INFINITESIMAL CALCULUS 1

COURSE NOTES

(with self-assesment exercises)

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Purpose of these notes

The aim of this text is to be an aid for the students of the subject Infinitesimal Calculus 1.

These notes contain the fundamental ideas presented in the theory classes, accompanied by solved examples and proposed exercises. By collecting an important part of the content of the sessions, they allow the students to dedicate more attention to the explanation, thus facilitating the understanding of the subject. The solved examples and proposed exercises also facilitate the subsequent personal study.

The notes are intended as a support to the lectures, but not as a substitute for them. In the classroom, the ideas contained herein are developed and commented, concepts are related, exercises are solved and some graphs are drawn to complement the notes. Therefore, class attendance is strongly recommended to facilitate the mastery of the subject. As far as possible, it is desirable to read the notes before the classes, which will help to know in advance the main aspects of the subject to be covered.

Contents

Unit I. The real number (14.12.2023)

1 Introduction

1.1 Necessary and sufficient condition

Implication. When from a fact (event, property...) A, we deduce another B, we say that "A implies B", which is denoted by the symbol $\Longrightarrow$.

$$\boxed{\text{A} \Longrightarrow \text{B}}$$

This is also stated as "if A, then B".

In this case, **A is a sufficient condition for B** (the occurrence of A is sufficient for the occurrence of B). And **B is a necessary condition for A** (B is necessarily deduced from A).

Example. In the sentence "if I step on a light bulb, it breaks", we assert that 1) the fact of stepping on the bulb is sufficient for it to break, so it is a sufficient condition; and 2) that the breakage necessarily occurs as a consequence, so it is a necessary condition. But it is not necessary to step on the bulb for it to break, because it can break for another reason (e.g. a hammer blow); and the fact that the bulb breaks is not sufficient to ensure that we have stepped on it, for the same reason. That is to say, in the relation

$$\text{If I step on the light bulb} \Longrightarrow \text{it breaks}$$

"stepping on the bulb" is a sufficient condition for breaking, but not a necessary condition; and "breaking" is a necessary condition of stepping on the bulb, but not a sufficient condition.

Equivalence. If "A implies B" and "B implies A" are given at the same time, then A is a necessary and sufficient condition of B and B is a necessary and sufficient condition of A. In this case we say that "A is equivalent to B" (or "B is equivalent to A") and is denoted by

$$\boxed{\text{A} \Longleftrightarrow \text{B}}$$

This is also stated as "A if and only if B".

Example. If the only way to pass a subject is to obtain a mark N greater than or equal to 6 in the coursework, then "$N \geq 6$" is the necessary and sufficient condition to pass. That is to say, if $N \geq 6$, you pass. And if you pass, it means that $N \geq 6$.

Opposite proposition. Given a proposition A, its opposite represents the complementary case or event to proposition A and is denoted by not(A).

Example. Given the numbers x and y, if proposition A is "x is greater than y", its opposite, not(A), is "x is less than or equal to y".

$$\big[\text{not}(x > y)\big] = \big[x \leq y\big]$$

If not(A) is the opposite proposition of A, it is immediate to see that the opposite proposition of not(A) will be A.

$$\boxed{\text{not}\big(\text{not(A)}\big) = \text{A}}$$

In the previous example, the opposite proposition of not(A) ("x is less than or equal to y") is "x is greater than y", i.e. A.

Relation between opposite propositions. If between A and B there is an implication, between their opposites there is an implication in the opposite direction and vice versa, i.e.

$$\boxed{\big[\text{A} \Longrightarrow \text{B}\big] \Longleftrightarrow \big[\text{not(B)} \Longrightarrow \text{not(A)}\big]}$$

Proof. (We reason it only from left to right).
If $\big[\text{A} \Longrightarrow \text{B}\big]$, then $\big[\text{not(B)} \Longrightarrow \text{not(A)}\big]$. Indeed, if not(A) does not follow from not(B), it means that A could happen. And if A occurs, B occurs, and we would have both not(B) and B, which does not make sense (something and its opposite cannot occur at the same time).

Example. If obtaining a mark N greater than or equal to 5 in the exam means passing a subject, then not passing the subject means that you have not obtained that mark, because if you had obtained it, you would have passed.
Note that the opposite of "obtaining a mark N greater than or equal to 5" is not "N is less than 5", but "not obtaining that mark". This may be due to getting a lower mark or no mark at all, e.g. due to not taking the exam.

1.2 Proof by *reductio ad absurdum* (reduction to the absurd)

To prove a property P by this method, we assume the opposite of P and reason from it. If we arrive at a contradiction (an absurd), we have proved the property. It is particularly useful when we have to prove something and we do not know from where to start.

Proof. We want to prove P. To do so, we assume not(P) and -reasoning- we arrive at R (false). From the relation between opposite propositions in **1.1**, we have

$$\big[\text{not(P)} \Longrightarrow \text{R}\big] \Longleftrightarrow \big[\text{not(R)} \Longrightarrow \text{not}\big(\text{not(P)}\big) = \text{P}\big]$$

that is, our reasoning - getting R (false) from not(P) - is equivalent to getting P from not(R) (true). So we have proved property P, as we wanted.

2 Successive extensions of the concept of number

We are going to review very briefly the different types of numbers (natural, integer, rational), which are assumed to be known. We will show that each new set gives a solution to an operation that did not exist in the previous set. This will be useful in the introduction of the real numbers.

2.1 Natural numbers

We designate with the symbol $\mathbb{N}$ the set of natural numbers, which allows us to count and enumerate things.

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

We see that it has an infinite number of elements and that it does not contain 0 (there are texts that include 0 among the naturals, but we will not do so here). We also observe that the sum and the product of elements of $\mathbb{N}$ are also elements of $\mathbb{N}$, that is to say that $+$ and $\cdot$ are "closed" operations.

Subsets of $\mathbb{N}$. The set $\mathbb{N}$ has subsets such as even numbers, multiples of 3 or powers of 5. We represent these subsets respectively as

$$\{2n\}_{n\in\mathbb{N}}\ , \quad \{3n\}_{n\in\mathbb{N}}\ , \quad \{5^n\}_{n\in\mathbb{N}}$$

and its elements are obtained by giving natural values to n in the expression between braces.

Comparing $\mathbb{N}$ with the set of even numbers, we can ask the question whether there are more naturals than even numbers. On the one hand it is clear that there is one even number for every two naturals. On the other hand, we can assign an even number to every natural, so that for every natural there is a even number and vice versa.

$$1 \longleftrightarrow 2,\ 2 \longleftrightarrow 4,\ 3 \longleftrightarrow 6, \ldots,\ n \longleftrightarrow 2n, \ldots$$

in which case it seems that there are the same number of even numbers as there are naturals. The conclusion from this apparent contradiction is that, being infinite sets, it is inadequate to say that they have more or fewer elements. What we can say is that there is a bijection between the two sets (a one to one relation between their elements) or that they have the same **cardinality**. The same happens between $\mathbb{N}$ and the multiples of 3 or the powers of 5, already mentioned.

Infinite set condition. We have just seen that $\mathbb{N}$ is bijective with a subset of it distinct from itself (proper subset). It can be proved that this is a sufficient condition for a set to be infinite.

2.2 Integer numbers

If, in the set of naturals, we try to solve the equation $3 + x = 8$, the solution is the natural number $x = 5$. But, if the equation is $8 + x = 3$, we see that no natural number satisfies it, which leads us to the concept of negative number and to the necessity of the extension of $\mathbb{N}$ to obtain the set of integers $\mathbb{Z}$, that is

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

In addition to including the neutral element 0, for every natural number n we have added its opposite (the element that added to n results in 0),

$$\forall n \in \mathbb{N}\ \exists\, (-n) \in \mathbb{Z} \,/\, n + (-n) = 0$$

which allows us to define the difference as **the sum of the opposite**. It is obvious that the naturals are a subset of the integers

$$\mathbb{N} \subset \mathbb{Z}$$

and we can describe $\mathbb{Z}$ as **an extension of $\mathbb{N}$ which allows us to define the subtraction**.

2.3 Rational numbers

If we now try to solve the equation $3x = 12$, the solution is the integer $x = 4$. But, if the equation is $12x = 3$, we see that there is no solution in $\mathbb{Z}$. This leads us to the concept of fractional number and to the need to extend $\mathbb{Z}$ to obtain the set of rationals $\mathbb{Q}$:

$$\mathbb{Q} = \left\{ \frac{p}{q},\ \text{being } p \in \mathbb{Z},\ q \in \mathbb{N} \right\}$$

where p/q and p'/q' represent the same number if $pq' = p'q$.

The operations of addition and product of fractions are known. And now we can find the inverse of any non-zero element (the product of an element by its inverse is 1).

$$\forall \frac{p}{q} \in \mathbb{Q}\ (p \neq 0)\ \exists \left(\frac{p}{q}\right)^{-1} \Big/ \ \frac{p}{q}\left(\frac{p}{q}\right)^{-1} = 1$$

which allows us to define the quotient as the product by the inverse.

It is easy to see that rational numbers of the form $p/1, p \in \mathbb{Z}$ behave like the integers, which allows us to say that the integers are a subset of the rationals.

$$\mathbb{Z} \subset \mathbb{Q}$$

and we can describe $\mathbb{Q}$ as **an extension of $\mathbb{Z}$ which allows us to define the quotient**.

2.4 Countable sets. Principle of induction

Definition. A set is countable if it is bijective with $\mathbb{N}$ or a subset of $\mathbb{N}$.

Examples. By the given definition, every subset of $\mathbb{N}$ is countable. It is also easy to show that $\mathbb{Z}$ is countable. To do so, it suffices to take first 0 and then positive and negative integers alternatively.

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

Thus, every integer appears in the list only once and we can find out the position of any of them. From this, the value of the integer that occupies a certain position, odd or even, can be deduced. It is useful, but not so easy, to obtain a single expression for the number in position n (see the supplementary document "General term for the sequence of integers").

Principle of induction. To prove a property P of the elements of a countable set, we can proceed as follows:

1) We check that the property is satisfied for the first element: $P(1)$ is true.

2) We prove that, if the property is satisfied for $n = k$, then it is satisfied for $n = k + 1$.

$$P(k) \text{ is true} \Longrightarrow P(k+1) \text{ is true}$$

From the above, we can assure that the property holds $\forall n \in \mathbb{N}$.

$$\boxed{1) + 2) \Longrightarrow P(n) \text{ is true } \forall n \in \mathbb{N}}$$

Example. We have to prove the formula for the sum of the n first natural numbers n:

$$1 + 2 + \cdots + n = \frac{n\,(n+1)}{2} \quad \forall n \in \mathbb{N}$$

1) We check it for $n = 1$.

$$1 = \frac{1\,(1+1)}{2} \Longrightarrow P(1) \text{ is true}$$

2) We assume that the formula is true for $n = k$

$$1 + 2 + \cdots + k = \frac{k\,(k+1)}{2}$$

To show that this is true for $n = k + 1$, we add $(k + 1)$ to the left-hand side and operate, taking into account the above equality.

$$1 + 2 + \cdots + k + (k+1) = \frac{k\,(k+1)}{2} + (k+1) = (k+1)\left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2}$$

which corresponds to the expression we have to prove, for $n = k+1$, i.e. $P(k+1)$, so that we have shown that the property is satisfied $\forall n \in \mathbb{N}$.

Exercises. Prove the following formulas:

1. The sum of the squares of the n first naturals: $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

2. The sum of the cubes of the n first naturals: $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$

3 Field structure. Ordered field

3.1 Definition of field

Let K be a set of two or more elements, in which two internal operations $+$ and $\cdot$ (addition or sum and multiplication or product) are defined. The set K has a field structure if and only if:

a) With respect to $+$, the properties associative, commutative, existence of the additive identity and the additive inverse are satisfied. That is, if x, y, z are any elements of K, then:

$$x+(y+z) = (x+y)+z; \quad x+y = y+x; \quad \exists\, 0 \in K /\, \forall x,\ x+0 = x; \quad \exists\,(-x) \in K /\, x+(-x) = 0$$

b) With respect to $\cdot$, the properties associative, commutative, existence of the multiplicative identity and the multiplicative inverse are satisfied. If x, y, z are any elements of K, then:

$$x\cdot(y\cdot z) = (x\cdot y)\cdot z; \quad x\cdot y = y\cdot x; \quad \exists\, 1 \in K /\, \forall x,\ x\cdot 1 = x; \quad \forall x \neq 0, \exists\, x^{-1} \in K /\, x\cdot\left(x^{-1}\right) = 1$$

c) The operation $\cdot$ is distributive with respect to $+$. That is, $\forall x, y, z \in K$ it is satisfied:

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

Example. It is easy to prove that $\mathbb{N}$ and $\mathbb{Z}$ do not have a field structure, but $\mathbb{Q}$ does.

3.2 Order

Given a set K, an order on K is a relation R between its elements which satisfies the following properties $\forall a, b, c \in K$:

a) Reflexive. Every element is related to itself: aRa.

b) Antisymmetric. If a is related to b and b to a, they coincide:

$$\left.\begin{matrix} a\,R\,b \\ b\,R\,a \end{matrix}\right\} \Longrightarrow a = b$$

c) Transitive. If a is related to b and b to c, a will be related to c:

$$\left.\begin{matrix} a\,R\,b \\ b\,R\,c \end{matrix}\right\} \Longrightarrow a\,R\,c$$

Order relations are usually denoted by the symbol $\leq$. We also use the symbol $\geq$:

$$a \geq b \Longleftrightarrow b \leq a$$

If a is related to b, but they are distinct, we write $a < b$ (strict order relation):

$$a \leq b,\ a \neq b \Longrightarrow a < b$$

Total order. The order is total if any two elements are related:

$$\forall a, b \in K \Longrightarrow a \leq b \ \text{ or } \ b \leq a$$

Otherwise, it is a partial order.

Compatible order. We say that the order is

- Compatible with the sum, if and only if:

$$a \leq b \Longrightarrow a + c \leq b + c, \, \forall c \in K$$

- Compatible with the product, if and only if:

$$a \leq b \Longrightarrow a \cdot c \leq b \cdot c, \, \forall c \in K, c > 0$$

Example 1. The relation $\leq$ between rationals is a total order, compatible with the sum and the product. It can be defined as

$$\text{Let } p, q \in \mathbb{Q} : \boxed{p \leq q \Longleftrightarrow \exists r \in \mathbb{Q}, \, r \geq 0 \big/ p + r = q}$$

Example 2. The inclusion relation between sets is a partial order. It is denoted by $\subset$.

