frac{\arg ch(1/\psi)}{(1/\psi)}$$

Representamos: $\mu = 2\psi \arg ch(1/\psi) \rightarrow \begin{cases} \text{Dado } \mu \text{ hay que hallar } \psi. \\ \rightarrow a = H \cdot \psi \end{cases}$

$L/H = 2\,(a/H)\,\text{argcosh}(H/a)$



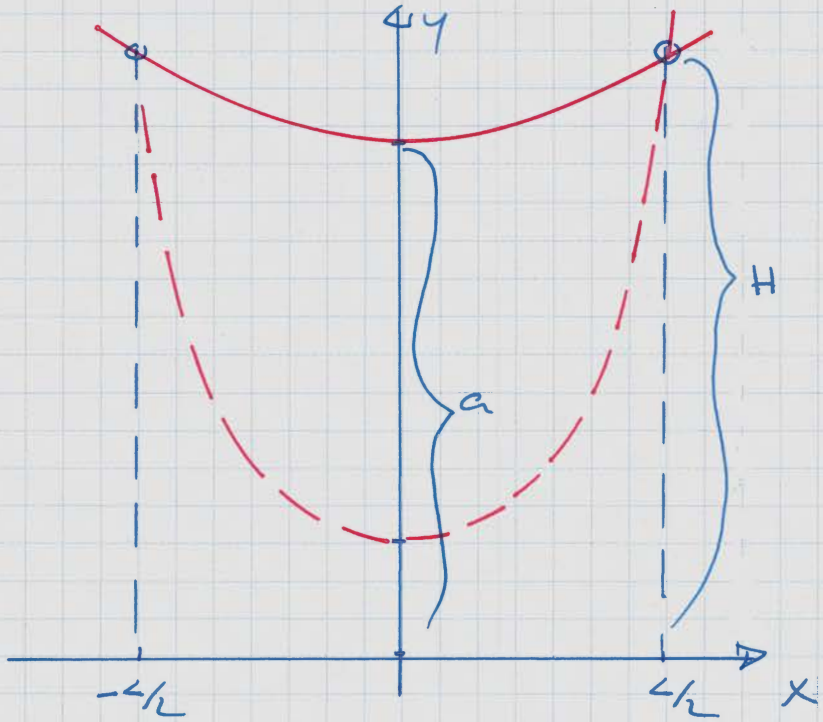
Observemos que para valores de $\mu = L/H$ inferiores a $\mu^* \simeq 1{,}32549$ hay dos valores de $\psi = a/H$ posibles.

Sin embargo, para $\mu = L/H > \mu^*$ no hay solución

$$\Rightarrow \boxed{L/H \leq 1{,}32549}$$

Notas: (Información complementaria)

1) Las catenarias en cuestión tienen el siguiente aspecto:



$$y = a\,\mathrm{ch}\,(x/a)$$

2) De las dos catenarias posibles sólo la superior tiene área mínima.

3) Se puede probar que para $\mu = L/H > 1{,}055394792$ la superficie de área mínima está formada por dos círculos de radio H (en $x=-L/2$ y $x=+L/2$, respectivamente)
(Solución de Goldschmidt)
El cálculo de variaciones no permite obtener este tipo de soluciones.

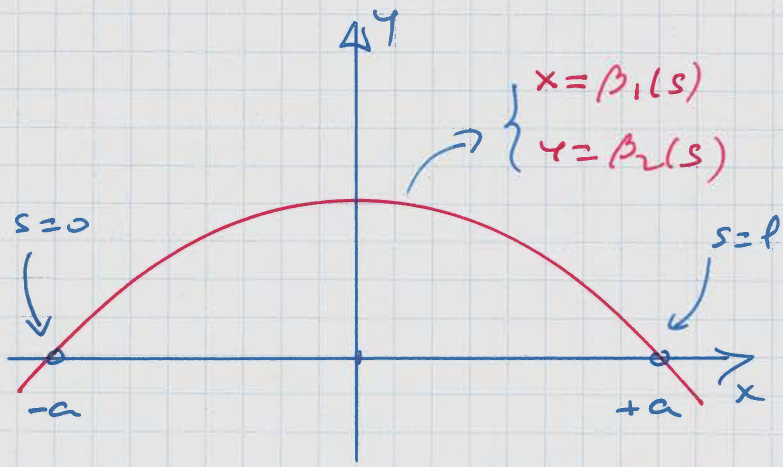
4) Por tanto, la solución será:

CATENOIDE

$$\begin{cases} L/H \leq 1.055394792 \Rightarrow \begin{cases} y/a = \mathrm{ch}\,(x/a) \\ \text{con } a = H\cdot\varphi \text{ (rama superior de la gráfica } \varphi-\mu) \end{cases} \\ L/H > 1{,}055394792 \rightarrow \text{Dos círculos separados} \end{cases}$$

SOL. DE GOLDSCHMIDT

FMC - P4 - E4



$$A=\int_{x=-a}^{x=a} y\,dx$$

$$\ell=\int_{x=-a}^{x=a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Planteamiento en función del arco

$$\left.\begin{array}{l} x=\beta_1(s)\\ y=\beta_2(s)\end{array}\right\} \Rightarrow \left\{\begin{array}{l} dx=\beta_1'(s)\,ds\\ dy=\beta_2'(s)\,ds\end{array}\right.$$

$$ds^2=dx^2+dy^2=\left((\beta_1')^2+(\beta_2')^2\right)ds^2 \Rightarrow (\beta_1')^2+(\beta_2')^2=1$$

