Introduction to changes of variable (06.07.2023)

1. **Explicit change of variable.**

   Consider a function $f : I \to \mathbb{R}$, continuous on $I = [a, b]$, whose primitive we want to obtain. Let $g : J \to \mathbb{R}$ be a function with continuous derivative and strictly monotone on $J = [c, d]$ (so that it admits inverse). If the image by g of the interval J is contained in I, then we can do the change of variable $x = g(t)$, resulting in

   $$\int f(x)\, dx = \int f\left(g(t)\right) g'(t)\, dt = \int H(t)\, dt\bigg|_{t=g^{-1}(x)}$$

   We will carry out the change if this integral is easier to solve than the initial one.

   Example. $\displaystyle\int \sqrt{1-x^2}\, dx$. Doing $x = \sin t$, the integral becomes $\displaystyle\int \cos^2 t\, dt$.

   Function $x = \sin t$ is strictly monotone on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ or on $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$.

2. **Implicit change of variable.**

   Sometimes it is useful to do $h(x) = t$ (or, equivalently, $x = h^{-1}(t)$), so $h'(x)\, dx = dt$.

   We write then the integral as

   $$\int f(x)\, dx = \int \frac{f(x)}{h'(x)}\, h'(x)\, dx,\ h'(x) \neq 0$$

   If $\dfrac{f(x)}{h'(x)}$ can be written in terms of t, the integral becomes $\displaystyle\int G(t)\, dt$.

   Function h must be, as before, strictly monotone and with a continuous derivative.

   Here again we will do the change if this integral is easier to solve than the initial one.

   Example. $\displaystyle\int x \cos x^2\, dx$. With $t = h(x) = x^2$, $h'(x) = 2x$, the integral becomes

   $$\int \frac{x \cos x^2}{2x}\, 2x\, dx = \int \frac{\cos t}{2}\, dt$$

   In practice there is no need to divide and multiply by $h'(x)$. The derivative of h is usually obtained by multiplying and dividing the integrand by an appropriate factor.

   $$\int x \cos x^2\, dx = \frac{1}{2} \int \cos x^2\, 2x\, dx \overset{x^2=t}{=} \frac{1}{2} \int \cos t\, dt$$

3. **Combination of both methods.**

   It is common to begin the change of variable as an implicit one by choosing $h(x)$, then solve for x $\left(x = h^{-1}(t)\right)$ and get dx from there.

   Example. $\displaystyle\int \frac{x\, dx}{2 - \sqrt[3]{x}}$. We do $2 - \sqrt[3]{x} = t \Longrightarrow x = (2-t)^3,\ dx = -3(2-t)^2 dt$.

   The integral becomes $\displaystyle\int \frac{-3(2-t)^5}{t}\, dt$.

Remark. We have imposed the condition of strict monotonicity, to be able to calculate the inverse function. But it is enough that this condition is satisfied piecewise: that is, that the interval J can be decomposed into subintervals, such that the condition is satisfied on all of them.

For example, the function $x = \sin t$ used in paragraph **1.** is not strictly monotone on its domain, but it is so on the indicated intervals of length π.

4. **Application.** It is proposed to solve the following cases by applying the suggested change. In the case **c)**, the exercise consists in modifying the integrand to be able to apply the change.

a) Explicit change.

1. $\displaystyle\int \sqrt{x^2+\alpha^2}\,dx \;\; (x = \alpha\sinh t)$. Sol: $\displaystyle\frac{x}{2}\sqrt{x^2+\alpha^2} + \frac{\alpha^2}{2}\ln\left|x+\sqrt{x^2+\alpha^2}\right| + C.$
2. $\displaystyle\int \frac{x^2}{\sqrt{x^2-\alpha^2}}\,dx \;\; (x = \alpha\cosh t)$. Sol: $\displaystyle\frac{x}{2}\sqrt{x^2-\alpha^2} + \frac{\alpha^2}{2}\ln\left|x+\sqrt{x^2-\alpha^2}\right| + C.$
3. $\displaystyle\int \frac{1}{x^2\sqrt{\alpha^2-x^2}}\,dx \;\; (x = \alpha\sin t)$. Sol: $\displaystyle -\frac{1}{\alpha^2}\frac{\sqrt{\alpha^2-x^2}}{x} + C.$

b) Implicit change.

1. $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx \;\; \left(1+\sqrt{x} = t\right)$. Sol: $2(1+\sqrt{x}) - 2\ln(1+\sqrt{x}) + C.$
2. $\displaystyle\int \frac{e^x}{(e^x+3)\sqrt{e^x-1}}\,dx \;\; \left(\sqrt{e^x-1} = t\right)$. Sol: $\displaystyle \arctan\frac{\sqrt{e^x-1}}{2} + C.$
3. $\displaystyle\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx \;\; \left(\sqrt{e^x+1} = t\right)$. Sol: $\displaystyle\frac{2}{3}\sqrt{(e^x+1)^3} - 2\sqrt{e^x+1} + C.$

c) Turn the integral into an immediate one, by using the suggested change.

1. $\displaystyle\int \frac{\sin x+\cos x}{\sqrt{\sin 2x}}\,dx \;\; (\sin x - \cos x = t)$. Sol: $\arcsin(\sin x - \cos x) + C.$
2. $\displaystyle\int \frac{x^2+1}{x\sqrt{-1+3x^2-x^4}}\,dx \;\; \left(x - \frac{1}{x} = t\right)$. Sol: $\displaystyle\arcsin\left(x-\frac{1}{x}\right) + C.$