Continuity, derivation and integration of a power series (16.04.2024)

Let $\sum a_n x^n$ be a power series with radius of convergence $r > 0$, and let $S(x)$ be its sum. It is satisfied:

a) $S(x)$ is continuous on $(-r, r)$.

b) $S(x)$ is differentiable on $(-r, r)$ being its derivative $S'(x) = \sum n a_n x^{n-1}$.

c) $S(x)$ is integrable on $[0, x], \forall x \in (-r, r)$. Its integral is $\displaystyle\int_0^x S(t)dt = \sum \frac{a_n}{n+1} x^{n+1}$.

Proof.

a) We prove it in two parts.

a.1) $\sum a_n x^n$ converges uniformly on every compact $[-\rho, \rho] \subset (-r, r)$.

Since $0 < \rho < r$, the series is absolutely convergent for $x = \rho$. That is, $\sum |a_n \rho^n|$ is convergent (Cauchy-Hadamard theorem).

Thus, $\forall x / |x| \leq \rho$, we have that $|a_n x^n| \leq |a_n \rho^n|$, so $\sum a_n x^n$ has a majorant which is a convergent numerical series of positive terms. Hence, by Weierstrass's theorem, it is uniformly convergent on $[-\rho, \rho]$.

a.2) For all $x \in (-r, r)$ we can find a ρ such that

$$-r < -\rho < x < \rho < r$$

(e.g., if $x > 0$, then $\rho = (x + r)/2$).

As we have just seen, the series $\sum a_n x^n$ converges uniformly on $[-\rho, \rho]$. Since the functions $a_n x^n$ are continuous $\forall x$, the sum $S(x)$ will be continuous $\forall x \in (-r, r)$ (section **2.4.**).

Remark: When studying Abel's Theorems, we will see that, if the series converges at $x = r$ or $x = -r$, $S(x)$ will also be continuous at those points, not only at $(-r, r)$.

b) Let $\sum n a_n x^{n-1}$ be the series of the derivatives of $\sum a_n x^n$. Since

$$\lim_{n\to\infty} \sqrt[n]{|n a_n|} = \lim_{n\to\infty} \sqrt[n]{n}\,\sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

its radius of convergence coincides with that of $\sum a_n x^n$, both series converge uniformly on the same intervals.

Since $\sum a_n x^n$ converges at least at $x = 0$ and its terms are differentiable functions, then the sum $S(x)$ of $\sum a_n x^n$ is differentiable on $(-r, r)$ and the derivative of the sum is the sum of the series of derivatives (section **2.6.**)

$$S'(x) = \sum n a_n x^{n-1}$$

c) Let $\sum \dfrac{a_n}{n+1} x^{n+1}$ be the series of integrals (between 0 and x). Since the radius of convergence of its derived series $\sum a_n x^n$ is r, its radius will also be r, as we have just seen.

Since $\sum a_n x^n$ converges uniformly on $[0, x], \forall x \in (-r, r)$, (section **a.1**) and its terms are integrable functions, then the sum function $S(x)$ is integrable on $[0, x], \forall x \in (-r, r)$ and it holds (section **2.5.**)

$$\int_0^x S(t)dt = \sum \frac{a_n}{n+1} x^{n+1}$$