Luego:

$$\left\{\begin{array}{l} A=\displaystyle\int_{s=0}^{s=\ell}\beta_2(s)\,\beta_1'(s)\,ds\\ \ell=\displaystyle\int_{s=0}^{s=\ell}ds=\ell \rightarrow \left\{\begin{array}{l}\text{Se cumple necesariamente}\\ \text{Sustituido por } (\beta_1')^2+(\beta_2')^2=1\end{array}\right.\end{array}\right.$$

a) Función Lagrangiana y planteamiento Variacional

$$\boxed{\mathcal{L}(s,y,p)=y\sqrt{1-p^2}}$$

Hallar $\beta_2(s)=u(s)$ que maximiza

$$\mathcal{J}[\beta_2(s)]=\int_{s=0}^{s=\ell}\left(\mathcal{L}(s,y,p)\Big|_{\substack{y=\beta_2(s)\\ p=\beta_2'(s)}}\right)ds$$

verificando: $\beta_2(0)=0$, $\beta_2(\ell)=0$

$$\beta_1'=\sqrt{1-(\beta_2')^2} \quad \text{con} \quad \left\{\begin{array}{l}\beta_1(0)=-a\\ \beta_1(\ell)=+a\end{array}\right.$$

CASO 1: Sin tener en cuenta $\beta_1(s)$

$$\boxed{\begin{array}{l}\text{Hallar } \beta_2(s) = u(s) \text{ que maximice} \\ \displaystyle J[\beta_2(s)] = \int_0^{\ell} \left(L(s,y,p) \Big|_{\substack{y=\beta_2(s) \\ p=\beta_2'(s)}} \right) ds \\ \text{verificando } \beta_2(0) = \beta_2(\ell) = 0 \end{array}}$$

b) Ecuación de Euler-Lagrange (Solución: $\beta_2(s) = u(s)$)

$$L(s,y,p) = y\,(1-p^2)^{1/2} \Longrightarrow \begin{cases} \dfrac{\partial L}{\partial y} = (1-p^2)^{1/2} \\[2mm] \dfrac{\partial L}{\partial p} = y\,(\cancel{1/2})(1-p^2)^{-1/2}(\cancel{-2}p) \end{cases}$$

$$\frac{\partial L}{\partial y}\Big|_{\substack{y=u \\ p=u'}} - \frac{d}{ds}\left(\frac{\partial L}{\partial p}\Big|_{\substack{y=u \\ p=u'}} \right) = 0$$

$$\Rightarrow (1-(u')^2)^{1/2} + \frac{d}{ds}\left((1-(u')^2)^{-1/2}\, u\, u' \right) = 0$$

$$(1-(u')^2)^{1/2} + (\cancel{-1/2})(1-(u')^2)^{-3/2}(\cancel{-2}u'u'')\,u\,u' + (1-(u')^2)^{-1/2}\left((u')^2 + u\,u''\right) = 0$$

$$(1-(u')^2)^{-3/2}\left\{ (1-(u')^2)^2 + u\,(u')^2 u'' + (1-(u')^2)\left((u')^2 + u u''\right) \right\} = 0$$

$$(1-(u')^2)^{-3/2}\left\{ 1 - 2(u')^2 + \cancel{(u')^4} + \cancel{u(u')^2 u''} + (u')^2 + u u'' - \cancel{(u')^4} - \cancel{u(u')^2 u''} \right\} = 0$$

$$(1-(u')^2)^{-3/2}\left\{ 1 - (u')^2 + u u'' \right\} = 0$$

$$\Rightarrow \boxed{1 - (u')^2 + u u'' = 0, \text{ con } u(0) = 0,\ u(\ell) = 0}$$

c) Identidad de Beltrami

$$\frac{\partial L}{\partial s} = 0 \Rightarrow \left(L - \left(\frac{\partial L}{\partial p}\right) p \right)\Big|_{\substack{y=u \\ p=u'}} = R = \text{constante}$$

$$\Rightarrow u\,(1-(u')^2)^{1/2} + \left(u\,(1-(u')^2)^{-1/2}\, u' \right) u' = R$$

$$\Rightarrow u\,(1-(u')^2)^{-1/2}\left[(1-\cancel{(u')^2}) + \cancel{(u')^2} \right] = R \Rightarrow \boxed{\begin{array}{c} u\,(1-(u')^2)^{-1/2} = R \\ \text{con } u(0)=0,\ u(\ell)=0 \end{array}}$$

d) Integración

$$u\left(1-(u')^2\right)^{-1/2} = R \Rightarrow u' = \sqrt{1-(u/R)^2}$$

Cambio: $u = R\cos\theta \Rightarrow \left|\begin{array}{l} u' = -R\,\text{sen}\,\theta\,\theta' \\ \sqrt{1-(u/R)^2} = \sqrt{1-\cos^2\theta} = \text{sen}\,\theta \end{array}\right.$

$$\Rightarrow -R\,\cancel{\text{sen}\,\theta}\,\theta' = \cancel{\text{sen}\,\theta} \Rightarrow -R\,d\theta = ds$$

Luego $\theta = -s/R + \varphi \Rightarrow u = R\cos(-s/R+\varphi) = R\cos(s/R-\varphi)$

$$\left.\begin{array}{l} u(0) = 0 \Rightarrow R\cos(\quad -\varphi) = 0 \\ u(\ell) = 0 \Rightarrow R\cos(\ell/R - \varphi) = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \varphi = \pi/2 \\ \ell/R = \pi \end{array}\right.$$

$$\boxed{\beta_2(s) = R\cos\left(s/R - \pi/2\right) \quad ; \ \ell = \pi R}$$

Ahora, $\beta_1' = \sqrt{1-(\beta_2')^2}$ con $\beta_1(0) = -a$, $\beta_1(\ell) = +a$