3.3 Ordered field

Definition. It is a field together with a total ordering of its elements that is compatible with the operations sum and product.

Properties. An ordered field possesses the following properties. The first three and **e)** are deduced from the definition of an ordered field; **f)** is proved from **e)**; **g)** and **h)** are proved from **f)**; several properties are used to prove **d)**.

a) $\boxed{a \leq b, \; c \leq d \Longrightarrow a + c \leq b + d}$

P: $a \leq b \Longrightarrow a + c \leq b + c; \quad c \leq d \Longrightarrow b + c \leq b + d.$ Then $a + c \leq b + d.$

b) $\boxed{a \leq 0 \Longrightarrow -a \geq 0}$

P: $a \leq 0 \Longrightarrow a + (-a) \leq 0 + (-a) \Longrightarrow 0 \leq -a \Longrightarrow -a \geq 0.$

c) $\boxed{a \leq b \Longrightarrow -a \geq -b}$

P: It is proposed as an exercise (see prop. **b**).

d) $\boxed{a \leq b, \; c < 0 \Longrightarrow ac \geq bc}$

P: It is proved from **b)**, **c)**, **f)** and the definition of ordered field.

e) $\boxed{a \cdot 0 = 0}$

P: $a\,b = a(b + 0) = a\,b + a \cdot 0 \Longrightarrow \overbrace{-(a\,b) + (a\,b)}^{0} = \overbrace{-(a\,b) + (a\,b)}^{0} + a \cdot 0 \Longrightarrow a \cdot 0 = 0.$

f) $\boxed{a(-1) = -a}$

P: $0 = a \cdot 0 = a(-1 + 1) = a(-1) + a \cdot 1 \Longrightarrow 0 = a(-1) + a \Longrightarrow a(-1) = -a.$

g) $\boxed{(-a)(-b) = ab}$

P: It is proposed as an exercise (see prop. **f**).

h) $\boxed{-a - b = -(a + b)}$

P: It is proposed as an exercise (see prop. **f**).

3.4 Bounds and intervals

Let A be a set and $\leq$ an order relation defined on A. Let $D \subset A$. We say that:

a) The element $M \in A$ is an **upper bound** of D if

$$M \geq x, \, \forall x \in D$$

in which case D is said to be bounded from above.

The least of the upper bounds (if it exists) is the **supremum**.

If the supremum belongs to D, it is called the **maximum**.

b) The element $m \in A$ is a **lower bound** of D if

$$m \leq x, \, \forall x \in D$$

in which case D is said to be bounded from below.

The greatest of the lower bounds (if it exists) is the **infimum**.

If the infimum belongs to D, it is called the **minimum**.

c) **Intervals.** In the set of rational numbers we define closed interval with endpoints a and b as:

$$[a, b] = \left\{ x \in \mathbb{Q} \,/\, a \leq x \leq b \right\}$$

Exercises.

1. Define the intervals (a, b), $[a, b)$, $(a, b]$, obtaining for each of them an upper and a lower bound, as well as the supremum, infimum, maximum and minimum.
2. Prove that the supremum, if it exists, is unique.

3.5 Absolute value

Definition. Let K be an ordered field and $K^+ = \left\{ x \in K \,/\, x > 0 \right\}$. We call absolute value to the following application of K to $K^+ \cup \{0\}$:

$$| \, | : K \to K^+ \cup \{0\} \,/\, |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Equivalent definition. The following definition is equivalent to the one given above. It is useful in some demonstrations.

$$|x| = \max \{x, -x\}$$

P: We will only prove the implication from left to right, i.e.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \implies |x| = \max \{x, -x\}$$

a) Let $x \geq 0$. From the definition of absolute value and property b) of an ordered field:

$$|x| = x; \quad -x \leq 0$$

Therefore

$$|x| = x \geq 0 \geq -x \implies |x| = \max \{x, -x\}$$

b) Let $x < 0$. From the definition of absolute value and property b) of an ordered field:

$$|x| = -x; \quad -x > 0$$

Therefore

$$|x| = -x > 0 > x \implies |x| = \max \{x, -x\}$$

Exercise. Prove the implication from right to left. That is,

$$|x| = \max\{x, -x\} \implies |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Properties of the absolut value.

a) $\boxed{|x| > 0,\ \forall x \neq 0;\ |0| = 0}$

P: If $x > 0 \implies |x| = x > 0$. If $x < 0 \implies |x| = -x > 0$. Then, $\forall x \neq 0, |x| > 0$.

If $x = 0 \implies |x| = x = 0$.

b) $\boxed{|x| < y \iff -y < x < y}$

P: Using the second definition of absolute value

$$|x| = \max\{x, -x\} \implies \begin{cases} |x| \geq x, \\ |x| \geq -x \end{cases}$$

Therefore:

$$\left.\begin{array}{r} y > |x| \geq x \implies x < y \\ y > |x| \geq -x \implies -y < x \end{array}\right\} \implies -y < x < y$$

c) $\boxed{|x\,y| = |x|\,|y|}$, from which, if $x \neq 0$, $\boxed{\left|x^{-1}\right| = |x|^{-1}}$

P: The first equality is proved using the properties of an ordered field in the four possible cases: $x, y > 0$; $x > 0, y < 0$; $x < 0, y > 0$; $x, y < 0$.

We prove the second one, $|x^{-1}| = |x|^{-1}$.

$$1 = |1| = \left|x\,x^{-1}\right| = |x|\left|x^{-1}\right| \implies \left|x^{-1}\right| = |x|^{-1}$$

d) $\boxed{|x + y| \leq |x| + |y|}$, from which $\boxed{|x - y| \geq ||x| - |y||}$

P: $|x| \geq x$, $|x| \geq -x$. Also $|y| \geq y$, $|y| \geq -y$. Therefore

$$|x| + |y| \geq x + y; \quad |x| + |y| \geq -x - y = -(x + y)$$

If $|x| + |y|$ is greater than or equal to $x + y$ and $-(x + y)$, it will be greater than or equal to the greater of the two, that is, $|x| + |y| \geq |x + y|$.

To prove the second inequality, we propose to follow the following method, doing the same with y and concluding.

$$|x| = |x - y + y| \leq |x - y| + |y| \implies |x - y| \geq |x| - |y|$$

Exercise. Find the values of x that are solutions of the inequation $|x| > k \in \mathbb{R}$ (it is solved in the self-assessment question of the unit). **Sol:** $x \in (-\infty, k) \cup (k, \infty)$.

4 Sequences on $\mathbb{Q}$

We will now give some basic ideas of sequences, which will be treated in more depth in unit III. The sequences will be useful when studying the properties and also the "gaps" or "missing points" of the set $\mathbb{Q}$, which will lead us to define the real numbers.

4.1 Definition

Informally, we can define a sequence on $\mathbb{Q}$ (rational sequence) as an ordered set of infinitely many rationals. Its terms are obtained by giving values to n in the general term $a_n \in \mathbb{Q}$ and the sequence is represented by its general term between braces:

$$\{a_n\}_{n\in\mathbb{N}} = a_1, a_2, a_3, \ldots$$

A particular case of interest is that of monotone sequences, which can be of two types, depending on the relation between each term and the previous one.

1. Monotone increasing, if $a_{n+1} \geq a_n \ \forall n \in \mathbb{N}$.
2. Monotone decreasing, if $a_{n+1} \leq a_n \ \forall n \in \mathbb{N}$.

In the case that each term is strictly greater than the previous one (or strictly less), the sequence is called strictly monotone. They can be:

1. Strictly increasing, if $a_{n+1} > a_n \ \forall n \in \mathbb{N}$.
2. Strictly decreasing, if $a_{n+1} < a_n \ \forall n \in \mathbb{N}$.

On the other hand, a sequence is bounded if its terms in absolute value do not exceed a certain value $k > 0$, i.e.

$$\exists k > 0 \big/ \ |a_n| \leq k \ \ \forall n \in \mathbb{N}$$

which is equivalent to say that they are between k and $-k$.

4.2 Convergent sequence

A sequence is convergent in $\mathbb{Q}$ if it has a finite limit, i.e. the elements of the sequence become closer and closer to some rational number called the limit of the sequence, taking sufficiently advanced values of n. This condition is expressed as

$$\lim_{n\to\infty} a_n = \alpha \Longleftrightarrow \forall \varepsilon > 0 \, \exists n \in \mathbb{N} \big/ \ |a_m - \alpha| < \varepsilon \ \forall m \geq n$$

Example. The sequence whose general term is a_n

$$a_n = \frac{2n+3}{n+1} \quad \left(= \frac{5}{2}, \frac{7}{3}, \frac{9}{4}, \frac{11}{5}, \ldots\right)$$

has a limit $L = 2$. This can be justified, for example, by decomposing a_n and noting that the second addend becomes as small as we want, by choosing the value of n

$$a_n = \frac{2n+3}{n+1} = \frac{2(n+1)+1}{n+1} = 2 + \frac{1}{n+1} \longrightarrow 2$$

In unit III we will study more general methods for the calculation of limits.

4.3 Cauchy sequences

A Cauchy sequence is a sequence whose terms become arbitrarily close together taking sufficiently advanced values of n. This condition is expressed as

$$\{a_n\} \text{ is a Cauchy seq. } \iff \forall\varepsilon > 0\, \exists n \in \mathbb{N} \big/ \ |a_p - a_q| < \varepsilon \ \forall p, q \geq n$$

Relation. Every convergent sequence is a Cauchy sequence.

Proof. We start from the definition of a convergent sequence, using $\varepsilon/2$

$$\lim_{n\to\infty} a_n = \alpha \Longrightarrow \forall\varepsilon > 0\, \exists n \in \mathbb{N} \big/ \ |a_m - \alpha| < \frac{\varepsilon}{2} \ \forall m \geq n$$

Thus, $\forall p, q \geq n$ we have

$$\left.\begin{aligned} |a_p - \alpha| < \varepsilon/2 \\ |a_q - \alpha| < \varepsilon/2 \end{aligned}\right\} \Longrightarrow |a_p - a_q| = |a_p - \alpha + \alpha - a_q| \leq |a_p - \alpha| + |\alpha - a_q| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and the two terms become as close to each other as we want.

5 Properties of $\mathbb{Q}$

5.1 $\mathbb{Q}$ is an ordered field

The set of rationals has a field structure (see **3.1**) and the relation $\leq$ defined in **3.2** satisfies the properties of a total order, compatible with $+$ and $\cdot$, so $\mathbb{Q}$ is an ordered field.

5.2 The order on $\mathbb{Q}$ is dense

We say that the order in a set is dense if, between two distinct elements in order relation, there exists another one, that is to say

$$\forall a, b \in \mathbb{Q},\ a < b,\ \exists c \in \mathbb{Q} \big/ \, a < c < b$$

To justify that the order on $\mathbb{Q}$ is dense, it suffices to find, for any pair of elements, an intermediate one c that satisfies the above condition. The simplest is the half-sum of both.

Proof. Let $a < b$. Since the order is compatible with the sum and the product,

$$a < b \Longrightarrow \left\{\begin{aligned} a + a < a + b \\ a + b < b + b \end{aligned}\right\} \Longrightarrow 2a < a + b < 2b \overset{\cdot\frac{1}{2}}{\Longrightarrow} a < \frac{a+b}{2} < b$$

Consequences.

1. Between the rationals a and b there is (at least) another c, so between a and c there will be a second one, c_1, and between c and b a third one, c_2. Between each two consecutive numbers of the previous ones there will be a new rational and so on. Since we can repeat the process indefinitely, we conclude that **between two distinct rationals there are infinitely many rationals**.

2. A second consequence of the dense order is that **there is no next rational to a given one** (on the real axis). If, given p, we could find the rational q immediately following p, being $p < q$ there would be infinitely many rationals between them (see 1.), then it is proved by *reductio ad absurdum* that there is no rational following p.

Exercise. Reason whether the usual order in $\mathbb{Z}$ is dense.

5.3 $\mathbb{Q}$ is a countable set

In **2.4** we proved that $\mathbb{Z}$ is countable, by starting with zero and taking alternatively positive and negative integers. But we know that between every two integers there are an infinite number of rationals, so it is not so immediate to prove the countability of $\mathbb{Q}$. For simplicity we will consider only positive rationals. First we build a table in which all of them appear.

$$
\begin{array}{ccccc}
\frac{1}{1}, & \frac{2}{1}, & \frac{3}{1}, & \frac{4}{1}, & \dots \\
\frac{1}{2}, & \frac{2}{2}, & \frac{3}{2}, & \frac{4}{2}, & \dots \\
\frac{1}{3}, & \frac{2}{3}, & \frac{3}{3}, & \frac{4}{3}, & \dots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Every element of the table corresponds to a rational number and any rational p/q has its place in row q, column p. Now, to put them in order we proceed by diagonals: the first one contains $1/1$, the second one $1/2$ and $2/1$, and so on. So the positive rationals are ordered as follows

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{1}{4}, \dots$$

Thus, each rational number has a unique place in the list and we have proved that the set of rationals is bijective with $\mathbb{N}$, therefore countable.

Exercise. Calculate the position occupied by $\frac{p}{q}$. **Sol:** $n_{p,q} = p + \frac{(p+q-1)(p+q-2)}{2}$

5.4 Decimal expression of a rational number

Any rational can be expressed in decimal form, taking one of the following representations:

a) Terminating decimal: all its digits are zero, except a finite number, e.g.: $7/2 = 3.5$

b) Repeating decimal. The decimal part has an infinite number of non-zero digits that can become periodic 1) just after the decimal dot or 2) after some non-periodic digits.

e.g.: b.1) $7/3 = 2.33333\ldots = 2.\widehat{3}$; b.2) $7/6 = 1.16666\ldots = 1.1\widehat{6}$

Then we can define a rational number as the limit of a monotone increasing sequence. For example, for $7/6$,

$$a_0 = 1,\ a_1 = 1.1,\ a_2 = 1.16,\ a_3 = 1.166,\ a_4 = 1.1666, \ldots \implies 7/6 = \lim_{n\to\infty} a_n$$

6 Extension of $\mathbb{Q}$. The real numbers

As we have just seen, the set $\mathbb{Q}$ solves different problems, such as the division of integers, but it has also some "shortcomings". We point out three, which will be justified below:

1. There are points on the line (x-axis) that do not correspond to a rational number.
2. There exist non-empty, upper bounded subsets of $\mathbb{Q}$, which do not have a supremum.
3. There exist Cauchy sequences without limit on $\mathbb{Q}$.

Before we start, we will prove that **the square root of 2 is not rational**.

