Repeated application of L'Hôpital's rule (19.07.2022)

If, when applying L'Hôpital's rule to a quotient, another indeterminate limit results, the method can be repeated if the conditions required for it continue to be met.

Example. Obtain the limit: $\displaystyle \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\ln\left(1+x^3\right)}{x - \operatorname{sen} x}$.

a) It is a limit of type $0/0$, which satisfies the conditions for the application of L'Hôpital's rule, so we calculate the limit of the quotient of derivatives:

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{3x^2/\left(1+x^3\right)}{1-\cos x} \overset{\text{if }\exists}{=} \lim_{x\to 0} \frac{3x^2}{1-\cos x} \lim_{x\to 0} \frac{1}{1+x^3} = \lim_{x\to 0} \frac{3x^2}{1-\cos x} \cdot 1$$

which is still indeterminate, of type $0/0$.

b) The numerator and denominator are still derivable functions and the derivative of the denominator does not cancel outside the origin, so we derive again (with this we are **not** calculating the second derivatives of f and g, since we have simplified the limit of f'/g').

$$\lim_{x\to 0} \frac{\left(3x^2\right)'}{(1-\cos x)'} = \lim_{x\to 0} \frac{6x}{\operatorname{sen} x}$$

c) To solve this indeterminate limit we derive a third time.

$$\lim_{x\to 0} \frac{(6x)'}{(\operatorname{sen} x)'} = \lim_{x\to 0} \frac{6}{\cos x} = 6$$

d) Thus, according to L'Hôpital's rule,

$$6 = \lim_{x\to 0} \frac{6x}{\operatorname{sen} x} = \lim_{x\to 0} \frac{3x^2}{1-\cos x} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f(x)}{g(x)}$$

e) The above can be summarized as:

$$\lim_{x\to 0} \frac{f(x)}{g(x)} \overset{\text{if }\exists}{=} \lim_{x\to 0} \frac{f'(x)}{g'(x)} \overset{\text{if }\exists}{=} \lim_{x\to 0} \frac{3x^2}{1-\cos x} \overset{\text{if }\exists}{=} \lim_{x\to 0} \frac{6x}{\operatorname{sen} x} \overset{\text{if }\exists}{=} \lim_{x\to 0} \frac{6}{\cos x} = 6$$

Non existence of the limit of the quotient of derivatives

L'Hôpital's rule says that, under certain conditions, if the limit of the quotient of derivatives exists, the limit of the quotient of functions exists and has the same value. But the limit of the quotient of derivatives may not exist and the limit of the quotient of functions may exist.

Example. Let be the functions $g(x) = \operatorname{sen} x$, $f(x) = \begin{cases} x^2 \operatorname{sen}(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

a) We calculate the limit of f'/g' at the origin.

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \cdots = \lim_{x\to 0} \Big(-\cos(1/x) \Big), \text{ which does not exist}$$

b) However, obtaining the limit of f/g directly, we get:

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x^2 \operatorname{sen}(1/x)}{\operatorname{sen} x} = \lim_{x\to 0} \underbrace{\left(\frac{x}{\operatorname{sen} x}\right)}_{\to 1} \underbrace{\Big(x \operatorname{sen}(1/x)\Big)}_{\to 0} = 0$$