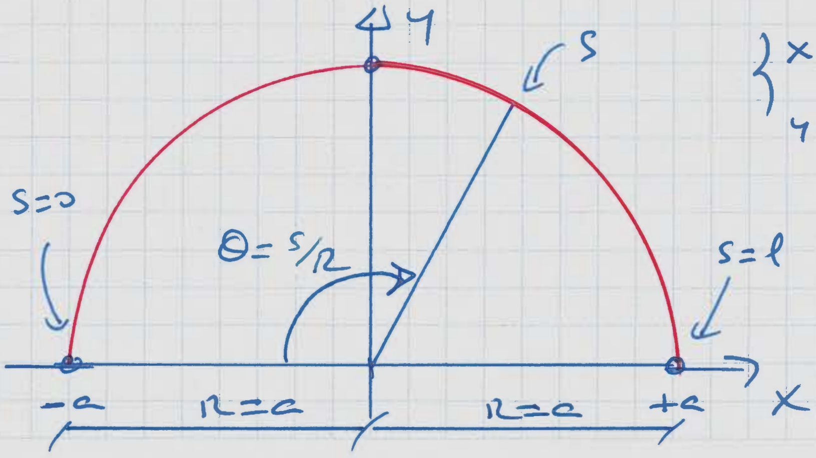
$$\beta_2' = -\text{sen}\left(s/R - \pi/2\right) \Rightarrow \beta_1' = \cos\left(s/R - \pi/2\right)$$

Luego $\beta_1(s) = R\,\text{sen}\left(s/R - \pi/2\right) + c$

$$\left\{\begin{array}{l} \beta_1(0) = -a \Rightarrow R\,\text{sen}(\quad -\pi/2) + c = -R + c = -a \\ \beta_1(\ell) = +a \Rightarrow R\,\text{sen}(\ell/R - \pi/2) + c = R\,\text{sen}(\pi - \pi/2) + c = R + c = +a \end{array}\right.$$

$$\boxed{\beta_1(s) = R\,\text{sen}\left(s/R - \pi/2\right) \quad ; \ R = a} \quad , \ c = 0$$

Se trata de una semi-circunferencia:



$$\left\{\begin{array}{l} x = R\,\text{sen}\left(\theta - \pi/2\right) \\ y = R\cos\left(\theta - \pi/2\right) \end{array}\right.$$

con $\theta = s/R$

$$\left\{\begin{array}{l} R = a \\ \ell = \pi R \end{array}\right.$$

CASO 2 : Teniendo en cuenta $\beta_1(s)$

La función $\beta_2(s)$ no es arbitraria, ya que

$$\beta_1'(s) = \sqrt{1-(\beta_2')^2} \quad \text{con} \quad \begin{cases} \beta_1(0) = -a \\ \beta_1(\ell) = +a \end{cases}$$

Luego $\beta_1(s) = \int_0^s \sqrt{1-(\beta_2')^2}\, ds + c$

$$\begin{cases} \beta_1(0) = c = -a \\ \beta_1(\ell) = \int_0^\ell \sqrt{1-(\beta_2')^2}\, ds + c = +a \end{cases}$$

Por tanto, $\beta_2(s)$ está sujeta a la restricción:

$$\boxed{\int_0^\ell \sqrt{1-(\beta_2')^2}\, ds = 2a}$$

Por tanto, el problema consiste en:

Hallar $\beta_2(s) = u(s)$ que maximiza

$$J[\beta_2(s)] = \int_0^s \left(L(s,y,p) \Big|_{\substack{y=\beta_2(s) \\ p=\beta_2'(s)}} \right) ds$$

verificando: $\int_0^\ell \left(\sqrt{1-p^2} \Big|_{\substack{y=\beta_2(s) \\ p=\beta_2'(s)}} \right) ds = 2a$

$$\beta_2(0) = 0 \ , \ \beta_2(\ell) = 0$$

Se trata de un problema de cálculo de variaciones con restricciones $\Rightarrow$ Exige introducir una función Lagrangiana modificada: $\boxed{\hat{L}(s,y,p,\lambda) = L(s,y,p) - \lambda\sqrt{1-p^2}}$

donde $\lambda \equiv$ multiplicador de Lagrange.

Luego el problema que hay que resolver es:

$$\boxed{\begin{array}{l} \text{Hallar } \beta_2(s) = u(s) \text{ que produce un estacionario de} \\ \mathcal{J}\{\beta_2(s)\} = \displaystyle\int_0^l \left(\hat{L}(s,y,p,\lambda) \Big|_{\substack{y=\beta_2(s) \\ p=\beta_2'(s)}} \right) ds \\ \text{verificando } \quad \beta_2(0)=0, \ \beta_2(l)=0 \end{array}}$$

b) Ecuación de Euler - Lagrange

$$\hat{L}(s,y,p,\lambda) = (y-\lambda)(1-p^2)^{1/2} \Rightarrow \left\{ \begin{array}{l} \dfrac{\partial \hat{L}}{\partial y} = (1-p^2)^{1/2} \\ \dfrac{\partial \hat{L}}{\partial p} = (y-\lambda)(\cancel{1/2})(1-p^2)^{-1/2}(-\cancel{2}p) \end{array} \right.$$

$$\frac{\partial \hat{L}}{\partial y}\Big|_{\substack{y=u \\ p=u'}} - \frac{d}{ds}\left(\frac{\partial \hat{L}}{\partial p}\Big|_{\substack{y=u \\ p=u'}} \right) = 0$$

$$\Rightarrow (1-(u')^2)^{1/2} + \frac{d}{ds}\left((1-(u')^2)^{-1/2}(u-\lambda)u' \right) = 0$$

Operando, se obtiene

$$\boxed{1-(u')^2 + (u-\lambda)u'' = 0 \quad , \quad \text{con } u(0)=0, \ u(l)=0}$$

c) Identidad de Beltrami

$$\frac{\partial \hat{L}}{\partial s} = 0 \Longrightarrow \left(\hat{L} - \left(\frac{\partial \hat{L}}{\partial p}\right) p \right)\Big|_{\substack{y=u \\ p=u'}} = R = \text{constante}$$

$$\Rightarrow (u-\lambda)(1-(u')^2)^{1/2} + \left((u-\lambda)(1-(u')^2)^{-1/2} u'\right) u' = R$$

$$\Rightarrow (u-\lambda)(1-(u')^2)^{-1/2}\left[(1-\cancel{(u')^2}) + \cancel{(u')^2} \right] = R$$

$$\Rightarrow \boxed{\begin{array}{l} (u-\lambda)(1-(u')^2)^{-1/2} = R \\ \text{con } u(0)=0, \ u(l)=0 \end{array}}$$

d) Integración

$$(u-\lambda)\left(1-(u')^2\right)^{-1/2} = R \Rightarrow u' = \sqrt{1-\left(\frac{u-\lambda}{R}\right)^2}$$