We say that b is the square root of a number a if the square of b is a, i.e.

$$\sqrt{a} = b \Longleftrightarrow b^2 = a$$

For example, the square root of 4 is ± 2, so there are two rationals whose square is 4. However, as we will see by *reductio ad absurdum*, there is no rational number whose square is equal to 2, so $\sqrt{2}$ is not rational.

Suppose that $\sqrt{2}$ is rational, i.e. it can be written as $p/q,\ p, q \in \mathbb{N}$, where p/q is an irreducible fraction. It turns out:

$$\frac{p}{q} = \sqrt{2} \Longrightarrow \left(\frac{p}{q}\right)^2 = 2 \Longrightarrow p^2 = 2q^2 \Longrightarrow p^2 \text{ is even}$$

then (as proved below) p is even and can be expressed as $p = 2m,\ m \in \mathbb{N}$. Substituting

$$p^2 = 2q^2 \overset{p=2m}{\Longrightarrow} p^2 = 4m^2 = 2q^2 \Longrightarrow q^2 = 2m^2$$

then q^2 is even, so q is even and p/q is not an irreducible fraction as we had supposed.

Remark. If the square p^2 of a natural number is even, p is also even. Otherwise it would be odd, so its square would also be odd. Indeed,

$$p = 2n - 1,\ n \in \mathbb{N} \Longrightarrow p^2 = 4n^2 - 4n + 1$$

Since $4n^2$ and $4n$ are even, p^2 would be odd, contrary to the hypothesis.

Justification of the 'shortcomings' of the set $\mathbb{Q}$. Let us analyse the previous statements.

1. There are points on the line (x-axis) that do not correspond to a rational number.

   For example, consider a right triangle with legs of length equal to 1, so that one of them lies on OX. We calculate the length of the hypotenuse:

   $$l^2 = 1^2 + 1^2 \Longrightarrow l = \sqrt{1^2 + 1^2} = \sqrt{2}$$

   On the other hand, taking this length on the x-axis, its end determines a point on the OX-axis. But $\sqrt{2}$ is not a rational number as has been shown, which means that there are points on the line that do not correspond to rational numbers.

2. There are non-empty, upper bounded subsets of $\mathbb{Q}$ which have no supremum.

   For example, if we consider the set A

   $$\mathrm{A} = \left\{x \in \mathbb{Q} \,/\, x^2 < 4\right\} = \left\{x \in \mathbb{Q} \,/\, |x| < 2\right\} \Longrightarrow \sup \mathrm{A} = 2$$

   we see that it has a supremum on $\mathbb{Q}$. If we now define the set B

   $$\mathrm{B} = \left\{x \in \mathbb{Q} \,/\, x^2 < 2\right\} = \left\{x \in \mathbb{Q} \,/\, |x| < \sqrt{2}\right\} \Longrightarrow \sup \mathrm{B} = \sqrt{2}$$

   it turns out that its supremum is $\sqrt{2}$, which is not rational, so B has no supremum on $\mathbb{Q}$.

3. There exist Cauchy sequences without limit on $\mathbb{Q}$.

   Let us consider the sequence of the default square roots of $\sqrt{2} = 1.4142135\ldots$, i.e.

   $$a_0 = 1,\ a_1 = 1.4,\ a_2 = 1.41,\ a_3 = 1.414,\ a_4 = 1.4142,\ a_5 = 1.41421, \ldots$$

   In this sequence we see that:

- Each term has one more decimal place than the previous one and a_n has n decimal places.

- The difference between a_n and any of the following ones is less than 10^{-n}, so it can be made as small as we want by taking n large enough. Therefore it is a Cauchy sequence.

- On the other hand, $\sqrt{2}$ lies between each default approximation and the corresponding excess approximation.

$$a_n < \sqrt{2} < a_n + \frac{1}{10^n} \Longrightarrow \boxed{\sqrt{2} - a_n < \frac{1}{10^n}}$$

so the difference between $\sqrt{2}$ and a_n will be less than 10^{-n}. Then the terms can approach $\sqrt{2}$ as close as we like and we have found a Cauchy sequence whose limit is $\sqrt{2}$, therefore without limit on $\mathbb{Q}$.

One possible way of defining irrational numbers. We have seen, then, that the square root of 2 is not a rational number, but it can be approximated as much as we like by a bounded monotone increasing sequence of rationals. This suggests that we could define $\sqrt{2}$ as the limit of that sequence (any other non-rational number, such as $e, \pi \ldots$, can also be approximated by an analogous sequence and so be defined as the limit of that sequence).

It then turns out that a bounded monotone increasing sequence of rationals will have a rational limit (as 7/6, in **5.4**) or a limit which is not rational (e.g. $\sqrt{2}, e, \pi, \ldots$). We can then assign an irrational (non-rational) number as a limit to these sequences which do not have it on $\mathbb{Q}$. And extending $\mathbb{Q}$ with the irrationals, we obtain the set $\mathbb{R}$ of the reals.
In this way, every bounded monotone increasing sequence of rationals will have a limit on $\mathbb{R}$, either rational or irrational.

Obviously the above is only an intuitive approximation to the real numbers. For a rigorous definition we would have to define them, for example, as the limit of bounded monotone increasing sequences, group the sequences that have the same limit, define the operations between these groupings and check that certain properties are satisfied.

Different mathematicians have followed this system, justifying the set $\mathbb{R}$ in different ways, such as Dedekind (method of cuts) or Cantor (method of Cauchy sequences). Depending on the method used, the fundamental property that characterises the set $\mathbb{R}$ (the property of **completeness**) is stated in different ways. Two of the most common are:

1. In $\mathbb{R}$, every non-empty upper bounded set has a supremum.
2. In $\mathbb{R}$, every Cauchy sequence has a limit.

7 Properties of $\mathbb{R}$

7.1 $\mathbb{R}$ is an ordered field

As with $\mathbb{Q}$, the set of real numbers, endowed with the sum, the product and the order relation $\leq$, is an ordered field, since its elements satisfy the corresponding properties.

7.2 $\mathbb{R}$ is an Archimedean field

The real numbers (as well as $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$) satisfy the Archimedean property, that is to say

$$\forall a, b \in \mathbb{R},\ a > 0\ \exists n \in \mathbb{N} \big/ na > b$$

This means that, given an $a > 0$, the sequence $a, 2a, 3a, 3a, \ldots$ is not bounded, since there is a n sufficiently large so that na exceeds b, for all b.

7.3 $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense on $\mathbb{R}$

In **5.2** we said that the order on $\mathbb{Q}$ is dense, since between two rationals there is always another one. On $\mathbb{R}$ the same thing happens; moreover, it can be proved that between two reals there is a rational and an irrational. Then it turns out:

1. Between two reals there is a rational. Since the rational is real, by repeatedly applying the property we conclude that between two reals there are infinitely many rationals. This is also expressed by saying that **the set of rationals is dense on** $\mathbb{R}$.

2. Between two reals there is an irrational (which is also a real), so by the above reasoning there are infinitely many irrationals. This is also expressed by saying that **the set of irrationals is dense on** $\mathbb{R}$.

7.4 $\mathbb{R}$ is complete

The completeness property has already been stated in two ways (see section **6**). A third way, equivalent to the previous ones, is that **there is a bijection between the real numbers and the points on the real line**.

7.5 $\mathbb{R}$ is not countable

As we have seen in the previous sections, $\mathbb{R}$ usually possesses the properties of $\mathbb{Q}$ (and some more). But there is a property of the rationals -countability- that is lost when we add the irrationals to the rationals to give rise to the reals. The reason is that the cardinal of $\mathbb{R} \setminus \mathbb{Q}$ is an infinite of higher order than that of $\mathbb{Q}$.

Proof. Let us see by *reductio ad absurdum* that the subset of the reals formed by the interval $(0, 1)$ is not countable. We start by assuming that it is countable, which means that we can write in an ordered list all the real numbers between 0 and 1, for instance

1. $0.a_{11}\, a_{12}\, a_{13}\, a_{14} \ldots$

2. $0.a_{21}\, a_{22}\, a_{23}\, a_{24} \ldots$

3. $0.a_{31}\, a_{32}\, a_{33}\, a_{34} \ldots$

4. $0.a_{41}\, a_{42}\, a_{43}\, a_{44} \ldots$

5. $\ldots$

From this list (in which we assume that all numbers between 0 and 1 are present), let us define another one and prove that it is not in the list. We define $X = 0.x_1\, x_2\, x_3\, x_4 \ldots$, its digits being

$$x_n = \begin{cases} 1, & \text{if } a_{nn} \neq 1 \\ 0, & \text{if } a_{nn} = 1 \end{cases}$$

The number X is between 0 and 1, but it is not in the table: it cannot be in position 1, since its first digit is different from the first of the first; it cannot be in position 2, since its second digit is different from the second of the second; it cannot be in position n, since its n-th digit is different from the n-th digit of the n-th number in the table.

So we have defined a number that differs from every number in the list by at least one digit. Therefore it cannot be in the table. Then the real numbers of the interval $(0, 1)$ are not numerable so neither is $\mathbb{R}$.

8 Operations on $\mathbb{R}$

Let $\alpha, \beta \in \mathbb{R}$. The following operations are defined (the first three coincide with those already defined for rationals):

a) Addition and subtraction (sum and difference): $\boxed{\alpha \pm \beta \in \mathbb{R}}$

b) Product and division: $\boxed{\alpha \cdot \beta \in \mathbb{R}}$; $\boxed{\alpha/\beta \in \mathbb{R},\ \forall \beta \neq 0}$

c) Power of integer exponent: α^m, $m \in \mathbb{Z}$. We distinguish three cases:

- If $m \in \mathbb{N}$, $\boxed{\alpha^m = \overset{\leftarrow \text{m factors} \rightarrow}{\alpha \cdot \alpha \cdot \cdots \cdot \alpha}}$
- If $m \in \mathbb{Z}^-$ $(\alpha \neq 0)$, $\alpha^m = \alpha^{-|m|} = \dfrac{1}{\alpha^{|m|}}$.
- If $m = 0$ $(\alpha \neq 0)$, $\boxed{\alpha^0 = 1}$ is defined (which is consistent with $\alpha^0 = \alpha^{n-n} = \alpha^n/\alpha^n = 1$).

d) n-th root, $n \in \mathbb{N}$: $\boxed{\sqrt[n]{\alpha} = x\ /\ x^n = \alpha}$ (if n is even, it is defined only for $\alpha \geq 0$).

Alternative form: $\sqrt[n]{\alpha} = \alpha^{\frac{1}{n}}$.

e) Power of rational exponent (irreducible fraction $m/n \in \mathbb{Q} \setminus \{0\}$).

- If $\dfrac{m}{n} \in \mathbb{Q}^+$, $\alpha^{\frac{m}{n}} = x\ /\ x^n = \alpha^m$ (if n is even, $\alpha \geq 0$).
- If $\dfrac{m}{n} \in \mathbb{Q}^-$ $(\alpha \neq 0)$, $\alpha^{\frac{m}{n}} = \alpha^{-\left|\frac{m}{n}\right|} = \dfrac{1}{\alpha^{\left|\frac{m}{n}\right|}}$ (If n is even, $\alpha > 0$).

Alternative form: $\alpha^{\frac{m}{n}} = \sqrt[n]{\alpha^m}$.

f) Power of base a positive number and of irrational exponent α^β, $\boxed{\alpha > 0}$, $\beta \notin \mathbb{Q}$: it is defined as the limit of a sequence.

- If we express β as the limit of a monotone sequence of rationals,

$$\beta = \lim_{n \to \infty} \beta_n$$

then the sequence $\{\alpha^{\beta_n}\}$ has a limit (as we will see in unit III) and α^β is defined as

$$\boxed{\alpha^\beta = \lim_{n \to \infty} \alpha^{\beta_n}}$$

- We can also consider the case $\boxed{\alpha = 0}$ with $\beta > 0$, taking a sequence of positive terms with β as a limit. Then the terms 0^{β_n} are null $\forall n$, so $\boxed{0^\beta = 0}$.
- In the other cases ($\alpha = 0$, with $\beta < 0$ and $\alpha < 0, \forall \beta$) the power of irrational exponent is not defined.

g) Logarithm. Given two real numbers $\alpha, \beta > 0$, $\beta \neq 1$, there exists a unique $\gamma \in \mathbb{R}$ such that $\beta^\gamma = \alpha$. This γ is called the logarithm in base β of α ($\gamma = \log_\beta \alpha$), i.e.

$$\boxed{\gamma = \log_\beta \alpha \Longleftrightarrow \beta^\gamma = \alpha}$$

The most common logarithms, based on the number e, are called natural. Decimal logarithms, of base 10, are also widely used.

9 Self-assesment exercises

9.1 True/False exercise

Decide whether the following statements are true or false.

1. The principle of induction can be used to prove a property of an infinite set, provided that an order relation is defined between its elements.
2. We say that A is a proper subset of B if it is strictly contained in B, that is: $A \subset B,\ A \neq B$. Therefore, a set can never be bijective with a proper subset.
3. The set of integers is countable even if it is not bounded from the right or from the left.
4. Between rationals, the "be less than" condition is expressed: $p/q < r/s \Longleftrightarrow ps < qr$.
5. Between sets, the relation "be included in" is a partial order.
6. On an ordered field K, the property $a \cdot 0 = 0,\ \forall a \in K$, is self-evident and needs no proof.
7. The set $\{x \in \mathbb{Q} \ / \ x^2 < 2\}$ has a supremum and a maximum in $\mathbb{R}$.
8. In a commutative field, the inverse of the inverse of any number x is x.
9. The sequence $\left\{ \frac{1}{n} \right\}_{n \in \mathbb{N}}$ is a Cauchy sequence.
10. $\mathbb{Q}$ is dense, so it is not countable.
11. $1.\widehat{9} = 1.999999...$ is the rational immediately before 2.
12. In $\mathbb{R}$, every non-empty set bounded from above has a supremum.
13. There is no 3/2 power of a negative number.
14. Any positive number can be the base of a system of logarithms.
15. Between two different real numbers there is always a rational, but there may not be any irrational.

9.2 Question

Find the values of x that are solutions to the inequation $|x| > k \in \mathbb{R}$.

9.3 Solution to the True/False exercise

1. **F**. The set must be countable.
2. **F**. $\mathbb{N}$ is bijective with an infinite number of proper subsets, for example even, odd, multiples of five...
3. **T**. We can take first 0, then 1, then -1, 2, -2, and so on.
4. **T**. Simply multiply both members of $p/q < r/s$ by the product qs
5. **T**. The relation satisfies the reflexive, antisymmetric and transitive properties. But given any two sets, one of them does not have to be contained in the other, therefore they do not have to be in order relation.

