Exercises with limits (29.12.2022)

1) Find the limit at $\vec{a}=(0,0)$ of $f(x,y)=\begin{cases}\dfrac{x^2+y^2}{x^2-y^2} & x^2\neq y^2\\ 0 & x^2=y^2.\end{cases}$

We calculate the directional limit, along the direction given by $\vec{\omega}$:

$$\lim_{\lambda\to 0} f(\vec{a}+\lambda\vec{\omega})=\lim_{\lambda\to 0} f(\lambda\omega_x,\lambda\omega_y)=\lim_{\lambda\to 0}\frac{\lambda^2({\omega_x}^2+{\omega_y}^2)}{\lambda^2({\omega_x}^2-{\omega_y}^2)}=\frac{1}{{\omega_x}^2-{\omega_y}^2}$$

which depends on the direction, so there is no functional limit.

2) Find the limit at $\vec{a}=(0,0)$ of $f(x,y)=\begin{cases}\dfrac{x^2}{y} & y\neq 0\\ 0 & y=0.\end{cases}$

Directional limit: $\displaystyle\lim_{\lambda\to 0} f(\lambda\omega_x,\lambda\omega_y)=\lim_{\lambda\to 0}\frac{\lambda^2{\omega_x}^2}{\lambda\,\omega_y}=0,\ \forall\omega_y\neq 0.$

For $\omega_y=0$ (OX axis) the function is null, therefore the limit is equal to 0. Then all the directional limits coincide, so there may exist a functional limit.

However, if we approach the origin by any parabola $y=kx^2,\ k\neq 0$, that is, along points (x,kx^2), with values of x tending to 0, the function takes the value $1/k\neq 0$ in all of them, so there is no functional limit.

3) Find the limit at $\vec{a}=(0,0)$ of $f(x,y)=\begin{cases}\dfrac{2x^2y}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (0,0).\end{cases}$

Directional limit: $\displaystyle\lim_{\lambda\to 0}\frac{2\lambda^2\omega_x^2\lambda\,\omega_y}{\lambda^2\omega_x^2+\lambda^2\omega_y^2}=\lim_{\lambda\to 0}\frac{2\lambda^3\omega_x^2\,\omega_y}{\lambda^2(\omega_x^2+\omega_y^2)}=\lim_{\lambda\to 0}2\lambda\omega_x^2\,\omega_y=0.$

The directional limits exist and they are null; therefore, if there exists a functional limit, it will also be null. To calculate it we use polar coordinates $x=\rho\cos\theta$, $y=\rho\sin\theta$. If $\rho\to 0$, we are approaching $(0,0)$ along an arbitrary path. We obtain

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{\rho\to 0} f(\rho\cos\theta,\rho\sin\theta)=\lim_{\rho\to 0}\frac{2\rho^3\cos^2\theta\sin\theta}{\rho^2}=\lim_{\rho\to 0}2\rho\cos^2\theta\sin\theta=0$$

then there is a functional limit and, as we expected, it is zero.

4) Find the limit at $\vec{a}=(0,0)$ of $f(x,y)=\begin{cases}\dfrac{y}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (0,0).\end{cases}$

Sol. There is no functional limit because there are no directionals, except along OX.

5) Find the limit at $\vec{a}=(0,0)$ of $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{y^2+x} & y^2+x\neq 0\\ 1 & y^2+x=0.\end{cases}$

Sol. Although the directionals exist and coincide, there is no functional limit (see ex. **2**).

6) Find the limit at $\vec{a}=(0,0)$ of $f(x,y)=\begin{cases}xy\left(\sin\dfrac{1}{x}+\sin\dfrac{1}{y}\right) & x\cdot y\neq 0\\ 0 & x\cdot y=0.\end{cases}$

Sol. It is null (see ex. **3**).