General term of the sequence of integers (16.07.2022)

There are sequences whose general term has different expressions for even or odd n, for example:

$$\frac{1}{1},\ 2,\ \frac{1}{3},\ 4, \ldots$$

where a_n is equal to n, for even n and $1/n$, for odd n.

In these cases we may want to obtain a unique expression for the general term. To do this, from the successive powers of -1, we define the sequences $\{i_n\}$ and $\{p_n\}$:

$$\{(-1)^n,\ n \in \mathbb{N}\} = -1, 1, -1, 1, \ldots \Longrightarrow \begin{cases} \{i_n\} = \left\{ \dfrac{1-(-1)^n}{2} \right\} = 1, 0, 1, 0, 1, 0, \ldots \\ \{p_n\} = \left\{ \dfrac{1+(-1)^n}{2} \right\} = 0, 1, 0, 1, 0, 1, \ldots \end{cases}$$

Now we consider the sequence obtained by rearranging the integers

$$0,\ 1,\ -1,\ 2, -2\,, 3,\ -3, \ldots$$

The proposed exercise is:

a) Obtain the expressions of the general term for even and odd n.

b) Get a single expression for the general term.

Solution.

a) Observing the sequence of integers, we see that:

- For n even ($n = 2, 4, 6\ldots$), the terms are $1, 2, 3\ldots$, then $a_n = \dfrac{n}{2}$.

- For n odd, the terms are $0, -1, -2, \ldots$. Its value increases by -1 when n increases by 2. Then a_n will be of the form $k - n/2$. Since $a_1 = 0$

$$k - \frac{n}{2} \overset{n=1}{=} k - \frac{1}{2} = 0 \Longrightarrow k = \frac{1}{2} \Longrightarrow a_n = \frac{1-n}{2}$$

This can be stated in a more general way: for both even and odd n, the increments of the value of a_n are constant with each increment of n, so a_n will be of the form $\alpha\, n + \beta$. Substituting in each case n by its two first values, we get two equations with two unknowns, which give us the values of α and β.

b) To obtain an expression of a_n that is valid $\forall n$, we will use the sequences $\{i_n\}$ and $\{p_n\}$ defined at the beginning.

- If we multiply by i_n the expression of a_n for n odd, the terms of the resulting sequence will take the correct value for n odd and will be equal to 0 otherwise.

- If we multiply by p_n the expression of a_n for even n, the terms of the resulting sequence will take the correct value for even n and will be equal to 0 otherwise.

- Adding both expressions, the result will give us the value of $a_n\,\forall n$.

$$a_n = \left(\frac{1-(-1)^n}{2}\right)\frac{1-n}{2} + \left(\frac{1+(-1)^n}{2}\right)\frac{n}{2} = \frac{1-n-(-1)^n+(-1)^n n}{4} + \frac{n+(-1)^n n}{4} =$$

$$= \frac{1+(-1)^n(-1+n+n)}{4} = \frac{2n-1}{4}(-1)^n + \frac{1}{4}$$