6. **F**. The 0 is the additive identity (element that, added to any one, does not modify it). In principle, we do not know the result of multiplying it by any other number. As demonstrated in class, this product is 0.

7. **F**. It has no maximum, since the supremum ($\sqrt{2}$) does not belong to the set

8. **F**. In a commutative field, the inverse of the inverse of any number $x \neq 0$ is x.

9. **T**. It is convergent (and its limit is 0). Therefore it is a Cauchy sequence.

10. **F**. It is dense and also countable (proved in class).

11. **F**. The difference between 1.999999... and 2 can be made as small as we want, just by taking a large enough number of digits therefore they represent the same rational number. The "repeating fraction" of 1.999999... is $\dfrac{19-1}{9} = 2$.

12. **T**. It is one of the ways to express the completeness of $\mathbb{R}$

13. **T**. $r < 0 \Longrightarrow r^3 = r \cdot r \cdot r < 0$. But $r^{3/2} = (r^3)^{1/2}$. Since $r^3 < 0$, its square root does not exist.

14. **F**. Any positive number –different from 1– can be the base of a system of logarithms.

15. **F**. Between two different real numbers there is always a rational (therefore infinitely many) and an irrational (therefore infinitely many). This is also expressed by saying that both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$ (see **7.3**).

9.4 Solution to the question

We distinguish two cases, $x \geq 0$ and $x < 0$. Then, from the definition of absolute value of a real number:

a) If $x \geq 0 \Longrightarrow x = |x| > k \Longrightarrow x \in (k, \infty)$.

b) If $x < 0 \Longrightarrow -x = |x| > k \Longrightarrow x < -k \Longrightarrow x \in (-\infty, -k)$.

Thus, the condition is satisfied by x belonging to the set $(-\infty, -k) \cup (k, \infty)$.

Note: nothing has been said about the sign of k. If $k \geq 0$, the solution set is the union of two disjoint intervals and can also be written as $\mathbb{R} \setminus [-k, k]$. This set is the complementary of the solution set of the condition $|x| \leq k$.

If, on the other hand, $k < 0$, the intervals $(-\infty, -k)$ and (k, ∞) overlap, so their union is all $\mathbb{R}$. This result is logical since, if $k < 0$, the condition is satisfied for all x, since $\forall x \in \mathbb{R},\ |x| \geq 0 > k$.

Unit II. Metric spaces (14.12.2023)

1 Distance

1.1 Definition

Let E be a non-empty set. We call **distance** or **metric** on E to any function

$$d : \mathrm{E} \times \mathrm{E} \to \mathbb{R}^+ \cup \{0\}$$

which assigns, to every pair (x, y) of elements of E, a real number $d(x, y) \geq 0$, such that the following properties are satisfied:

1. Positivity: $d(x, y) > 0, \forall x \neq y;\quad d(x, x) = 0.$
2. Symmetry: $d(x, y) = d(y, x), \forall x, y \in \mathrm{E}.$
3. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in \mathrm{E}.$

   (The distance between two elements is less than or equal to the sum of the distances between those elements and any other element).

The set E endowed with distance d is called a **metric space**, which is denoted by (E, d). The elements $x \in \mathrm{E}$ are called **points**.

1.2 Most common metrics

Among the many functions that satisfy the distance properties, we distinguish the following.

a) On $\mathbb{R}$ we define the natural metric (or absolute value metric) as

$$d(x, y) = |x - y|$$

b) Metrics on $\mathbb{R}^n$. Let $\vec{x} = (x_1, \dots, x_n)$ and $\vec{y} = (y_1, \dots, y_n)$ be elements of $\mathbb{R}^n$.

b.1. "Taxicab metric" (or "Manhattan distance").

$$d_1(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i| = |x_1 - y_1| + \cdots + |x_n - y_n|$$

b.2. Euclidean distance.

$$d_2(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

b.3. Supremum distance (or infinity distance).

$$d_\infty(\vec{x}, \vec{y}) = \sup |x_i - y_i|_{i=1,\dots,n} = \sup\big(|x_1 - y_1|, \dots, |x_n - y_n|\big)$$

c) Discrete metric. In this case there are only two possible values for the distance between the elements of E.

$$d(x, y) = 1\ \forall x \neq y;\quad d(x, x) = 0$$

2 Balls and neighborhoods

We now define the concepts of ball and neighborhood, that will be used to characterize the different types of points and sets in a metric space.

2.1 Open ball

Let $a \in \mathrm{E}$, $r > 0$. An open ball of centre a and radius r is the set of points of E whose distance to the point a is less than r.

$$\boxed{B(a,r) = \left\{x \in \mathrm{E} \,/\, d(a,x) < r\right\}}$$

2.2 Closed ball

Let $a \in \mathrm{E}$, $r \geq 0$. A closed ball of centre a and radius r is the set of points of E whose distance to the point a is less than or equal to r.

$$\boxed{\overline{B}(a,r) = \left\{x \in \mathrm{E} \,/\, d(a,x) \leq r\right\}}$$

Examples with the Euclidean distance

1. In $\mathbb{R}$, an open ball is an open interval and a closed ball is a closed interval.
2. In $\mathbb{R}^2$, an open ball is a circle without the circumference; and a closed ball, a circle including the circumference.

Balls obtained with other metrics. With metrics other than Euclidean, the shape of the balls can change. Obtain in $\mathbb{R}^2$ the closed balls with center at the origin and radius $r = 1$, being the distance: **a)** d_1; **b)** d_∞.

Solution. The balls are the areas bounded by the squares whose vertices are respectively:
a) $A(1,0)$, $B(0,1)$, $C(-1,0)$, $D(0,-1)$; **b)** $A(1,1)$, $B(-1,1)$, $C(-1,-1)$, $D(1,-1)$.

Punctured ball. It is the ball that does not contain its center. The punctured open ball of center a and radius r is

$$\boxed{B^*(a,r) = \left\{x \in \mathrm{E} \,/\, 0 < d(a,x) < r\right\}}$$

2.3 Neighborhood

Definition. A neighborhood of a is any set U_a containing an open ball of center a.

$$\boxed{\mathrm{U}_a \text{ is a neighborhood of } a \Longleftrightarrow \exists r > 0 \,/\, B(a,r) \subset \mathrm{U}_a}$$

The ball $B(a,r)$ is contained in itself, so an open ball of center a is a neighborhood of a.

Properties.

1. If a set V contains a neighborhood of a, V is a neighborhood of a.
2. The intersection of two neighborhoods of a is a neighborhood of a.
3. Two distinct points have disjoint neighborhoods.

3 Notable points in a metric space

Let (E, d) be a metric space and a set $\mathrm{A} \subset \mathrm{E}$. We define various types of points according to their relation to A. The examples refer to $\mathrm{C} = [0, 2) \cup \{3\}$, in the metric space $(\mathbb{R}, ||)$.

3.1 Closure point

The point $x \in \mathrm{E}$ is a **closure** point (or **adherent point**) of A if and only if every open ball of center x contains some point of A; that is, if

$$\boxed{\forall r \in \mathbb{R}^+, B(x, r) \cap \mathrm{A} \neq \phi}$$

We can say that x is a closure point of A if it has points of A as close as we want.

Consequence. The points of A are closure points of A.

Example. The points $\{0, 1, 2, 3\}$ -among others- are closure points of C.

3.2 Accumulation point

The point $x \in \mathrm{E}$ is an **accumulation** point (or **limit** point) of A if and only if every open ball of center x contains some point of A, other than x. That is, using the punctured ball,

$$\boxed{\forall r \in \mathbb{R}^+,\ B^*(x, r) \cap \mathrm{A} \neq \phi}$$

We can say that x is an accumulation point of A if it has points of A -distinct from itself- as close as we want.

Consequences. 1) Every accumulation point is a closure point. 2) Every closure point of A either belongs to A or is an accumulation point of A.

Example. The points $\{0, 1, 2\}$ -among others- are accumulation points of C.

3.3 Isolated point

The point x is an **isolated** point of A if and only if it belongs to A and there exists an open ball of center x that does not contain any other point of A; that is, if

$$\boxed{\exists r \in \mathbb{R}^+ / B(x, r) \cap \mathrm{A} = \{x\}}$$

Consequence. A closure point x is an accumulation point, unless there is some ball centered on x that contains no other point of A than x, in which case x is an isolated point. Therefore, every closure point is either an accumulation point or an isolated point.

Example. The only isolated point of C is $\{3\}$.

3.4 Interior point

The point $x \in \mathrm{E}$ is an **interior** point of A if and only if there exists an open ball of center x contained in A; that is, if

$$\boxed{\exists r \in \mathbb{R}^+ / B(x, r) \subset \mathrm{A}}$$

Example. The points of the interval $[1, 2)$ -among others- are interior points of C.

3.5 Exterior point

The point $x \in \mathrm{E}$ is an **exterior** point of A if and only if there exists an open ball of center x that does not contain any point of A; that is, if

$$\boxed{\exists r \in \mathbb{R}^+ / B(x,r) \cap \mathrm{A} = \phi}$$

Example. The points of the interval $[4,5]$ -among others- are exterior points of C.

3.6 Boundary point

Given a set A, the **complementary** of A is the set of points of E that do not belong to A. It is denoted by $\mathrm{E} \setminus \mathrm{A}$. This expression is read "E minus A" and will be justified in section **4.3**.

The point $x \in \mathrm{E}$ is a **boundary** point of A if and only if every open ball of center x contains points of A and of its complementary; that is, if

$$\boxed{\forall r \in \mathbb{R}^+ \begin{cases} B(x,r) \cap \mathrm{A} \neq \phi \\ B(x,r) \cap \mathrm{E} \setminus \mathrm{A} \neq \phi \end{cases}}$$

We can say that x is a boundary point of A if it has points of A and $\mathrm{E} \setminus \mathrm{A}$ as close as we want.

Consequencies. 1) A boundary point of A is a closure point of A and of its complementary $\mathrm{E} \setminus \mathrm{A}$. 2) A boundary point satisfies neither the interior point nor the exterior point condition. 3) In $(\mathbb{R}, ||)$, it can be proved that the isolated points are boundary points.

Example. The points $\{0,2,3\}$ are the boundary points of C.

Exercise. Prove that an exterior point of A is an interior point of $\mathrm{E} \setminus \mathrm{A}$ and that an interior point of A is an exterior point of $\mathrm{E} \setminus \mathrm{A}$.

4 Notable sets in a metric space

Starting from the different type of points in a metric space (E, d), referred to a set $\mathrm{A} \subset \mathrm{E}$, we now study the sets formed by the groupings of these points.

4.1 Closure of A

Definition. The closure (or adherence) of A, $\overline{\mathrm{A}}$, is the set of the closure points of A.

Properties.

a) $\mathrm{A} \subset \overline{\mathrm{A}}$. By convention, $\overline{\mathrm{E}} = \mathrm{E}$ and $\overline{\phi} = \phi$.

b) $\mathrm{A} \subset \mathrm{B} \Longrightarrow \overline{\mathrm{A}} \subset \overline{\mathrm{B}}$.

c) $\overline{\mathrm{A} \cup \mathrm{B}} = \overline{\mathrm{A}} \cup \overline{\mathrm{B}}$.

d) $\overline{\mathrm{A} \cap \mathrm{B}} \subset \overline{\mathrm{A}} \cap \overline{\mathrm{B}}$. Equality is not satisfied, for example, in the case of sets without common points, such that their closures do have them, as the intervals $(0,1)$ and $(1,2)$.

4.2 Set of the isolated points

The set formed by the isolated points of A is denoted by $Isol(\mathrm{A})$.

4.3 Derived set of A

Definition. The derived set of A, A', is the set of the accumulation points of A.

Set subtraction. If the sets A and B are disjoint and we call C their union, then

$$\mathrm{A} \cup \mathrm{B} = \mathrm{C}; \quad \mathrm{A} \cap \mathrm{B} = \phi$$

In this case we say that "A is equal to C minus B".

$$\mathrm{A} = \mathrm{C} \setminus \mathrm{B}$$

The subtraction (or difference) of C and B can also be called the relative complement of B in C, as A is formed by the elements of C that do not belong to B.

$$\mathrm{A} = \mathrm{C} \setminus \mathrm{B} = \big\{x \in \mathrm{C} \,/\, x \notin \mathrm{B}\big\}$$

Notice that in this definition we require that both A and B are disjoint subsets of C; therefore to subtract B from C, the former must be a subset of the latter. Other definitions can be given for subtraction. The one used here will be useful to obtain the derived set.

Application. From this definition and from what we have seen in **3.3**, between the closure, the derived set and the isolated points of A, the following relation exists

$$\boxed{\mathrm{A}' = \overline{\mathrm{A}} \setminus \mathit{Isol}(\mathrm{A})}$$

which gives us the derived set of A by subtracting the set of isolated points from the closure.

Proof.

1. Both the accumulation points and the isolated points are closure points, hence the union of A' and $\mathit{Isol}(\mathrm{A})$ is contained in $\overline{\mathrm{A}}$.

   On the other hand, every closure point is either an accumulation point or an isolated one, so $\overline{\mathrm{A}}$ is contained in $\mathrm{A}' \cup \mathit{Isol}(\mathrm{A})$.

   As a consequence, the two sets coincide: $\overline{\mathrm{A}} = \mathrm{A}' \cup \mathit{Isol}(\mathrm{A})$.

2. From the respective definitions, a point cannot be both of accumulation and isolated, so the intersection of A' and $\mathit{Isol}(\mathrm{A})$ is empty.

3. Hence we can state that $\mathrm{A}' = \overline{\mathrm{A}} \setminus \mathit{Isol}(\mathrm{A})$.

4.4 Interior of A

Definition. The interior of A, $\mathring{\mathrm{A}}$, is the set of its interior points.

Properties.

a) $\mathring{\mathrm{A}} \subset \mathrm{A}$. By convention, $\mathring{\mathrm{E}} = \mathrm{E}$ and $\mathring{\phi} = \phi$.

b) $\mathrm{A} \subset \mathrm{B} \Longrightarrow \mathring{\mathrm{A}} \subset \mathring{\mathrm{B}}$.

c) $(\mathrm{A} \cap \mathrm{B})^\circ = \mathring{\mathrm{A}} \cap \mathring{\mathrm{B}}$.

d) $(\mathrm{A} \cup \mathrm{B})^\circ \supset \mathring{\mathrm{A}} \cup \mathring{\mathrm{B}}$. Equality is not satisfied, for example, in the case of sets with common points, such that their interiors do not have them, as the intervals $[0, 1]$ and $[1, 2]$.