Cambio: $u-\lambda = R\cos\theta \rightarrow \begin{cases} u' = -R\,\text{sen}\,\theta \\ \sqrt{1-\left(\frac{u-\lambda}{R}\right)^2} = \sqrt{1-\cos^2\theta} = \text{sen}\,\theta \end{cases}$

$\Rightarrow -R\,\text{sen}\,\theta\;\theta' = \text{sen}\,\theta \Rightarrow -R\,d\theta = ds$

Luego $\theta = -s/R + \varphi \rightarrow u-\lambda = R\cos(-s/R+\varphi) = R\cos(s/R-\varphi)$

$$\left.\begin{aligned} u(0)=0 &\Rightarrow R\cos(\;\;-\varphi) = -\lambda \\ u(\ell)=0 &\Rightarrow R\cos(\ell/R-\varphi) = -\lambda \end{aligned}\right\} \Rightarrow \begin{cases} \lambda = -R\cos\varphi \\ \ell/R = 2\varphi \end{cases}$$

$$\boxed{\beta_2(s) = R\cos(s/R-\varphi) - R\cos\varphi \quad ; \quad \ell = 2\varphi R}$$

Ahora, $\beta_1' = \sqrt{1-(\beta_2')^2}$ con $\beta_1(0) = -a$, $\beta_1(\ell) = +a$

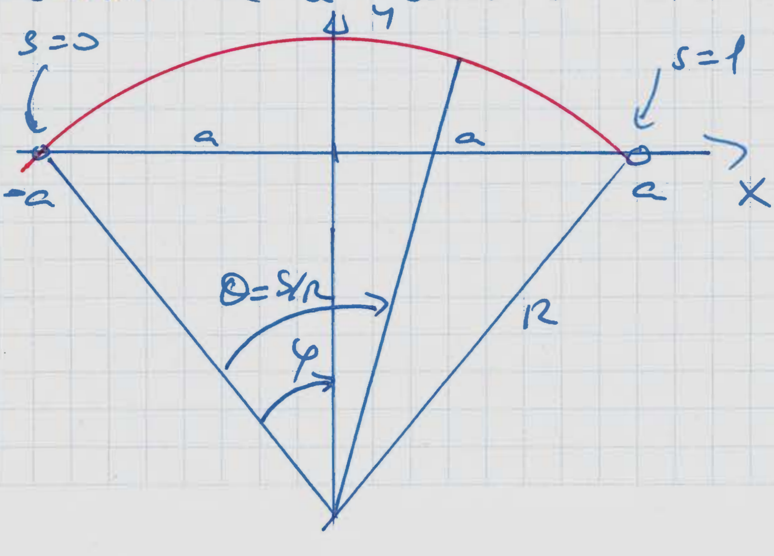
$\beta_2' = -\text{sen}(s/R-\varphi) \Rightarrow \beta_1' = \cos(s/R-\varphi)$

Luego $\beta_1(s) = R\,\text{sen}(s/R-\varphi) + c$

$$\begin{cases} \beta_1(0) = -a \Rightarrow R\,\text{sen}(\;\;-\varphi)+c = -R\,\text{sen}\,\varphi + c = -a \\ \beta_1(\ell) = +a \Rightarrow R\,\text{sen}(\ell/R-\varphi)+c = R\,\text{sen}\,\varphi + c = a \end{cases}$$

$$\boxed{\beta_1(s) = R\,\text{sen}(s/R-\varphi) \quad ; \quad R\,\text{sen}\,\varphi = a} \quad , c=0$$

Se trata de un arco de circunferencia:



$$\begin{cases} x = R\,\text{sen}(\theta-\varphi) \\ y = R\cos(\theta-\varphi) - R\cos\varphi \end{cases}$$

con $\theta = s/R$

$$\begin{cases} R\varphi = \ell/2 \\ R\,\text{sen}\,\varphi = a \end{cases}$$

Notas:

El caso 1 es más restrictivo que el caso 2, ya que la longitud l no puede adquirir cualquier valor, y tiene que cumplirse necesariamente $l = \pi a$.

Lo que sucede es que al no tenerse en cuenta $\beta_1(s)$ en el planteamiento variacional, se obtiene la función $\beta_2(s)$ que proporciona el mejor resultado posible. Es decir, la solución del caso 1 es la mejor de las soluciones del caso 2. Si calculamos el área de cada solución del caso 2 para un valor fijo de "l" en función de "a", observamos que el área es máxima para $a = l/\pi$.

El problema resuelto en el caso 2 se llama "problema isoperimétrico especial". El correspondiente al caso 1 recibe el nombre de "problema isoperimétrico" o "problema de Dido" (en referencia a la leyenda de la Reina Dido de Cartago).

El problema correspondiente al caso 2 puede plantearse y resolverse directamente mediante multiplicadores de Lagrange.

