# 2.5. Sufficient conditions of integrability (23.10.2022)

**a) Every monotonous function on $[a, b]$ is integrable on $[a, b]$.**

**Proof:** Suppose that $f$ is monotone increasing and $f(b) > f(a)$ (if $f(b) = f(a)$, the function would be constant and its integrability would be immediate).

Then, in each subinterval $[x_{i-1}, x_i]_{\,i=1,2\ldots n}$, it holds

$$m_i = \inf f(x) = f(x_{i-1})\,; \quad M_i = \sup f(x) = f(x_i)$$

For all $\varepsilon > 0$, we take a partition such that $\Delta x_i < \dfrac{\varepsilon}{f(b) - f(a)}$ and use this value in the Darboux sums. It turns out:

$$S(P) - s(P) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i < \frac{\varepsilon}{f(b) - f(a)} \sum_{i=1}^{n} (M_i - m_i) =$$
$$\frac{\varepsilon}{f(b) - f(a)}\,[f(x_n) - f(x_{n-1}) + f(x_{n-1}) - f(x_{n-2}) + \cdots + f(x_1) - f(x_0)] = \varepsilon$$

Thus $S(P) - s(P) < \varepsilon$ and the condition of integrability is verified.

**b) Every continuous function on $[a, b]$ is integrable on $[a, b]$.**

**Proof:** If $f$ is continuous on $[a, b]$, it is uniformly continuous on $[a, b]$ (Heine's theorem). Then, given $\varepsilon > 0$, it exists a $\delta$ such that

$$|x - x'| < \delta \Longrightarrow |f(x) - f(x')| < \frac{\varepsilon}{b - a}, \quad \text{being } x, x' \in [a, b] \qquad (1)$$

Let $P(x_0, x_1, \ldots, x_n)$ be a partition of $[a, b]$ such that $\Delta x_i < \delta$ $(i = 1, 2 \ldots n)$. Since $f$ is continuous, in each interval $[x_{i-1}, x_i]$ it will reach a maximum value $M_i$ and a minimum value $m_i$, such that

$$m_i = f(x_i'); \; M_i = f(x_i''), \text{ being } x_i', x_i'' \in [x_{i-1}, x_i]$$

Since $x_i - x_{i-1} = \Delta x_i < \delta$, we will also have that

$$x_i' - x_i'' < \Delta x_i < \delta$$

So, due to condition (**??**),

$$M_i - m_i = |M_i - m_i| = |f(x_i'') - f(x_i')| < \frac{\varepsilon}{b - a}$$

Therefore, using partition $P$, it results

$$S(P) - s(P) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i < \frac{\varepsilon}{b - a} \sum_{i=1}^{n} \Delta x_i = \frac{\varepsilon}{b - a}(b - a) = \varepsilon$$

and the condition of integrability is satisfied.

**c) Every piecewise continuous function on $[a, b]$ is integrable on $[a, b]$.**

We say that a function is piecewise continuous if it has a finite number of points of discontinuity, in they exist one-sided limits, and they are finite.

**Proof**: It can be seen in J. Burgos, pg. 296.