Example of a semidefinite cuadratic form (29.12.2022)

We study the extrema of $f(x,y) = x^2 + ky^4$, depending on the sign of k.

a) Applying the N.C. of extremum, $\nabla f = \vec{0}$, we obtain the critical point $P(0,0)$.

b) We calculate the second differential at $P(0,0)$. The only non-zero second derivative (f''_{xx}) is equal to 2, while $f''_{xy} = f''_{yx} = f''_{yy} = 0$. Hence $d^2f(0,0) = 2dx^2 \geq 0$ and the second differential of f turns out to be a positive semidefinite quadratic form.

c) However, depending on the sign we give to k, we get different results:

c.1. For $k > 0$, the function reaches a relative (and absolute) **minimum** at $P(0,0)$.

c.2. $k = 0$, the function becomes $z = x^2$ (a parabolic cylinder), which has a **minimum** along the line $x = 0$, i.e. the OY axis (the axis corresponds to a generatrix of the cylinder).

c.3. For $k < 0$, the function becomes $z = x^2 - |k|\, y^4$. In this case:

- If, from $P(0,0)$, we move along the X axis, f increases.
- If we do it along the Y axis, the function decreases.

Therefore, at $P(0,0)$, there is neither a maximum nor a minimum (it is a **saddle point**).

The following figures represent, in this order, the cases $k = 1$, $k = 0$, $k = -1$.