Solución directa en explícitas: $y = f(x)$

$$A = \int_{-a}^{a} y\, dx \quad ; \quad \ell = \int_{-a}^{a} \sqrt{1+(y')^2}\, dx$$

a) Planteamiento Variacional

$$\boxed{\begin{array}{l} \text{Hallar } f(x) = u(x) \text{ que maximiza} \\ \mathcal{J}[f(x)] = \displaystyle\int_{-a}^{a} \left(L(x,y,p)\Big|_{\substack{y=f(x)\\ p=f'(x)}} \right) dx, \text{ con } L(x,y,p) = y \\ \text{verificando: } \displaystyle\int_{-a}^{a} \left((1+p^2)^{1/2}\Big|_{\substack{y=f(x)\\ p=f'(x)}} \right) dx = \ell \\ f(-a) = f(+a) = 0 \end{array}}$$

Lagrangiana modificada:

$$\hat{L}(x,y,p,\lambda) = L(x,y,p) - \lambda\,(1+p^2)^{1/2}$$

b) Ecuaciones de Euler-Lagrange

$$\hat{L}(x,y,p,\lambda) = y - \lambda(1+p^2)^{1/2} \Rightarrow \begin{cases} \dfrac{\partial \hat{L}}{\partial y} = 1 \\ \dfrac{\partial \hat{L}}{\partial p} = -\lambda(\cancel{1/2})(1+p^2)^{-1/2}\,\cancel{2}p \end{cases}$$

$$\Rightarrow \boxed{1 + \frac{d}{dx}\left(\lambda\left(1+(u')^2\right)^{-1/2} u'\right) = 0}$$

c) Beltrami

$$\left(u - \lambda(1+(u')^2)^{1/2}\right) + \left(\lambda\,(1+(u')^2)^{-1/2} u'\right)u' = c$$

$$\Rightarrow u' = \frac{1}{\sqrt{\left(\frac{u-c}{\lambda}\right)^2 - 1}}$$

d) Integración

Se obtiene $\begin{cases} \boxed{x^2 + (y + R\cos\varphi)^2 = R^2} \longrightarrow \text{arco de circunferencia} \\ \text{donde } R\varphi = \ell/2, \quad R\,\mathrm{sen}\,\varphi = a \end{cases}$

Solución directa en paramétrica $\bar r = \bar\alpha(\tau)$

$$A = \int_{\tau_I}^{\tau_F} \alpha_2(\tau)\,\alpha_1'(\tau)\,d\tau, \quad \ell = \int_{\tau_I}^{\tau_F} \sqrt{(\alpha_1'(\tau))^2 + (\alpha_2'(\tau))^2}\,d\tau$$

a) Planteamiento Variacional

$$\boxed{\begin{array}{l}
\text{Hallar } \bar\alpha(\tau) = \bar u(\tau) \text{ que maximice} \\
\mathcal{J}[\bar\alpha(\tau)] = \displaystyle\int_{\tau_I}^{\tau_F} \left(L(\tau,\bar r,\bar p)\Big|_{\substack{\bar r = \bar\alpha(\tau) \\ \bar p = \bar\alpha'(\tau)}} \right) d\tau, \\
\text{con } L(\tau,\bar r,\bar p) = r_2\, p_1 \\
\text{Verificando } \displaystyle\int_{\tau_I}^{\tau_F} \left((p_1^2 + p_2^2)^{1/2}\Big|_{\substack{\bar r = \bar\alpha(\tau) \\ \bar p = \bar\alpha'(\tau)}} \right) d\tau = \ell \\
\bar\alpha(\tau_I) = \begin{Bmatrix} -a \\ 0 \end{Bmatrix}, \quad \bar\alpha(\tau_F) = \begin{Bmatrix} a \\ 0 \end{Bmatrix}
\end{array}}$$

Lagrangiano modificado:

$$\hat L(\tau,\bar r,\bar p,\lambda) = L(\tau,\bar r,\bar p) - \lambda\,(p_1^2 + p_2^2)^{1/2}$$

b) Ecuaciones de Euler-Lagrange

$$\hat L(\tau,\bar r,\bar p,\lambda) = r_2\, p_1 - \lambda\,(p_1^2 + p_2^2)^{1/2}$$

$$\begin{cases}
\dfrac{\partial \hat L}{\partial \bar r} = \left[\dfrac{\partial \hat L}{\partial r_1} \quad \dfrac{\partial \hat L}{\partial r_2} \right] = \left[0 \qquad p_1 \right] \\[2ex]
\dfrac{\partial \hat L}{\partial \bar p} = \left[\dfrac{\partial \hat L}{\partial p_1} \quad \dfrac{\partial \hat L}{\partial p_2} \right] = \left[\left(r_2 - \lambda(\cancel{1/2})(p_1^2+p_2^2)^{-1/2}\cancel{2}p_1 \right) \quad \left(-\lambda(\cancel{1/2})(p_1^2+p_2^2)^{-1/2}\cancel{2}p_2 \right) \right]
\end{cases}$$

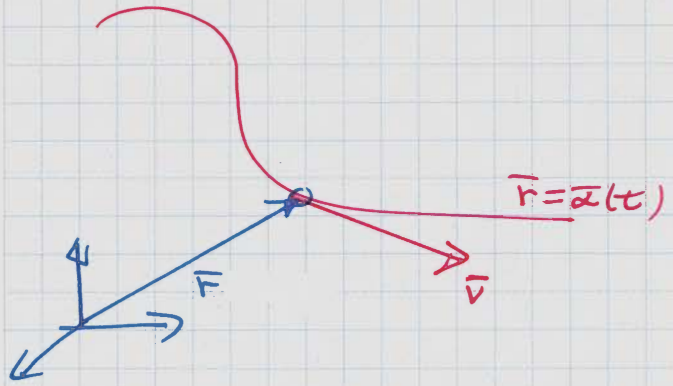
$$\Rightarrow \boxed{\begin{array}{l}
0 - \dfrac{d}{d\tau}\left(u_2 - \lambda\left((u_1')^2 + (u_2')^2\right)^{-1/2} u_1' \right) = 0 \\[2ex]
u_1' - \dfrac{d}{d\tau}\left(-\lambda\left((u_1')^2 + (u_2')^2\right)^{-1/2} u_2' \right) = 0
\end{array}}$$

d) Integración

Se obtiene $\boxed{x^2 + (y + R\cos\varphi)^2 = R^2}$ $\longrightarrow$ arco de circunferencia

donde $R\varphi = \ell/2$, $R\sin\varphi = a$

FMC - P4 - E5



$$\begin{cases} T = \frac{1}{2} m |\bar{v}|^2 \\ V = m\, \mathcal{U}(\bar{r}) \end{cases}$$