4.5 Exterior of A

Definition. The exterior of A, $Ext(\mathrm{A})$, is the set of its exterior points.

The exercise at the end of section **3** consists of proving that an exterior point of a set is interior to its complementary and that an interior point of a set is exterior to its complementary. Having this into account:

- If x is exterior of A, it is interior of $\mathrm{E} \setminus \mathrm{A}$.
- If x is interior of $\mathrm{E} \setminus \mathrm{A}$, it is exterior of the complementary of $\mathrm{E} \setminus \mathrm{A}$, that is of A.

Thus x is exterior of A if and only if it is interior of $\mathrm{E} \setminus \mathrm{A}$. The set $Ext(\mathrm{A})$ can then be obtained as the interior of the complementary of A.

$$\boxed{Ext(\mathrm{A}) = \big(\mathrm{E} \setminus \mathrm{A}\big)^{\circ}}$$

In practice, we will usually obtain the exterior set of A by means of the following simpler expression:

$$\boxed{Ext(\mathrm{A}) = \mathrm{E} \setminus \overline{\mathrm{A}}}$$

Proof. To prove it, let us see that the inclusion of sets is satisfied in both directions.

($\rightarrow$) If x is an exterior point of A, there exists a ball centered at x that does not contain points of A, then x is not a closure point of A, so x belongs to the complementary of $\overline{\mathrm{A}}$.

$$\forall x \in Ext(\mathrm{A})\, \exists r \,/ B(x,r) \cap \mathrm{A} = \phi \Longrightarrow x \notin \overline{\mathrm{A}} \Longrightarrow x \in \mathrm{E} \setminus \overline{\mathrm{A}} \Longrightarrow Ext(\mathrm{A}) \subset \mathrm{E} \setminus \overline{\mathrm{A}}$$

($\leftarrow$) If x belongs to the complementary of $\overline{\mathrm{A}}$, it is not a closure point of A, then there exists some ball centered at x that does not contain points of A, so x is exterior of A.

$$\forall x \in \mathrm{E} \setminus \overline{\mathrm{A}} \Longrightarrow x \notin \overline{\mathrm{A}} \Longrightarrow \exists r / B(x,r) \cap \mathrm{A} = \phi \Longrightarrow x \in Ext(\mathrm{A}) \Longrightarrow \mathrm{E} \setminus \overline{\mathrm{A}} \subset Ext(\mathrm{A})$$

4.6 Boundary of A

Definition. The boundary of A, $\partial(\mathrm{A})$, is the set of its boundary points.

We know (**3.6**) that a point is a boundary point if and only if it is a closure point of A and of its complementary; thus we can obtain the boundary of A as the intersection of both closures:

$$\boxed{\partial(\mathrm{A}) = \overline{\mathrm{A}} \cap \overline{\mathrm{E} \setminus \mathrm{A}}}$$

As before, it is easier to obtain the boundary set by means of the following expression:

$$\boxed{\partial(\mathrm{A}) = \overline{\mathrm{A}} \setminus \mathring{\mathrm{A}}}$$

Proof. The double inclusion is satisfied.

($\rightarrow$) Let $x \in \partial(\mathrm{A})$. Then $x \in \overline{\mathrm{A}}$ and $x \in \overline{\mathrm{E} \setminus \mathrm{A}}$. Since x belongs to $\overline{\mathrm{E} \setminus \mathrm{A}}$, every open ball centered at x contains points of $\mathrm{E} \setminus \mathrm{A}$, so no ball is contained in A, then x does not fulfill the condition of interior point, $x \notin \mathring{\mathrm{A}}$. Since $x \in \overline{\mathrm{A}}$ and $x \notin \mathring{\mathrm{A}}$, and furthermore $\mathring{\mathrm{A}} \subset \overline{\mathrm{A}}$, it follows that $x \in \overline{\mathrm{A}} \setminus \mathring{\mathrm{A}}$.

($\leftarrow$) Let $x \in \overline{\mathrm{A}} \setminus \mathring{\mathrm{A}}$. Then $x \in \overline{\mathrm{A}}$ and $x \notin \mathring{\mathrm{A}}$. If $x \notin \mathring{\mathrm{A}}$, there is no open ball centered at x contained in A, so every ball contains points of the complementary of A, $\mathrm{E} \setminus \mathrm{A}$. Then x belongs to the closure of $\mathrm{E} \setminus \mathrm{A}$. Since it also belongs to the closure of A, it belongs to the intersection of both, so it belongs to the boundary.

Example. As in section **3**, we calculate the notable sets referred to $\mathrm{C} = [0,2) \cup \{3\}$ in the metric space $(\mathbb{R}, ||)$.

1. Closure: $\overline{\mathrm{C}} = [0,2] \cup \{3\}$.
2. Set of isolated points: $Isol(\mathrm{C}) = \{3\}$.
3. Derived set: $\mathrm{C}' = \overline{\mathrm{C}} \setminus Isol(\mathrm{C}) = [0,2]$.
4. Interior: $\mathring{\mathrm{C}} = (0,2)$.
5. Exterior: $Ext(\mathrm{C}) = \mathbb{R} \setminus \overline{\mathrm{C}} = (-\infty,0) \cup (2,3) \cup (3,\infty)$.
6. Boundary: $\partial(\mathrm{C}) = \overline{\mathrm{C}} \setminus \mathring{\mathrm{C}} = \{0,2,3\}$.

5 Closed, open and compact sets

Let E be a metric space and the set $\mathrm{A} \subset \mathrm{E}$. We are going to study the conditions that must be satisfied to affirm that A is closed, open or compact.

5.1 Closed set

Definition. A set is closed if and only if it coincides with its closure, that is

$$\boxed{\mathrm{A} = \overline{\mathrm{A}}}$$

Example. In $(\mathbb{R}, ||)$, the interval $[a,b]$ is a closed set.

Equivalent conditions: There are two other equivalent closed set conditions: "to contain its accumulation points" and "to contain its boundary". They are proved below.

a) A is closed if and only if it contains its accumulation points.

$$\boxed{\mathrm{A} = \overline{\mathrm{A}} \Longleftrightarrow \mathrm{A}' \subset \mathrm{A}}$$

($\rightarrow$) The accumulation points are closure points, then $\mathrm{A}' \subset \overline{\mathrm{A}}$. Since A is closed, $\overline{\mathrm{A}}$ coincides with A. Then $\mathrm{A}' \subset \mathrm{A}$.

$$\mathrm{A}' \subset \overline{\mathrm{A}} = \mathrm{A} \Longrightarrow \mathrm{A}' \subset \mathrm{A}$$

($\leftarrow$) The closure $\overline{\mathrm{A}}$ is the union of A' and $Isol(\mathrm{A})$. Since both are contained in A, their union is also contained in A, so $\overline{\mathrm{A}} \subset \mathrm{A}$. Since furthermore $\mathrm{A} \subset \overline{\mathrm{A}}$, it is satisfied that $\overline{\mathrm{A}} = \mathrm{A}$.

$$\mathrm{A}', Isol(\mathrm{A}) \subset \mathrm{A} \Longrightarrow \overline{\mathrm{A}} = \mathrm{A}' \cup Isol(\mathrm{A}) \subset \mathrm{A} \overset{\mathrm{A} \subset \overline{\mathrm{A}}}{\Longrightarrow} \overline{\mathrm{A}} = \mathrm{A}$$

b) A is closed if and only if it contains its boundary.

$$\boxed{\mathrm{A} = \overline{\mathrm{A}} \Longleftrightarrow \partial(\mathrm{A}) \subset \mathrm{A}}$$

($\rightarrow$) The boundary is contained in the closure. Since A coincides with its closure, the boundary is contained in A.

$$\partial(\mathrm{A}) \subset \overline{\mathrm{A}} = \mathrm{A} \Longrightarrow \partial(\mathrm{A}) \subset \mathrm{A}$$

($\leftarrow$) The closure $\overline{\mathrm{A}}$ is the union of $\mathring{\mathrm{A}}$ and $\partial(\mathrm{A})$. Since $\mathring{\mathrm{A}}$ is contained in A and $\partial(\mathrm{A})$ also (by hypothesis), $\overline{\mathrm{A}}$ is contained in A. Since $\mathrm{A} \subset \overline{\mathrm{A}}$, it is satisfied that $\overline{\mathrm{A}} = \mathrm{A}$.

$$\mathring{\mathrm{A}} \subset \mathrm{A}, \partial(\mathrm{A}) \subset \mathrm{A} \Longrightarrow \overline{\mathrm{A}} = \mathring{\mathrm{A}} \cup \partial(\mathrm{A}) \subset \mathrm{A} \overset{\mathrm{A} \subset \overline{\mathrm{A}}}{\Longrightarrow} \overline{\mathrm{A}} = \mathrm{A}$$

Properties.

a) $\overline{\mathrm{A}}$ is a closed set, hence the closure of $\overline{\mathrm{A}}$ is equal to $\overline{\mathrm{A}}$: $\overline{(\overline{\mathrm{A}})} = \overline{\mathrm{A}}$.

b) E and ϕ are closed sets, since each of them coincides with its adherence (by convention).

c) The intersection of closed sets is a closed set.

d) The finite union of closed sets is a closed set (the infinite union may not be).

5.2 Open set

Definition. A set is open if and only if it coincides with its interior, that is

$$\boxed{\mathrm{A} = \mathring{\mathrm{A}}}$$

Example. In $(\mathbb{R}, ||)$, the interval (a, b) is an open set.

Equivalent condition. A is an open set if and only if it has an empty intersection with its boundary (i.e. none of its points is a boundary point).

$$\boxed{\mathrm{A} = \mathring{\mathrm{A}} \Longleftrightarrow \partial(\mathrm{A}) \cap \mathrm{A} = \phi}$$

Proof.

($\rightarrow$) $\partial(\mathrm{A}) = \overline{\mathrm{A}} \setminus \mathring{\mathrm{A}}$ (see **4.6**), so the intersection of $\mathring{\mathrm{A}}$ and $\partial(\mathrm{A})$ is empty. Since A is open, it coincides with its interior $\mathring{\mathrm{A}}$, then the intersection of A and its boundary is empty.

($\leftarrow$) If the intersection of A and $\partial(\mathrm{A})$ is empty, no point of A belongs to its boundary. Since A and its boundary are contained in $\overline{\mathrm{A}}$, then every point of A belongs to $\overline{\mathrm{A}} \setminus \partial(\mathrm{A}) = \mathring{\mathrm{A}}$, so $\mathrm{A} \subset \mathring{\mathrm{A}}$. Since furthermore $\mathring{\mathrm{A}} \subset \mathrm{A}$, it follows that $\mathrm{A} = \mathring{\mathrm{A}}$.

Properties.

1) $\mathring{\mathrm{A}}$ is an open set, hence the interior of $\mathring{\mathrm{A}}$ is equal to $\mathring{\mathrm{A}}$: $(\mathring{\mathrm{A}})^\circ = \mathring{\mathrm{A}}$.

2) E and ϕ are open sets, since each of them coincides with its interior (by convention).

3) The union of open sets is an open set.

4) The finite intersection of open sets is an open set (the infinite intersection may not be).

Exercise. The union of an infinite number of closed sets can be a non closed set. And the intersection of an infinite number of open sets can be a non open set. Study the two following cases:

$$\bigcup_{n\in\mathbb{N}} \left[0\,, \frac{n-1}{n}\right] \quad ; \quad \bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$$

5.3 Relation between open and closed sets

Theorem. *A set is open if and only if its complementary is closed.*

$$\boxed{\mathrm{A} = \mathring{\mathrm{A}} \Longleftrightarrow \mathrm{E} \setminus \mathrm{A} = \overline{\mathrm{E} \setminus \mathrm{A}}}$$

Proof.

(→) Let A = $\mathring{A}$. We want to prove that E \ A = $\overline{\text{E} \setminus \text{A}}$. Since E \ A ⊂ $\overline{\text{E} \setminus \text{A}}$, it suffices to show that $\overline{\text{E} \setminus \text{A}}$ ⊂ E \ A.

Let $x \in \overline{\text{E} \setminus \text{A}}$. Then every open ball centered at x contains points of E \ A, so none is contained in A, so x does not belong to $\mathring{A}$. Since A = $\mathring{A}$, x does not belong to A, then it belongs to its complementary E \ A. Therefore $\overline{\text{E} \setminus \text{A}}$ ⊂ E \ A.

(←) Let E \ A = $\overline{\text{E} \setminus \text{A}}$. We want to prove that A = $\mathring{A}$. Since $\mathring{A}$ ⊂ A, it suffices to show that A ⊂ $\mathring{A}$.

Let $x \in$ A. Then $x \notin$ E \ A. Since E \ A = $\overline{\text{E} \setminus \text{A}}$, x does not belong to $\overline{\text{E} \setminus \text{A}}$. So there exists some open ball centered at x that does not contain points of E \ A, so it is contained in A, then $x \in \mathring{A}$. Therefore A ⊂ $\mathring{A}$.

5.4 Compact set

Definition. A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Examples. In $\mathbb{R}$, any interval $[a, b]$ or a finite union of closed intervals are compact. The intervals (a, b) or $[a, \infty)$ are not compact.

Case of $(\mathbb{R}^n, d_2)$. In $\mathbb{R}^n$, with the Euclidean distance, we use the same definition as in $\mathbb{R}$.

Remark. The bounded set condition that we use in R^n $(n > 1)$ is different from the one we use in $\mathbb{R}$. We say that a set A is bounded if and only if it is contained in a ball of finite radius.

$$\text{A} \subset \mathbb{R}^n \text{ bounded} \iff \exists\, \vec{a} \in \mathbb{R}^n,\ \exists r \in \mathbb{R}^+ \cup \{0\} \,/\, \text{A} \subset B(\vec{a}, r)$$

Example. The set $\left\{(x, y) \in \mathbb{R}^2 \,/\, (x-3)^2 + (y-2)^2 \leq 1\right\}$ is compact.

6 The metric space $(\mathbb{R}, ||)$

We now study a particular metric space: the set of real numbers, endowed with the absolute value metric. First we will verify that the different distances studied (except the discrete one) coincide. Then we define the open and closed sets and state the Bolzano-Weierstrass theorem.

6.1 Distance

It is immediate to verify that the distances d_1, d_2, and d_∞ reduce to that of the absolute value (natural metric). Indeed, since the points x and y have only one coordinate, it follows:

a) $d_1(x, y) = |x - y|$.

b) $d_2(x, y) = \sqrt{(x-y)^2} = |x - y|$.

c) $d_\infty(x, y) = |x - y|$.

