Between two real numbers there is a rational (16.07.2022)

The statement is proved below. From it follows that between each of the two real numbers and the rational between them (which is also real), there are two new rational numbers. Repeating the process again an again, we conclude that **between two reals there are infinite rational numbers**. This can also be expressed by saying that $\mathbb{Q}$ is dense on $\mathbb{R}$ (J. Burgos, pg. 58).

Proof: Let $x, y \in \mathbb{R},\ x < y$.

- Since $x < y \Longrightarrow y - x > 0$. We take $\dfrac{1}{y-x}$. The set $\mathbb{N}$ is unbounded, so we can ensure that

$$\exists n \in \mathbb{N} \ / \ n > \frac{1}{y-x}; \quad \text{that is } \underbrace{y - x > \frac{1}{n}}_{(1)}.$$

- Let $p \in \mathbb{Z}$ be the integer part of nx. Then

$$p \le nx < p + 1 \Longrightarrow \underbrace{\frac{p}{n} \le x}_{(2)} \text{ and } \underbrace{x < \frac{p+1}{n}}_{(3)}.$$

- Applying the relations (1), (2) and (3) it is satisfied

$$y = x + (y - x) \overset{(2+1)}{>} \frac{p}{n} + \frac{1}{n} = \frac{p+1}{n} \overset{(3)}{>} x$$

  That is,

$$\boxed{x < \frac{p+1}{n} < y}, \text{ being } \frac{p+1}{n} \text{ a rational number.}$$

Between two real numbers there is an irrational

The statement is proved below. From it follows that between each of the two real numbers and the irrational between them (which is also real), there are two new irrational numbers. Repeating the process again an again, we conclude that **between two reals there are infinite irrational numbers**. This can also be expressed by saying that $\mathbb{R} \setminus \mathbb{Q}$ is dense on $\mathbb{R}$ (J. Burgos, pg. 58).

Proof: Let $x, y \in \mathbb{R},\ x < y$.

- By the properties of the order relation it will be satisfied

$$\frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}.$$

- From the previous demostration

$$\exists r \in \mathbb{Q} \ / \ \frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$$

  from where

$$x < \sqrt{2}\, r < y, \text{ being } \sqrt{2}\, r \text{ an irrational number.}$$