Coordenadas generalizadas: $\begin{cases} \bar{q} = \bar{r} \\ \dot{\bar{q}} = \bar{v} \end{cases}$

a) $\mathcal{L}(t, \bar{q}, \bar{p}) = T - V = \frac{1}{2} m |\bar{v}|^2 - m\, \mathcal{U}(\bar{r})$

$$= \frac{1}{2} m\, \bar{p}^T \bar{p} - m\, \mathcal{U}(\bar{q})$$

F. Lagrangiana : $\boxed{\mathcal{L}(t, \bar{q}, \bar{p}) = \frac{1}{2} m\, \bar{p}^T \bar{p} - m\, \mathcal{U}(\bar{q})}$

b) Principio de Hamilton (acción estacionaria)

La trayectoria $\bar{q} = \bar{\alpha}(t)$ es la solución $\bar{f}(t) = \bar{\alpha}(t)$ que hace estacionaria la "acción"

$$S[\bar{f}(t)] = \int_{t_0}^{t_1} \mathcal{L}(t, \bar{q}, \bar{p}) \Big|_{\substack{\bar{q} = \bar{f}(t) \\ \bar{p} = \bar{f}'(t)}} dt$$

verificando $\bar{f}(t_0) = \bar{q}_0$, $\bar{f}'(t_0) = \dot{\bar{q}}_0$

c) Ecuaciones del Movimiento de Lagrange

$$\frac{\partial \mathcal{L}}{\partial \bar{q}} \Big|_{\substack{\bar{q} = \bar{\alpha}(t) \\ \bar{p} = \bar{\alpha}'(t)}} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \bar{p}} \Big|_{\substack{\bar{q} = \bar{\alpha}(t) \\ \bar{p} = \bar{\alpha}'(t)}} \right) = \bar{0}^T$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \bar{q}} = -m \dfrac{d\mathcal{U}}{d\bar{q}} \\ \dfrac{\partial \mathcal{L}}{\partial \bar{p}} = \cancel{1/2}\, m\, \cancel{2}\, \bar{p}^T = m \bar{p}^T \end{cases}$$

$$\Rightarrow \quad -m\, \overline{\text{grad}}^T \left(\mathcal{U}(\bar{q}) \Big|_{\bar{q} = \bar{\alpha}(t)} \right) - \frac{d}{dt}\left(m\, \bar{\alpha}'(t) \right) = \bar{0}^T$$

Luego:

$$\boxed{m \frac{d^2}{d\tau^2}\left(\bar{\alpha}(\tau)\right) = -m\, \overline{\text{grad}}\, (U)\Big|_{\bar{\alpha}(\tau)}} \qquad (*)$$

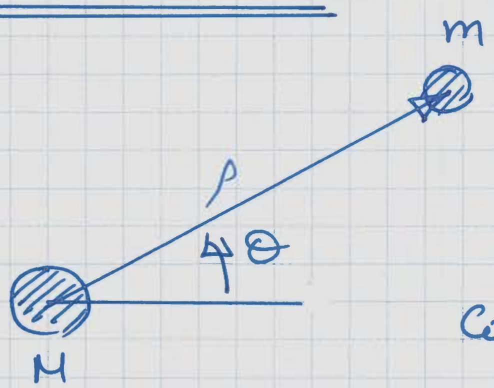
d) <u>Leyes de Newton</u>:

$$\bar{F} = m\bar{a} \quad \left\{ \begin{array}{l} \bar{F} = -m\, \overline{\text{grad}}(U)\Big|_{\bar{\alpha}(\tau)} \\ \bar{a} = \frac{d^2}{d\tau^2}\left(\bar{\alpha}(\tau)\right) \end{array} \right\} \Rightarrow \text{CONCUERDAN } (*)$$

(*) Además, hay que tener en cuenta las condiciones iniciales:

$$\left\{ \begin{array}{l} \bar{\alpha}(\tau_0) = \bar{r}_0 \\ \frac{d}{d\tau}\bar{\alpha}(\tau_0) = \bar{v}_0 \end{array} \right.$$

FMC - P4 - E6

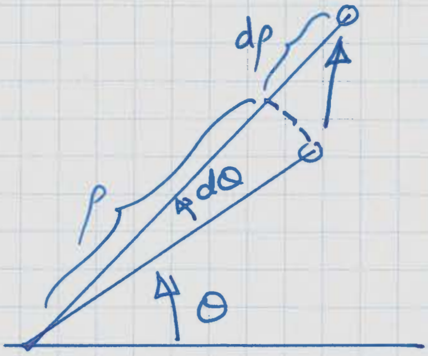


$$\begin{cases} T = \frac{1}{2} m v^2 = \frac{1}{2} m |\bar{v}|^2 \\ V = -\dfrac{G m M}{\rho} \end{cases}$$

Coordenadas generalizadas: $\bar{q} = \begin{Bmatrix} \rho \\ \theta \end{Bmatrix} \rightarrow \dot{\bar{q}} = \begin{Bmatrix} \dot{\rho} \\ \dot{\theta} \end{Bmatrix}$

a) Función Lagrangiana

$$\mathcal{L}(\tau, \bar{q}\ \bar{p}) = T - V = \frac{1}{2} m |\bar{v}|^2 + \frac{G m M}{\rho}$$



$$|\bar{v}| = \left(\left(\rho \frac{d\theta}{d\tau} \right)^2 + \left(\frac{d\rho}{d\tau} \right)^2 \right)^{1/2} = \sqrt{(\rho\dot{\theta})^2 + (\dot{\rho})^2}$$

$$\mathcal{L}(\tau, \underbrace{\rho, \theta}_{\bar{q}}, \underbrace{\dot{\rho}\ \dot{\theta}}_{\bar{p}}) = \frac{1}{2} m \left(\sqrt{(\rho\dot{\theta})^2 + (\dot{\rho})^2} \right)^2 + \frac{G m M}{\rho}$$

$$\boxed{\begin{array}{c} \mathcal{L}(\tau, \bar{q}, \bar{p}) = \frac{1}{2} m \left((\rho\dot{\theta})^2 + (\dot{\rho})^2 \right) + \dfrac{G m M}{\rho} \\ \text{con } \bar{q} = \begin{Bmatrix} \rho \\ \theta \end{Bmatrix} \quad \bar{p} = \dot{\bar{q}} = \begin{Bmatrix} \dot{\rho} \\ \dot{\theta} \end{Bmatrix} \end{array}}$$

b) Principio de Hamilton (acción estacionaria)