Example. The interval (a, b), in the form of a ball, is $B(c, r)$, being $c = \dfrac{a+b}{2}$, $r = \dfrac{b-a}{2}$.

Exercise. Write the expression of the ball $B(x, \delta)$ in the form of an interval.

6.2 Open and closed sets

From the definition of open set ($\mathrm{A} = \mathring{\mathrm{A}}$) and the fact that every closed set is the complementary of an open set (theorem **5.3**), it follows:

1. In $(\mathbb{R}, ||)$, an open set is any set that can be expressed as a union of open intervals. We justify it below:

   ($\rightarrow$) A is open if and only if it coincides with its interior, then all its points must be interior. That $x \in \mathrm{A}$ is interior means that it is the center of an open ball contained in A, so A can be expressed as the union of all such balls (each contains one of the points of A and is contained in A). But, in $(\mathbb{R}, ||)$, an open ball is an open interval, so if A is open, it can be expressed as a union of open intervals.

   ($\leftarrow$) If A can be expressed as a union of open intervals, it will be open, since an open interval is an open set and the union of open sets is an open set.

2. We define closed set as the complementary of an open set.

Examples. $(a, b) \cup (c, d)$ is an open set. The complementary of the closed set $[p, q]$ is $\mathbb{R} \backslash [p, q] = (-\infty, p) \cup (q, \infty)$, which is a union of open sets, therefore open.

Exercise. Reason whether the set $\{c\}$, formed by a point, is closed.

6.3 Bolzano-Weierstrass theorem

Every bounded subset $\mathrm{S} \subset \mathbb{R}$*, with infinitely many points, has at least one accumulation point.*

Proof. Since S is bounded, we can find some values $a, b \in \mathbb{R} \,/\, \mathrm{S} \subset [a, b]$. We divide $[a, b]$ into two equal parts and take one that contains infinitely many points (at least one of them does, otherwise S would be finite).

We divide the chosen half-interval into two equal parts, taking again one containing infinitely many points. We repeat the operation again and again, obtaining –after n divisions– the interval I_n of length

$$l_n = \frac{b - a}{2^n}$$

which tends to 0, when $n \to \infty$. But I_n still contains infinitely many points.

Thus, we end up defining a point α, such that it has infinite points of S in all its neighborhoods, so it fulfills the condition of accumulation point of S. This point may not belong to S, in which case it will belong to its boundary.

Examples. We study the sets: a) $\mathrm{S}_1 = \left\{ \dfrac{n+1}{n} \right\}$; b) $\mathrm{S}_2 = \left\{ \dfrac{(-1)^n}{n} \right\}$; c) $\mathrm{S}_3 = (a, b)$.

In a) the accumulation point is $\alpha = 1$. In b) $\alpha = 0$. In case c), any point in the interval $[a, b]$ is an accumulation point of S_3.

Exercises.

a) Apply the theorem to a proper subset of $\mathbb{N}$ (i.e. a subset of $\mathbb{N}$, which is distinct from it).

b) Apply the theorem to $\mathrm{S} = [0, 1] \cap \mathbb{Q}$.

c) Justify that, if the accumulation point α does not belong to S, it belongs to its boundary.

7 Self-assesment exercises

7.1 True/False exercise

Decide whether the first ten statements are true or false and find the sets proposed in the last paragraph.

1. $\forall \overline{x}, \overline{y} \in \mathbb{R}^2$ it holds: $d_\infty(\overline{x}, \overline{y}) \leq d_1(\overline{x}, \overline{y}) \quad (d_1,$ "taxicab metric").
2. In $\mathbb{R}^2$ a closed ball is a circle, regardless of the metric used.
3. A boundary point is neither interior nor exterior.
4. $\mathring{\mathrm{A}} \subset \mathring{\mathrm{B}} \Longrightarrow \mathrm{A} \subset \mathrm{B}$.
5. $Ext(\mathrm{A}) = \mathrm{E} \setminus \overline{\mathrm{A}}$.
6. $\mathrm{A} = \overline{\mathrm{A}} \Longleftrightarrow \mathrm{A}' \subset \mathrm{A} \Longleftrightarrow Fr(\mathrm{A}) \subset \mathrm{A}$.
7. In $(\mathbb{R}, ||)$ every closed set is a closed interval.
8. The interval [2,3] is a neighborhood of all its points.
9. If a bounded set $S \subset \mathbb{R}$ does not have any accumulation points, then it has only a finite number of points.
10. Between the elements of $\mathbb{R}^2$ there is no order relation, so in $\mathbb{R}^2$ there are no bounded sets.
11. Given the set $\mathrm{A} = \left\{ \dfrac{3n^2}{3n^2 - 1} \right\}_{n \in \mathbb{N}}$, obtain:

    11.1 $\overline{\mathrm{A}}$:

    11.2 A':

    11.3 $\mathring{\mathrm{A}}$:

    11.4 $\partial(\mathrm{A})$:

    11.5 $Isol(\mathrm{A})$:

7.2 Question

Define interior point and write the mathematical condition it must satisfy. Reason if an interior point can be a boundary point.

7.3 Solution to the True/False exercise

1. **T**. $d_\infty(\overline{x}, \overline{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \leq |x_1 - y_1| + |x_2 - y_2| = d_1(\overline{x}, \overline{y})$.
2. **F**. A closed ball in $\mathbb{R}^2$, with different distances, can take different shapes: it is a rhombus with d_1 and a square with d_∞.
3. **T**. In every ball centered on the boundary point there are points of the set and of the complementary, so no ball is contained in the set (interior point condition) nor in the complementary (exterior point condition).
4. **F**. Example: $\mathrm{A} = [0, 1]$, $\mathrm{B} = (0, 1)$. The statement is true with the implication in the opposite direction: it is one of the properties of the interior sets.

5. **T**. See section **4.5**.

6. **T**. They are three equivalent definitions of a closed set (proved in the course notes).

7. **F**. Every closed interval is a closed set, but not vice versa. The union of two disjoint closed intervals is a closed set, and yet it is not an interval.

8. **F**. For the boundary points (2 and 3) there is no ball centered on them and contained in the interval.

   In fact, being a neighborhood of all its points is an alternative definition of an open set. As the closed interval [2,3] is a closed set, it cannot satisfy this condition.

9. **T**. By the Bolzano-Weierstrass theorem, if S is bounded and has infinite points, "there exists at least one point of $\mathbb{R}$, which is of accumulation of S". Then, if it does not have accumulation points, it is not bounded or it does not have infinite points. Since, by hypothesis, it is bounded, then it must have a finite number of points.

10. **F**. We say that a set $\mathrm{A} \subset \mathbb{R}^n$ is bounded if it is contained in a ball of finite radius. This condition does not require an order relation between the elements of $\mathbb{R}^n$.

11. $\mathrm{A} = \left\{ \frac{3}{2}, \frac{12}{11}, \frac{27}{26}, \frac{48}{47}, \frac{75}{74} \cdots \right\}$. The sequence has limit 1. Then

    11.1 $\overline{\mathrm{A}} = \mathrm{A} \cup \{1\}$.

    11.2 $\mathrm{A}' = \{1\}$.

    11.3 $\mathring{\mathrm{A}} = \phi$.

    11.4 $\partial(\mathrm{A}) = \overline{\mathrm{A}} \setminus \mathring{\mathrm{A}} = \overline{\mathrm{A}}$.

    11.5 $\mathit{Isol}(\mathrm{A}) = \mathrm{A}$.

7.4 Solution to the question

In a metric space (E, d), a point $x \in \mathrm{E}$ is an interior point of $\mathrm{A} \subset \mathrm{E}$ if and only if there is an open ball of center x, contained in A. That is

$$x \text{ is an interior point of A} \iff \exists r \in \mathbb{R}^+ \,/\, \mathring{B}(x, r) \subset \mathrm{A}$$

The condition that x must satisfy to be a boundary point of A, is that in every open ball centered at x there are points of A and of its complementary. Thus, if x is a boundary point of A, no ball contains only points of A, so an interior point cannot be a boundary point.

Unit III. Sequences on $\mathbb{R}$ (14.12.2023)

1 Definition and types of sequences

1.1 Sequences on $\mathbb{R}$

A sequence of real numbers (real sequence) is the image of an application (or function) φ from $\mathbb{N}$ to $\mathbb{R}$. To each natural number n (index) corresponds the term a_n of the sequence.

$$\boxed{\varphi : \mathbb{N} \to \mathbb{R}, \quad \varphi(n) = a_n \in \mathbb{R}, \quad \varphi(\mathbb{N}) = \{a_n\}_{n\in\mathbb{N}}}$$

1.2 Concept of limit

The limit of a sequence $\{a_n\}$ is a number a which is approximated by the terms of the sequence as closely as we want, taking a sufficiently advanced term. That is:

$$\boxed{\lim_{n\to\infty} a_n = a \Longleftrightarrow \forall\varepsilon > 0 \; \exists n \in \mathbb{N} \,\big/\, |a_m - a| < \varepsilon, \; \forall m \geq n}$$

Graphic interpretation. The sequence has a limit a if, for any $\varepsilon > 0$, the terms of $\{a_n\}$ starting from a given one are contained in the interval $(a - \varepsilon, a + \varepsilon)$.

1.3 Types of sequences

a) In relation to the existence of a limit, a sequence can converge, diverge or oscillate.

- **Convergent sequence.** A sequence converges if it has a finite limit.

$$\boxed{\{a_n\} \text{ is convergent} \Longleftrightarrow \lim_{n\to\infty} a_n = a \in \mathbb{R}}$$

- **Divergent sequence.** A sequence is divergent if its terms in absolute value become as large as we want, taking a sufficiently advanced term.

$$\boxed{\{a_n\} \text{ is divergent} \Longleftrightarrow \forall A > 0 \; \exists n \in \mathbb{N} \,\big/\, |a_m| > A, \; \forall m \geq n}$$

  which can also be written as $\boxed{\lim_{n\to\infty} |a_n| = \infty}$ (∞ = infinity)

  If, in addition, the terms are positive starting from a given one, we say that $\boxed{\lim_{n\to\infty} a_n = +\infty}$

  (if they are negative starting from a given one, we say that $\lim_{n\to\infty} a_n = -\infty$)

- **Oscillating sequence.** Sequences that neither converge nor diverge are called oscillating.

  From this definition it follows that any sequence is convergent, divergent or oscillating.

b) In relation to the boundedness of its terms, a sequence can be:

- Bounded from above $\Longleftrightarrow \exists k \in \mathbb{R} \,\big/\, a_n \leq k, \; \forall n \in \mathbb{N}$.
- Bounded from below $\Longleftrightarrow \exists k \in \mathbb{R} \,\big/\, a_n \geq k, \; \forall n \in \mathbb{N}$.
- Bounded $\Longleftrightarrow \exists k \in \mathbb{R}^+ \,\big/\, |a_n| \leq k, \; \forall n \in \mathbb{N}$.

Exercise 1. Determine whether the following sequences are convergent, divergent or oscillating, also indicating whether they are bounded.

$$\left\{\frac{1}{\sqrt{n}}\right\} \; ; \; \{2^n\} \; ; \; \left\{1, \frac{1}{2}, 3, \frac{1}{4}, \dots\right\} \; ; \; \left\{\frac{n-1}{n+1}\right\} \; ; \; \left\{1, \sqrt{2}, 3, \sqrt{4}, \dots\right\} \; ; \; \{1, -2, 3, -4, \dots\}$$

Exercise 2. Find an example of a sequence:
a) Oscillating and bounded ; **b)** Oscillating unbounded ; **c)** Bounded non convergent ;
d) Bounded non oscillating ; **e)** Neither bounded nor divergent.

2 Properties of the limits

1) *If a sequence has a limit, it is unique.*

P: By *reductio ad absurdum.* Suppose there are two limits $\alpha < \beta$.

We take $\varepsilon > 0 \; / \; 2\varepsilon < \beta - \alpha$. Since both are limits, there will exist n_α, n_β such that

$$|a_m - \alpha| < \varepsilon, \; \forall m \geq n_\alpha \;\; \text{and} \;\; |a_m - \beta| < \varepsilon, \; \forall m \geq n_\beta$$

Then, $\forall m \geq \max(n_\alpha, n_\beta)$, it will be fulfilled that

$$\beta - \alpha = |\beta - \alpha| = |\beta - a_m + a_m - \alpha| \leq |a_m - \alpha| + |a_m - \beta| < 2\varepsilon$$

That is, $\beta - \alpha < 2\varepsilon$, against the hypothesis.

2) *If a sequence has a limit $a > c$, its terms starting from a given one are greater than c.*

$$\boxed{\{a_n\} \to a > c \Longrightarrow \exists n \in \mathbb{N} \; / \; a_m > c, \; \forall m \geq n}$$

P: Let us take $\varepsilon > 0 \; / \; \varepsilon < a - c \Longrightarrow c < a - \varepsilon$. Since a is the limit,

$$\exists n \; / \; \forall m \geq n, \;\; |a_m - a| < \varepsilon \Longrightarrow -\varepsilon < a_m - a < \varepsilon \Longrightarrow a - \varepsilon < a_m < a + \varepsilon$$

That is, $a_m > a - \varepsilon > c$.

3) *If a sequence has a limit $a < c$, its terms starting from a given one are less than c.*

$$\boxed{\{a_n\} \to a < c \Longrightarrow \exists n \in \mathbb{N} \; / \; a_m < c, \; \forall m \geq n}$$

P: Analogous to the previous one, taking $\varepsilon > 0 \; / \; \varepsilon < c - a$.

4) *If a sequence has a limit $a \neq 0$, its terms starting from a given one have the sign of a.*

P: If $a > 0$, from the property 2: $\exists n \; / \; a_m > 0, \; \forall m \geq n$.

If $a < 0$, from the property 3: $\exists n \; / \; a_m < 0, \; \forall m \geq n$.

That is, if $a \neq 0$, the terms starting from a given one have the sign of a.

5) *If the limit of $\{a_n\}$ is less than that of $\{b_n\}$, the terms of the former are less than those of the latter, starting from a given one.*

$$\boxed{\{a_n\} \to a,\ \{b_n\} \to b,\ a < b \Longrightarrow \exists n \in \mathbb{N} \ / \ a_m < b_m,\ \forall m \geq n}$$

P: We take $c = (a + b)/2$, so that $a < c < b$, and apply properties 2 and 3.

