Jacobian of the inverse function (29.12.2022)

Real functions of one variable. The practical method that we use to obtain the derivative of the function f^{-1}, inverse of f, is the following:

$$y = f^{-1}(x) \Longrightarrow x = f(y) \Longrightarrow 1 = f'(y)\, y' \Longrightarrow y' = \frac{df^{-1}(x)}{dx} = \left.\frac{1}{f'(y)}\right|_{y=f^{-1}(x)}$$

That is, "the derivative of the function f^{-1}, inverse of f, is the inverse of the derivative of f".

Vector functions. We consider a function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^n$. Its inverse will be $\vec{y} = \vec{f}^{-1}(\vec{x})$, where $\vec{x} = \vec{f}(\vec{y})$. We derive both members with respect to $\vec{x}$. The first one is the identity function $\vec{\phi}(\vec{x}) = \vec{x}$, whose derivative with respect to $\vec{x}$ will be the unit matrix:

$$\frac{d\vec{\phi}}{d\vec{x}} = \begin{pmatrix} \dfrac{\partial x_1}{\partial x_1} & \cdots & \dfrac{\partial x_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial x_n}{\partial x_1} & \cdots & \dfrac{\partial x_n}{\partial x_n} \end{pmatrix} = I$$

We apply the chain rule to the second member, obtaining $\dfrac{d\vec{f}(\vec{y})}{d\vec{x}} = \dfrac{d\vec{f}}{d\vec{y}} \dfrac{d\vec{f}^{-1}}{d\vec{x}}$, so

$$I = \frac{d\vec{f}}{d\vec{y}} \frac{d\vec{f}^{-1}}{d\vec{x}} \Longrightarrow \boxed{\frac{d\vec{f}^{-1}}{d\vec{x}} = \left[\frac{d\vec{f}}{d\vec{y}}\right]^{-1}_{\vec{y}=\vec{f}^{-1}(\vec{x})}}$$

That is, "the Jacobian of the inverse function of $\vec{f}$ is the inverse matrix of the Jacobian of $\vec{f}$".

Example. Polar and cartesian coordinates. Consider the function $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{Bmatrix} \rho \\ \theta \end{Bmatrix} = \vec{f}(x,y) \quad \text{with} \quad \begin{cases} \rho = f_1(x,y) = \sqrt{x^2+y^2} \\ \\ \theta = f_2(x,y) = \operatorname{arctg} \dfrac{y}{x} \end{cases}$$

Let its inverse function $\vec{f}^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$, be given by $\begin{Bmatrix} x \\ y \end{Bmatrix} = \vec{f}^{-1}(\rho,\theta)$, with $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$

The Jacobian matrices of $\vec{f}$ and f^{-1} are inverse of each other. The Jacobian of $\vec{f}^{-1}$ is

$$\frac{d\vec{f}^{-1}}{d(\rho,\theta)} = \begin{pmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix}$$

And the Jacobian of f is

$$\frac{d\vec{f}}{d(x,y)} = \begin{pmatrix} \dfrac{\partial \rho}{\partial x} & \dfrac{\partial \rho}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{pmatrix}$$

Writing it as a function of the variables (ρ,θ), becomes $\begin{pmatrix} \cos\theta & \sin\theta \\ -\dfrac{\sin\theta}{\rho} & \dfrac{\cos\theta}{\rho} \end{pmatrix}$. It is immediate to verify that the product of both is the unit matrix.