La órbita $\bar{q} = \begin{Bmatrix} \rho(\tau) \\ \theta(\tau) \end{Bmatrix}$ es la solución

$\bar{f}(\tau) = \begin{Bmatrix} \rho(\tau) \\ \theta(\tau) \end{Bmatrix}$ que hace estacionaria la "acción"

$$S[\bar{f}(\tau)] = \int_{\tau_0}^{\tau_1} \left(\mathcal{L}(\tau, \bar{q}, \bar{p}) \Big|_{\substack{\bar{q} = \bar{f}(\tau) \\ \bar{p} = \bar{f}'(\tau)}} \right) d\tau$$

verificando $\bar{f}(\tau_0) = \bar{q}_0$, $\bar{f}'(\tau_0) = \dot{\bar{q}}_0$

c) Ecuaciones del Movimiento de Lagrange

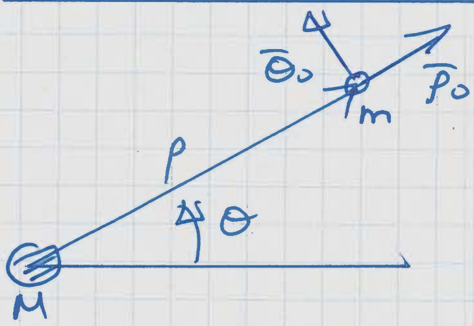
$$\left(\frac{\partial L}{\partial \bar q}\Bigg|_{\substack{\bar q = \{\rho(\tau),\ \theta(\tau)\} \\ \bar p = \{\dot\rho(\tau),\ \dot\theta(\tau)\}}}\right) - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \bar p}\Bigg|_{\substack{\bar q = \{\rho(\tau),\ \theta(\tau)\} \\ \bar p = \{\dot\rho(\tau),\ \dot\theta(\tau)\}}}\right) = \bar 0^T$$

$$\begin{cases} \dfrac{\partial L}{\partial \bar q} = \left[\dfrac{\partial L}{\partial \rho} \quad \dfrac{\partial L}{\partial \theta}\right] = \left[\left(\cancel{\tfrac{1}{2}} m \cancel{2} (\rho\dot\theta)\dot\theta - \dfrac{GmM}{\rho^2}\right) \quad 0\right] \\[2ex] \dfrac{\partial L}{\partial \bar p} = \left[\dfrac{\partial L}{\partial \dot\rho} \quad \dfrac{\partial L}{\partial \dot\theta}\right] = \left[\left(\cancel{\tfrac{1}{2}} m \cancel{2} \dot\rho\right) \quad \left(\cancel{\tfrac{1}{2}} m \cancel{2} (\rho\dot\theta)\rho\right)\right] \end{cases}$$

Luego:
$$\begin{cases} \left(m\,\rho(\dot\theta)^2 - \dfrac{GmM}{\rho^2}\right) - \dfrac{d}{d\tau}(m\dot\rho) = 0 \\[2ex] 0 - \dfrac{d}{d\tau}(m\rho^2\dot\theta) = 0 \end{cases}$$

$$\iff \boxed{\begin{aligned} \frac{d}{d\tau}(m\dot\rho) - m\rho(\dot\theta)^2 &= -\frac{GmM}{\rho^2} \\ \frac{d}{d\tau}(m\rho^2\dot\theta) &= 0 \end{aligned}}$$

d) Ecuaciones de Newton



En polares (ejes móviles):

$$\begin{cases} \bar r = \rho\,\bar p_0 \quad ; \bar\Omega = \dot\theta\,\bar k \\ \bar v = \dot\rho\,\bar p_0 + \bar\Omega \wedge \bar r = \dot\rho\,\bar p_0 + \rho\dot\theta\,\bar k \\ \bar a = (\ddot\rho\,\bar p_0 + (\dot\rho\dot\theta + \rho\ddot\theta)\,\bar\theta_0) + \bar\Omega\wedge\bar v \\ \quad = (\ddot\rho - \rho(\dot\theta)^2)\,\bar p_0 + (2\dot\rho\dot\theta + \rho\ddot\theta)\,\bar\theta_0 \end{cases}$$

$\bar F = m\bar a$ con $\bar F = -\dfrac{GmM}{\rho^2}\,\bar p_0 \Rightarrow$

$$\Rightarrow \left.\begin{cases} m(\ddot\rho - \rho(\dot\theta)^2) = -\dfrac{GmM}{\rho^2} \\ m(2\dot\rho\dot\theta + \rho\ddot\theta) = 0 \end{cases}\right\}$$

CONCUERDAN, pues $\dfrac{d}{d\tau}(m\rho^2\dot\theta) = m(2\rho\dot\rho\dot\theta + \rho^2\ddot\theta)$

d) Integración

$$\left\{\begin{array}{l} \frac{d}{dt}(\cancel{m}\dot\rho) - \cancel{m}\rho(\dot\theta)^2 = -\frac{G\cancel{m}M}{\rho^2} \\ \frac{d}{dt}(\cancel{m}\rho^2\dot\theta) = 0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \ddot\rho - \rho(\dot\theta)^2 = -\frac{GM}{\rho^2} \\ \frac{d}{dt}(\rho^2\dot\theta) = 0 \end{array}\right.$$

Luego: $\boxed{\rho^2\dot\theta = \ell = cte}$ (*) $\Rightarrow \dot\theta = \ell/\rho^2$

$$\Rightarrow \ddot\rho - \rho(\ell/\rho^2)^2 = -\frac{GM}{\rho^2} \Rightarrow \ddot\rho - \ell^2/\rho^3 = -\frac{GM}{\rho^2}$$