6) *If $a_n < b_n < c_n,\ \forall n \geq n_0$ and the sequences $\{a_n\}$ and $\{c_n\}$ have limit α, then the sequence $\{b_n\}$ has a limit α.*

P: $\lim\limits_{n\to\infty} a_n = \alpha \Longrightarrow \exists n_1 \ / \ |a_m - \alpha| < \varepsilon \Longrightarrow \boxed{\alpha - \varepsilon < a_m} < \alpha + \varepsilon,\ \forall m \geq n_1.$

$\lim\limits_{n\to\infty} c_n = \alpha \Longrightarrow \exists n_2 \ / \ |c_m - \alpha| < \varepsilon \Longrightarrow \alpha - \varepsilon < \boxed{c_m < \alpha + \varepsilon},\ \forall m \geq n_2.$

As, in addition, $\boxed{a_n < b_n < c_n}$, $\forall n \geq n_0$, then, $\forall m \geq \max(n_0, n_1, n_2)$,

$$\alpha - \varepsilon < a_m < b_m < c_m < \alpha + \varepsilon \Longrightarrow \boxed{\lim_{n\to\infty} b_n = \alpha}$$

7) A **subsequence** of $\{a_n\}$ is obtained by taking **infinitely many terms** of it, **without altering the relative order** between them. If the places that these elements occupy in $\{a_n\}$ are $n_1 < n_2 < \cdots < n_i < \ldots$, the subsequence will be: $\{a_{n_i}\} = a_{n_1}, a_{n_2}, \ldots a_{n_i}, \ldots$ For example, even numbers or multiples of 3 are subsequences of the naturals.

Property. *If $\{a_n\}$ has a limit, every subsequence of it has the same limit.*

P: If a is the limit, $\forall\varepsilon > 0,\ \exists n \ / \ |a_m - a| < \varepsilon,\ \forall m \geq n.$ In this case, also $|a_{m_i} - a| < \varepsilon,\ \forall m_i \geq n$, for $\{a_{n_i}\}_{i\in\mathbb{N}} \subset \{a_n\}_{n\in\mathbb{N}}$.

8) *Every convergent sequence is bounded.*

P: For being convergent, $\forall\varepsilon > 0\ \exists n \big/ \ |a_m - a| < \varepsilon,\ \forall m \geq n.$ Then, by the properties of the absolute value

$$|a_m| - |a| \leq |a_m - a| < \varepsilon \Longrightarrow |a_m| < |a| + \varepsilon,\ \forall m \geq n$$

Thus, the bound is $k = (\max)\left(|a_1|, |a_2|, \ldots |a_{n-1}|, |a| + \varepsilon\right)$.

3 Monotone sequences

3.1 Definitions

With respect to the relation between the value of its terms a numerical sequence $\{a_n\}$ is:

- Monotone increasing, if and only if $a_{n+1} \geq a_n\ \forall n \in \mathbb{N}$.
- Monotone decreasing, if and only if $a_{n+1} \leq a_n\ \forall n \in \mathbb{N}$.
- Strictly increasing, if and only if $a_{n+1} > a_n\ \forall n \in \mathbb{N}$.
- Strictly decreasing, if and only if $a_{n+1} < a_n\ \forall n \in \mathbb{N}$.

Exercise. Indicate the type of monotonicity of the following sequences:

a) $1, 2, 2, 3, 3, 3, 4, \ldots$; **b)** $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$; **c)** $1, 1, 1, 1, \ldots$; **d)** $-1, 0, 1, -2, 0, 2, -3, 0, 3, \ldots$

3.2 Theorem of monotone sequences

Every monotone increasing (decreasing) sequence bounded from above (below) has a limit.

Proof. We prove it for the increasing case, starting from the property of the supremum.

a) The sequence $\{a_n\}$ is a set of upper bounded real numbers, so it has supremum α (unit I, section 6). Since α is the smallest of the upper bounds, any value less than α must be exceeded by some term of $\{a_n\}$, that is:

$$\forall \varepsilon > 0 \ \exists n_0 / \boxed{\alpha - \varepsilon < a_{n_0}}$$

otherwise there would exist some ε such that $\alpha - \varepsilon \geq a_{n_0}$, $\forall n_0$, in which case the supremum would be $\alpha - \varepsilon$, contrary to the hypothesis.

b) Since $\{a_n\}$ is monotone increasing, $\boxed{a_{n_0} \leq a_m}$, $\forall m \geq n_0$. And, since α is the supremum, $\boxed{a_m \leq \alpha}$.

c) Then, from a) and b),

$$\forall \varepsilon > 0 \ \exists n_0 \,/\, \alpha - \varepsilon < a_{n_0} \leq a_m \leq \alpha \Longrightarrow \alpha - \varepsilon < a_m \leq \alpha, \ \forall m \geq n_0$$

But $\alpha - \varepsilon < a_m \Longrightarrow \alpha - a_m < \varepsilon$ and $a_m \leq \alpha \Longrightarrow 0 \leq \alpha - a_m$. That is to say

$$0 \leq \alpha - a_m < \varepsilon \Longrightarrow \boxed{|\alpha - a_m| < \varepsilon \ , \forall m \geq n_0}$$

which is the limit condition for $\{a_n\}$. Thus **the limit of the sequence is the supremum**.

3.3 Sequences of nested intervals

A sequence of nested intervals is a sequence of real intervals, such that each one is included in the previous one, $I_n \subset I_{n-1}$, $\forall n \in \mathbb{N}$. If the intervals are closed and nonempty, i.e.

$$I_n = [a_n, b_n], \quad a_n \leq b_n, \ \forall n \in \mathbb{N}$$

then its intersection is the interval $[a, b]$, $a \leq b$, being

$$a = \lim_{n\to\infty} a_n, \qquad b = \lim_{n\to\infty} b_n$$

If, in addition, the lengths of the intervals tend to zero, i.e. $(b_n - a_n) \to 0$, then there is a single real number α common to all the intervals and it holds that

$$\boxed{\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \alpha \in \mathbb{R}}$$

Proof. (For a more complete proof see J. Burgos, p. 64). Let the sequence $\{a_n\}$ be monotone increasing and $\{b_n\}$ monotone decreasing, such that $a_n < b_n$, $\forall n \in \mathbb{N}$. Then

$$a_1 \leq a_2 \leq \cdots \leq a_{n-1} \leq a_n < b_n \leq b_{n-1} \leq \cdots \leq b_2 \leq b_1$$

so

$$[a_n, b_n] \subset [a_{n-1}, b_{n-1}] \subset [a_{n-2}, b_{n-2}] \ldots$$

then $\{[a_n, b_n]\}$ is a sequence of nested intervals. The sequence $\{a_n\}$ is upper bounded by any b_i and $\{b_n\}$ is lower bounded by any a_i, hence both sequences are monotone and bounded, so they have limit, which we will call respectively a and b. If $(b_n - a_n) \to 0$, these limits coincide, as we will see in **4.1**,

$$0 = \lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n \Longrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

4 Operations with limits

4.1 Addition and subtraction

a) If two sequences have a **finite** limit, the limit of the sum (difference) is the sum (difference) of the limits.

$$\boxed{\{a_n\} \to a, \ \{b_n\} \to b \Longrightarrow \{a_n \pm b_n\} \to a \pm b}$$

P: We prove it for the sum. Since a and b are limits, it holds that

$$\forall \varepsilon > 0 \begin{cases} \exists n_a \in \mathbb{N} \ / \ |a_m - a| < \varepsilon/2, \ \forall m \geq n_a \\ \exists n_b \in \mathbb{N} \ / \ |b_m - b| < \varepsilon/2, \ \forall m \geq n_b \end{cases}$$

Then, $\forall m \geq \max(n_a, n_b)$, both conditions will be fulfilled, so

$$|a_m + b_m - (a+b)| = |(a_m - a) + (b_m - b)| \leq |a_m - a| + |b_m - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

b) If $\{a_n\}$ is divergent and $\{b_n\}$ bounded, then $\{a_n \pm b_n\}$ is divergent.

c) If $\{a_n\} \to \pm\infty$ and $\{b_n\} \to \pm\infty$, then $\{a_n + b_n\} \to \pm\infty$.

d) If $\{a_n\} \to \pm\infty$ and $\{b_n\} \to \mp\infty$, then $\{a_n - b_n\} \to \pm\infty$.

e) If $\{a_n\} \to \pm\infty$ and $\{b_n\} \to \pm\infty$, then $\{a_n - b_n\}$ is an **indeterminate case**.

4.2 Product

a) If two sequences have a **finite** limit, the limit of the product is the product of the limits.

$$\boxed{\{a_n\} \to a, \ \{b_n\} \to b \Longrightarrow \{a_n \, b_n\} \to a \, b}$$

P: We must prove that $\forall \varepsilon > 0 \ \exists n \in \mathbb{N} \ / \ |a_m b_m - ab| < \varepsilon, \forall m \geq n.$ $\quad (*)$

Since $\{a_n\}$ is convergent, it is bounded: $|a_n| \leq K, \ \forall n \in \mathbb{N}$. We study two options:

1) Let $b \neq 0$. Then $|a_m b_m - ab| = |a_m b_m - a_m b + a_m b - ab| =$

$$|(a_m b_m - a_m b) + (a_m b - ab)| \leq |a_m| \, |b_m - b| + |b| \, |a_m - a| \leq K \, |b_m - b| + |b| \, |a_m - a| \, .$$

Since both are convergent, $\begin{cases} \forall \dfrac{\varepsilon}{2\,|b|} > 0 \ \exists n_a \ / \ |a_m - a| < \dfrac{\varepsilon}{2\,|b|}, \ \forall m \geq n_a \\ \\ \forall \dfrac{\varepsilon}{2K} > 0 \ \exists n_b \ / \ |b_m - b| < \dfrac{\varepsilon}{2K}, \ \forall m \geq n_b \end{cases}$

Then, $\forall m \geq \max(n_a, n_b), \ \ |a_m b_m - ab| < K \dfrac{\varepsilon}{2K} + |b| \dfrac{\varepsilon}{2\,|b|} = \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon.$

2) If $b = 0$, $\forall \dfrac{\varepsilon}{K} > 0 \ \exists n \ / \ |b_m - 0| < \dfrac{\varepsilon}{K}, \ \forall m \geq n$ and, replacing in $(*)$, we obtain

$$\forall m \geq n, \ \ |a_m b_m - ab| = |a_m b_m - 0| \leq |a_m| \, |b_m| \leq K \, |b_m| < K \frac{\varepsilon}{K} = \varepsilon$$

b) If $\{a_n\} \to 0$ and $\{b_n\}$ is bounded, then $\{a_n \, b_n\} \to 0$.

c) If $\{a_n\} \to \pm\infty$ and $b_n \geq k > 0 \, \forall n_0 \geq 0$, then $\{a_n \, b_n\} \to \pm\infty$.

d) If $\{a_n\} \to \pm\infty$ and $\{b_n\} \to 0$, then $\{a_n \, b_n\}$ is an **indeterminate case**.

4.3 Inverse

a) If a sequence has a limit which is **finite and non-zero**, the limit of the inverse is the inverse of the limit.

$$\boxed{\{a_n\} \to a \neq 0 \Longrightarrow \left\{\frac{1}{a_n}\right\} \to \frac{1}{a}}$$

P: We have to prove that $\forall \varepsilon > 0 \; \exists n \in \mathbb{N} \,/\, \left|\frac{1}{a_m} - \frac{1}{a}\right| < \varepsilon, \; \forall m \geq n.$

We operate: $\left|\frac{1}{a_m} - \frac{1}{a}\right| = \frac{|a - a_m|}{|a_m|\,|a|} = \frac{1}{|a|}\frac{1}{|a_m|}\,|a - a_m|$. In this expression:

1) $\frac{1}{|a_m|}$ is bounded. Indeed, if $\{a_n\} \to a$, then $\{|a_n|\} \to |a|$ (seen in practice sessions).

Since $\left|\frac{a}{2}\right| < |a|$, hence $\exists n_1 \,/\, |a_m| > \left|\frac{a}{2}\right| \Longrightarrow \frac{1}{|a_m|} < \frac{2}{|a|}, \; \forall m \geq n_1.$

2) The term $|a - a_m|$ can be made as small as we want, because $\{a_n\} \to a$.

So $\forall \varepsilon > 0 \; \exists n_2 \,/\, |a - a_m| < \frac{1}{2}\,\varepsilon\,|a|^2, \; \forall m \geq n_2.$

Therefore $\forall m \geq \max(n_1, n_2)$, $\left|\frac{1}{a_m} - \frac{1}{a}\right| = \frac{1}{|a|}\frac{1}{|a_m|}\,|a - a_m| < \frac{1}{|a|}\frac{2}{|a|}\frac{1}{2}\,\varepsilon|a|^2 = \varepsilon.$

b) If $\{a_n\}$ is divergent, then $\left\{\frac{1}{a_n}\right\} \to 0$. And if $\{a_n\} \to 0$, then $\left\{\frac{1}{a_n}\right\}$ is divergent.

4.4 Division

a) If two sequences have a **finite** limit , the limit of the quotient is the quotient of the limits <u>if the denominator does not tend to 0</u>.

$$\boxed{\{a_n\} \to a, \; \{b_n\} \to b \neq 0 \Longrightarrow \left\{\frac{a_n}{b_n}\right\} \to \left\{\frac{a}{b}\right\}}$$

P: $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} a_n \frac{1}{b_n} \overset{\text{If } \exists}{=} \lim\limits_{n\to\infty} a_n \lim\limits_{n\to\infty} \frac{1}{b_n} = a\,\frac{1}{b} = \frac{a}{b}$

b) If both $\{a_n\}$ and $\{b_n\}$ tend to 0 or to $\pm\infty$, then $\left\{\frac{a_n}{b_n}\right\}$ is an **indeterminate case**.

