Sequence of continuous functions (16.04.2024)

Property. "*Let $\{f_n\}$ be a sequence of functions f_n, defined on I. If $\{f_n\}$ converges uniformly to f on I and the f_n are continuous on I, then f is continuous on I*".

Proof. We have to prove that, for every point a of I, it is satisfied

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon \tag{1}$$

- From the uniform convergence of the sequence $\{f_n\}_{n \in \mathbb{N}}$ on I we have that

$$\forall \varepsilon > 0 \ \exists n_0(\varepsilon) \ / \ |f_m(x) - f(x)| < \frac{\varepsilon}{3}, \ \forall m \geq n_0, \forall x \in I \tag{2}$$

- Since the functions f_n are continuous on I, we know that, $\forall a \in I$,

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - a| < \delta \Longrightarrow |f_m(x) - f_m(a)| < \varepsilon/3, \ \forall m \in \mathbb{N} \tag{3}$$

- So, given ε, we obtain $n_0(\varepsilon)$ at (2) and choose m such that $m \geq n_0$. Now, from ε and the continuity condition (3) of f_m at a, we obtain δ.

- Thus, given ε, there exist m and δ such that both conditions are satisfied at the same time. As a result, the condition (1) is satisfied.

  Indeed, if $0 < |x - a| < \delta$, then

$$\begin{aligned}&|f(x) - f(a)| = |(f(x) - f_m(x)) + (f_m(x) - f_m(a)) + (f_m(a) - f(a))| \leq \\ &\underbrace{|f(x) - f_m(x)|}_{(1)} + \underbrace{|f_m(x) - f_m(a)|}_{(2)} + \underbrace{|f_m(a) - f(a)|}_{(3)} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

  where

  - (1) and (3) are less than $\varepsilon/3$ by the uniform continuity of $\{f_n\}$.
  - (2) is less than $\varepsilon/3$ by the continuity of f_m at a.