Cambio: $u = 1/\rho \Longleftrightarrow \rho = 1/u \Rightarrow \dot\theta = \ell u^2$

$$\left\{\begin{array}{l} \dot\rho = \frac{d}{dt}(1/u) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\dot\theta = -\frac{1}{\cancel{u^2}}\frac{du}{d\theta}(\ell\cancel{u^2}) = -\ell\frac{du}{d\theta} \\ \ddot\rho = \frac{d}{dt}(\dot\rho) = -\ell\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -\ell\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\dot\theta = -\ell\frac{d^2u}{d\theta^2}(\ell u^2) = -\ell^2u^2\frac{d^2u}{d\theta^2} \end{array}\right.$$

Luego: $-\ell^2u^2\frac{d^2u}{d\theta^2} - \ell^2u^3 = -GMu^2$

$$\Rightarrow \frac{d^2u}{d\theta^2} = -u + \frac{GM}{\ell^2} \quad ; \quad \delta = \ell^2/(GM)$$

$$\Rightarrow \frac{d^2u}{d\theta^2} = -u + 1/\delta$$

Solución: $u(\theta) = 1/\delta + A\cos(\theta - \theta_0)$

$$\Rightarrow \delta u = 1 + \underbrace{(\delta A)}_{\varepsilon}\cos(\theta-\theta_0) \quad ; \quad \varepsilon = \delta A$$

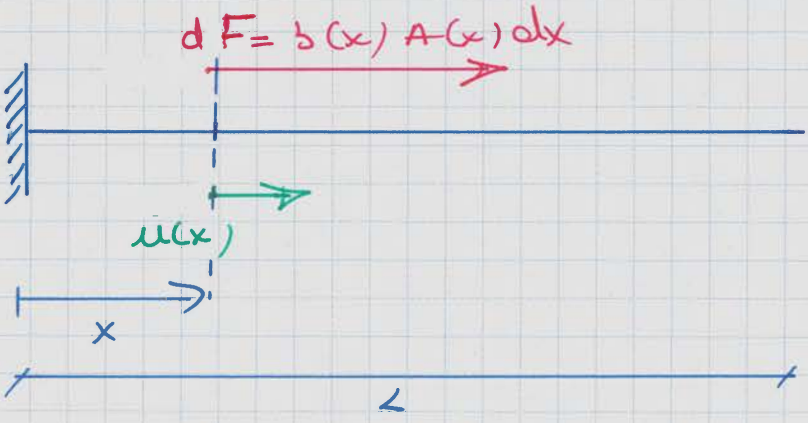
$$\Rightarrow \delta u = 1 + \varepsilon\cos(\theta-\theta_0) \Rightarrow \rho = \frac{\delta}{1+\varepsilon\cos(\theta-\theta_0)}$$

$$\boxed{\rho = \frac{\ell^2G^{-1}M^{-1}}{1+\varepsilon\cos(\theta-\theta_0)}}$$

(*) La velocidad areolar $\frac{dA}{dt} = 1/2\,\rho^2\dot\theta$ es el área barrida por el vector de posición, por unidad de tiempo.
Luego $1/2\,\ell = cte. \Longleftrightarrow$ 2ª Ley de Kepler.
También, $m\ell = cte \Longleftrightarrow$ Conservación del momento angular.

FMC - P4 - E7

Supongamos que la barra está empotrada en $x=0$ y tiene un extremo libre en $u=L$



a) Problema Variacional

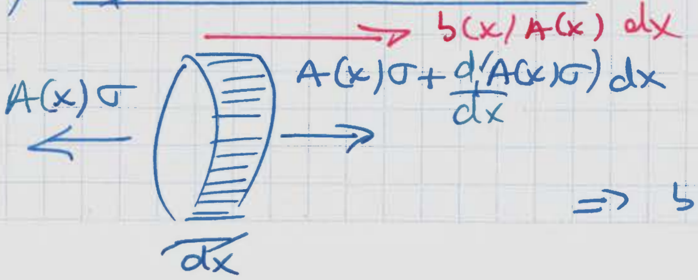
$$\boxed{\begin{array}{l} \text{Hallar } f(x)=u(x) \text{ que minimice el funcional} \\ \mathcal{E}[f(x)] = \displaystyle\int_0^L \left(L(x,y,p)\Big|_{\substack{y=f(x)\\ p=f'(x)}} \right) dx \quad \text{con} \\ L(x,y,p) = \tfrac{1}{2} E A(x)\, p^2 - b(x) A(x)\, y \\ \text{verificando: } \quad f(0)=0 \;, \quad f'(L)=0 \end{array}}$$

b) Ecuación de Euler-Lagrange : (solución $f(x)=u(x)$)

$$\left(\frac{\partial L}{\partial y}\Big|_{\substack{y=u(x)\\ p=u'(x)}} \right) - \frac{d}{dx}\left(\frac{\partial L}{\partial p}\Big|_{\substack{y=u(x)\\ p=u'(x)}} \right) = 0$$

$$\left.\begin{array}{l} \dfrac{\partial L}{\partial y} = -b(x)A(x) \\ \dfrac{\partial L}{\partial p} = \cancel{\tfrac{1}{2}}\, E A(x)\, \cancel{2}\, p \end{array}\right\} \Rightarrow \boxed{+b(x)A(x) + \frac{d}{dx}\left(E A(x) \frac{du}{dx} \right) = 0}$$

c) Equilibrio de la sección



$$\left\{\begin{array}{l} b(x)A(x) + \dfrac{d}{dx}\left(A(x)\sigma \right) = 0 \\ \sigma = E\varepsilon \;;\; \varepsilon = \dfrac{du}{dx} \end{array}\right\}$$

$$\Rightarrow bA(x) + \frac{d}{dx}\left(E A(x) \frac{du}{dx} \right) = 0 \Rightarrow \text{COINCIDE}$$