Principle of induction: example (16.07.2022)

We want to prove the following property, which we will call $P(n)$: "The sum of the first n cubes of the natural numbers is equal to the square of the sum of these n numbers"

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

For this we apply the Principle of induction, that is:

a) We show that $P(1)$ is true: indeed $1^3 = 1^2$, then $P(1)$ holds.

b) We assume $P(k)$ is true: $1^3 + 2^3 + \cdots + k^3 = (1 + 2 + \cdots + k)^2$.

c) We prove $P(k+1)$ is true: $1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = (1 + 2 + \cdots + k + k + 1)^2$.

To do it we add $(k+1)^3$ to both members of the equality **b)** and operate on the second member, taking into account the expression for the sum of k consecutive natural numbers:

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = (1 + 2 + \cdots + k)^2 + (k+1)^3 =$$

$$\left[\frac{(1+k)}{2}\,k\right]^2 + (k+1)^3 = \frac{(k+1)^2k^2}{4} + (k+1)^3 = (k+1)^2\left[\frac{k^2}{4} + k + 1\right] =$$

$$(k+1)^2\left[\frac{k^2+4k+4}{4}\right] = \left[\frac{(k+1)(k+2)}{2}\right]^2 = (1 + 2 + \cdots + k + k + 1)^2.$$

Another approach to the proof (which in this case involves performing the same operations) is to write the first member of the equality **c)** that we want to prove and apply $P(k)$, replacing the sum of the k first cubes by the square of the sum of the k first natural. Operating as before, the desired equality is obtained.

Note. It is proposed to solve the exercise by proving $P(k+1)$ from right to left, starting from the second member of the equality to be proved.