Implicit function theorem. Examples (01.01.2023)

1.- Given the function $g(x,y,z) = z^3 + 2xyz + x$, we want to prove that the equation $g = 0$ defines z as an implicit function $z = \psi(x,y)$ on a neighborhood of $P(1,-1,1)$, as well as to obtain the equation of the tangent plane to the surface $z = \psi(x,y)$ at this point.

a) Since g is a polynomial, the function and its derivatives are continuous:

$$\frac{\partial g}{\partial x} = 2xy + 1, \quad \frac{\partial g}{\partial y} = 2xz, \quad \frac{\partial g}{\partial z} = 3z^2 + 2xy.$$

b) The function verifies $g(1,-1,1) = 0$ and $\dfrac{\partial g}{\partial z}(1,-1,1) = 1 \neq 0$.

Hence there exists $z = \psi(x,y)$, differentiable on a neighborhood of P (see section **10.2**).

From the derivatives of g, we obtain the derivatives of ψ with respect to x and y:

$$\left.\frac{\partial g}{\partial x}\right|_P = -1, \quad \left.\frac{\partial g}{\partial y}\right|_P = 2 \implies \left.\frac{\partial \psi}{\partial x}\right|_P = -\left(\frac{\partial g}{\partial z}\right)^{-1} \left.\frac{\partial g}{\partial x}\right|_P = 1, \quad \left.\frac{\partial \psi}{\partial y}\right|_P = -\left(\frac{\partial g}{\partial z}\right)^{-1} \left.\frac{\partial g}{\partial y}\right|_P = -2,$$

which allow us to find the equation of the tangent plane:

$$z = \psi|_P + \frac{1}{1!}\left(\left.\frac{\partial \psi}{\partial x}\right|_P (x-1) + \left.\frac{\partial \psi}{\partial y}\right|_P (y+1)\right) = 1 + (x-1) - 2(y+1).$$

2.- Given $g_1(x,y,z) = x^2 + xy + z$ and $g_2(x,y,z) = x + y^2 - z^2$, we want to prove that the equation $\vec{g} = \vec{0}$ defines the implicit functions $y = \psi_1(x)$, $z = \psi_2(x)$ on a neighborhood of $P(-1,1,0)$. We will also find the derivatives at P of ψ_1 and ψ_2 with respect to x.

a) P. derivatives: $\dfrac{\partial g_1}{\partial x} = 2x + y, \ \dfrac{\partial g_1}{\partial y} = x, \ \dfrac{\partial g_1}{\partial z} = 1, \ \dfrac{\partial g_2}{\partial x} = 1, \ \dfrac{\partial g_2}{\partial y} = 2y, \ \dfrac{\partial g_2}{\partial z} = -2z.$

b) Since g_1 and g_2 are polynomials, they and their partial derivatives with respect to x, y and z are continuous.

c) It holds that: $\vec{g}|_P = \vec{0}$; $\left|\dfrac{\partial \vec{g}}{\partial(y,z)}\right|_P = \begin{vmatrix} \dfrac{\partial g_1}{\partial y} & \dfrac{\partial g_1}{\partial z} \\ \dfrac{\partial g_2}{\partial y} & \dfrac{\partial g_2}{\partial z} \end{vmatrix}_P = \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} = -2 \neq 0.$

Then there exists the implicit function $\vec{\psi}(x)$, differentiable (section **10.3**), and it holds:

$$\left\{\begin{matrix} \dfrac{d\psi_1}{dx} \\ \dfrac{d\psi_2}{dx} \end{matrix}\right\}_P = -\left[\frac{\partial \vec{g}}{\partial(y,z)}\right]^{-1} \left\{\frac{\partial \vec{g}}{\partial x}\right\}_P = -\begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}^{-1} \left\{\begin{matrix} -1 \\ 1 \end{matrix}\right\} = \left\{\begin{matrix} -1/2 \\ 1/2 \end{matrix}\right\}.$$

Remark. This result can also be obtained by solving a system of equations:

$g_1\big(x, \psi_1(x), \psi_2(x)\big) = \phi_1(x) = 0$; $g_2\big(x, \psi_1(x), \psi_2(x)\big) = \phi_2(x) = 0$. Deriving ϕ_1 and ϕ_2:

$$\frac{d\phi_1}{dx} = \frac{\partial g_1}{\partial x} + \frac{\partial g_1}{\partial y}\frac{d\psi_1}{dx} + \frac{\partial g_1}{\partial z}\frac{d\psi_2}{dx} = (2x+y) + x\,\frac{d\psi_1}{dx} + 1\,\frac{d\psi_2}{dx} = 0$$

$$\frac{d\phi_2}{dx} = \frac{\partial g_2}{\partial x} + \frac{\partial g_2}{\partial y}\frac{d\psi_1}{dx} + \frac{\partial g_2}{\partial z}\frac{d\psi_2}{dx} = 1 + 2y\,\frac{d\psi_1}{dx} - 2z\,\frac{d\psi_2}{dx} = 0.$$

At P: $\; -1 - \left.\dfrac{d\psi_1}{dx}\right|_P + \left.\dfrac{d\psi_2}{dx}\right|_P = 0; \;\; 1 + 2\left.\dfrac{d\psi_1}{dx}\right|_P - 0 = 0 \Rightarrow \boxed{\left.\dfrac{d\psi_1}{dx}\right|_P = -\dfrac{1}{2}; \;\; \left.\dfrac{d\psi_2}{dx}\right|_P = \dfrac{1}{2}}$