4.5 Logarithm

If $\{a_n\}$ has a limit which is **finite and positive**, the limit of the logarithm of $\{a_n\}$ is the logarithm of the limit.

$$\boxed{\{a_n\} \to a > 0 \Longrightarrow \{\log_b a_n\} \to \log_b a} \quad (b > 0, \, b \neq 1)$$

4.6 Exponential

If $\{a_n\} \to a$ and $b > 0$, the limit of the exponential of a_n is the exponential of the limit.

$$\boxed{\{a_n\} \to a, \, b > 0 \Longrightarrow \{b^{a_n}\} \to b^a}$$

4.7 Potential-exponential

If $\{a_n\} \to a > 0$ and $\{b_n\} \to b$, then the limit of $\left\{a_n^{b_n}\right\}$ is a^b.

$$\boxed{\{a_n\} \to a > 0,\ \{b_n\} \to b \Longrightarrow \left\{a_n^{b_n}\right\} \to a^b}$$

P: $\lim\limits_{n\to\infty} a_n^{b_n} = \lim\limits_{n\to\infty} e^{\ln a_n^{b_n}} = \lim\limits_{n\to\infty} e^{b_n \ln a_n} \overset{\text{If}\,\exists}{=} e^{\lim b_n \ln a_n} \overset{\text{If}\,\exists}{=} e^{\lim b_n\ \lim \ln a_n} = e^{b \ln a} = a^b$

Remark. In the previous demonstration, if $a > 0$ the terms of $\{a_n\}$ are greater than zero, starting from a given term (property 4 of the limits). So the expression $a_n^{b_n}$ will also be greater than zero and we can calculate its logarithm.

4.8 Indeterminate forms

The indeterminate cases (sections **4.1**, **4.2** and **4.4**) and certain cases of null or infinite limit in the base or exponent (section **4.7**) produce the following indeterminate forms:

$$\boxed{\infty - \infty;\quad \infty \cdot 0;\quad \frac{\infty}{\infty};\ \frac{0}{0};\quad 0^0;\quad \infty^0;\quad 1^\infty}$$

Remark. In the case 1^∞ of the table, 1 means a sequence of limit 1.

5 Convergence criteria

5.1 Stolz theorem

Let $\{a_n\}$ and $\{b_n\}$ be two sequences. We assume that the following conditions are fulfilled:

a) $\{b_n\}$ is strictly monotone.

b) Either $\{b_n\} \to \pm\infty$ or $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$.

Then

$$\boxed{\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = l \Longrightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = l} \quad (l \in \mathbb{R} \text{ or } l = \infty)$$

A proof can be seen in J. Burgos, p. 45.

Application. Indeterminate forms of the type $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$.

5.2 Arithmetic mean criterion

If the sequence $\{x_n\}$ converges, the sequence formed by the arithmetic means of its terms has the same limit. That is:

$$\boxed{\lim_{n\to\infty} x_n = l \Rightarrow \lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = l} \quad (l \in \mathbb{R} \text{ or } l = \infty)$$

Proof. We define the following sequences $a_n = x_1 + x_2 + \cdots + x_n$ y $b_n = n$ (which fulfill the Stolz conditions). Then $a_n - a_{n-1} = x_n$ and $b_n - b_{n-1} = 1$, so

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = \lim_{n\to\infty} \frac{a_n}{b_n} \overset{\text{If}\,\exists}{=} \lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_{n\to\infty} \frac{x_n}{1} = l.$$

5.3 Geometric mean criterion

Let $x_n > 0\ \forall n$. If the sequence $\{x_n\}$ converges, the sequence formed by the geometric means of its terms has the same limit. That is to say

$$\boxed{\lim_{n\to\infty} x_n = l \Longrightarrow \lim_{n\to\infty} \sqrt[n]{x_1 x_2 \cdots x_n} = l}$$

Proof. The criterion is valid for $l \geq 0$ or $l = +\infty$. We prove it in the case $l > 0$.

$$\lim_{n\to\infty} \sqrt[n]{x_1 \cdots x_n} = \lim_{n\to\infty} e^{\ln \sqrt[n]{(x_1 \cdots x_n)}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln(x_1\cdots x_n)} = \lim_{n\to\infty} e^{\frac{\ln x_1 + \cdots + \ln x_n}{n}} \overset{\text{If}\,\exists}{=} e^{\lambda},$$

where

$$\lambda = \lim_{n\to\infty} \frac{\ln x_1 + \cdots + \ln x_n}{n} \overset{\text{If}\,\exists}{=} \lim_{n\to\infty} \ln x_n \overset{\text{If}\,\exists}{=} \ln\left(\lim_{n\to\infty} x_n\right) = \ln l$$

then

$$\lim_{n\to\infty} \sqrt[n]{x_1 x_2 \cdots x_n} = e^{\ln l} = l$$

5.4 Root law

Consider the sequence $\{a_n\} \big/ a_n > 0\ \forall n$. It holds that

$$\boxed{\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = l \Longrightarrow \lim_{n\to\infty} \sqrt[n]{a_n} = l}$$

Proof. We define a new sequence $x_1 = a_1, x_2 = \dfrac{a_2}{a_1}, \ldots, x_n = \dfrac{a_n}{a_{n-1}}$. Then:

$$l = \lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \lim_{n\to\infty} x_n \overset{\text{G.M.}}{=} \lim_{n\to\infty} \sqrt[n]{x_1 x_2 \cdots x_n} = \lim_{n\to\infty} \sqrt[n]{a_1 \frac{a_2}{a_1} \cdots \frac{a_n}{a_{n-1}}} = \lim_{n\to\infty} \sqrt[n]{a_n}$$

That is, if the limit of the quotient exists, the limit of the root exists and they coincide. Although it may happen that the first limit does nor exist and the second does.

6 Infinites and infinitesimals

6.1 Definitions

Let $\{a_n\}$ be a sequence. We say that:

a) $\{a_n\}$ is an infinite if $\boxed{\lim_{n\to\infty} a_n = \pm\infty}$

b) $\{a_n\}$ is an infinitesimal if $\boxed{\lim_{n\to\infty} a_n = 0}$

If the limit of a sequence is 0, we also call it a null sequence.

Example.

1. Sequences $\{a_n\}$, of general term n^2, $\sqrt{n}$, -2^n, $\ln n$, are infinites.
2. Sequences $\{a_n\}$, of general term $\dfrac{1}{n^2}$, $\dfrac{1}{\sqrt{n}}$, $\dfrac{-1}{2^n}$, $\dfrac{(-1)^n}{\ln n}$ $(n \neq 1)$, are infinitesimals.

6.2 Comparison

Let $\{a_n\}$ and $\{b_n\}$ be infinites (infinitesimals):

a) If $\left\{\dfrac{a_n}{b_n}\right\}$ is an infinite (infinitesimal), $\{a_n\}$ **is of higher order than** $\{b_n\}$.

b) If $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = k \in \mathbb{R},\ k \neq 0$, $\{a_n\}$ and $\{b_n\}$ **are of the same order**. Then

- Let $\{a_n\}$ be an infinite. If

$$\lim_{n\to\infty}\frac{a_n}{n^p} = k \neq 0,\ p > 0$$

we say that $\{a_n\}$ **is of order** p **and** $\boxed{kn^p}$ **is its principal part**.

- Let $\{a_n\}$ be an infinitesimal. If

$$\lim_{n\to\infty}\frac{a_n}{(1/n)^p} = k \neq 0,\ p > 0$$

we say that $\{a_n\}$ is **of order** p **and** $\boxed{k/n^p}$ **is its principal part**.

c) If $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = 0$, we say that $\{a_n\}$ is **negligible** compared to $\{b_n\}$.

Examples. From definitions a), b) and c) above, we observe the following (we omit the braces indicating sequence for simplicity):

a.1) The infinite n^3 is of higher order than n and of lower order than n^5.

a.2) The infinitesimal $\dfrac{1}{n^3}$ is of higher order than $\dfrac{1}{n}$ and of lower order than $\dfrac{1}{n^5}$.

b.1) $3n^2$ is an infinite of the same order than n^2.

b.2) $\dfrac{3}{n}$ is an infinitesimal of the same order than $\dfrac{1}{n}$.

b.3) $3n^2 + n$ is an infinite of order 2 and its principal part is $3n^2$.

b.4) $\dfrac{3}{n} + \dfrac{1}{n^2}$ is an infinitesimal of order 1 and its principal part is $\dfrac{3}{n}$.

c.1) $3n$ is negligible compared to n^2.

c.2) $\dfrac{1}{n^2}$ is negligible compared to $\dfrac{3}{n}$.

6.3 Relation between types of infinite

It is shown (J. Burgos, p. 50) that every logarithmic infinite is negligible compared to any potential infinite; every potential compared to any exponential; and every exponential compared to any potential-exponential. We represent it with the symbol $\ll$:

$$\boxed{\underset{p>1,\ a>0}{\left(\log_p n\right)^a} \ll \underset{b>0}{n^b} \ll \underset{c>1}{c^n} \ll \underset{d>0}{n^{dn}}}$$

Example. It holds that: $(\ln n)^{1000} \ll \sqrt{n};\quad n^{1000} \ll 2^n;\quad 1000^n \ll n^n.$

7 Equivalent sequences

7.1 Definition

Let $\{a_n\}$ and $\{a'_n\}$ be sequences with the same limit, finite or infinite. We say that they are equivalent if the limit of their quotient is 1.

$$\boxed{a_n \sim a'_n \Longleftrightarrow \lim_{n\to\infty} \frac{a_n}{a'_n} = 1}$$

Then two equivalent sequences have the same limit (by definition). And if two sequences have the same limit a, finite and non-zero, they are equivalent. Indeed

$$\lim_{n\to\infty} \frac{a_n}{a'_n} = \frac{a}{a} = 1$$

7.2 Properties

If $a_n \sim a'_n$, it holds that

$$\boxed{\lim_{n\to\infty} \frac{a_n - a'_n}{a_n} = \lim_{n\to\infty} \frac{a_n - a'_n}{a'_n} = 0}$$

Proof. We divide the difference by a'_n (the same result is obtained by dividing by a_n).

$$\lim_{n\to\infty} \frac{a_n - a'_n}{a'_n} = \lim_{n\to\infty} \frac{a_n}{a'_n} - 1 = 0$$

Then:

a) The difference between two equivalent infinitesimals is another, of higher order than both.

b) The difference between two equivalent infinites is negligible compared to both.

7.3 Equivalence to the principal parts

a) Let a_n be an infinite. We know (**6.2**) that kn^p is the principal part of a_n if

$$\lim_{n\to\infty} \frac{a_n}{n^p} = k \in \mathbb{R},\ k \neq 0$$

Then

$$\lim_{n\to\infty} \frac{a_n}{kn^p} = \frac{1}{k} \lim_{n\to\infty} \frac{a_n}{n^p} = \frac{k}{k} = 1 \Longrightarrow a_n \sim kn^p$$

thus **any infinite is equivalent to its principal part**.

b) Let a_n be an infinitesimal: k/n^p is its principal part if the following is fulfilled

$$\lim_{n\to\infty} \frac{a_n}{(1/n)^p} = k \in \mathbb{R},\ k \neq 0$$

Then

$$\lim_{n\to\infty} \frac{a_n}{k/n^p} = \frac{1}{k} \lim_{n\to\infty} \frac{a_n}{(1/n)^p} = \frac{k}{k} = 1 \Longrightarrow a_n \sim k/n^p$$

thus **any infinitesimal is equivalent to its principal part**.

Example. Consider $a_n = 2n^3 + 3n$ (infinite) and $b_n = 2/n^3 + 3/n$ (infinitesimal). Each of them will be equivalent to its principal part, so $a_n \sim 2n^3,\ b_n \sim 3/n$.

8 Substitution by equivalent sequences

8.1 Product and quotient

Let $a_n \sim a'_n$, $b_n \sim b'_n$. The following relations hold (in each case the position in a table of equivalences, which is at the end of section **9** is indicated).

a) If $\exists \lim_{n\to\infty} a'_n b'_n \Longrightarrow \boxed{a_n b_n \sim a'_n b'_n}$ (A.1)

P: Let us see that both products have the same limit and the limit of the quotient is 1.

a.1. $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} \left[\frac{a_n\, b_n}{a'_n\, b'_n}\, a'_n b'_n\right] = \lim_{n\to\infty} \left[\left(\frac{a_n}{a'_n}\right)\left(\frac{b_n}{b'_n}\right)(a'_n b'_n)\right] = \lim_{n\to\infty} a'_n b'_n.$

because the expressions in the first two parentheses tend to 1.

a.2. $\lim_{n\to\infty} \frac{a_n b_n}{a'_n b'_n} = \lim_{n\to\infty} \frac{a_n}{a'_n} \lim_{n\to\infty} \frac{b_n}{b'_n} = 1.$

So either the products $a_n b_n$ and $a'_n b'_n$ have the same limit or both lack it (from a.1). And, if the limit exists, both products are equivalent (from a.2).

b) If $\exists \lim_{n\to\infty} \frac{a'_n}{b'_n} \Longrightarrow \boxed{\frac{a_n}{b_n} \sim \frac{a'_n}{b'_n}}$ (A.2)

Thus, **when substituting in products or quotients** any of the terms by an equivalent sequence, **the resulting expression is equivalent** to the first one.

Example.

1. $\lim_{n\to\infty} \frac{4n^2+\sqrt{n}}{3n^2-n} = \lim_{n\to\infty} \frac{4n^2}{3n^2} = \frac{4}{3}.$
2. $\left(\frac{1}{n}+\frac{1}{3n^2}\right)\left(\frac{1}{2n}-\frac{1}{4n^2}\right) \sim \frac{1}{n}\,\frac{1}{2n} = \frac{1}{2n^2}.$

In the first case we are looking for the limit of an indeterminate quotient. In the second we want to calculate the infinitesimal equivalent to the product (it is immediate that the limit is zero).

8.2 Logarithm

Let $\{a_n\}$ be a sequence of positive terms (starting from a given one), such that its limit is $a \geq 0$, $a \neq 1$ (a can be $+\infty$). It holds that

$$\boxed{a_n \sim a'_n \Longrightarrow \log a_n \sim \log a'_n} \quad \text{(A.3)}$$

That is to say, **replacing the argument of a logarithm** by an equivalent sequence, **the resulting expression is equivalent** to the first one, provided that the argument has not 1 as a limit. To prove this, let us see that:

a) $\log a_n$ and $\log a'_n$ have the same limit.

$$\lim_{n\to\infty} \log a_n = \lim_{n\to\infty} \log\left[\frac{a_n}{a'_n}\, a'_n\right] = \lim_{n\to\infty}\left[\log\overbrace{\left(\frac{a_n}{a'_n}\right)}^{\to 1} + \log a'_n\right] = \lim_{n\to\infty} \log a'_n.$$

Note that this limit always exists, since $\log a'_n$ tends to $\log a$, if $a \in \mathbb{R}^+$; it tends to $+\infty$, if $a_n \to \infty$; and it tends to $-\infty$ if $a_n \to 0$.

b) The limit of the quotient of $\log a_n$ by $\log a'_n$ is 1, because

$$\log a_n - \log a'_n = \log \Bigg(\overbrace{\frac{a_n}{a'_n}}^{\to 1}\Bigg) = \varepsilon_n \Longrightarrow \log a_n = \log a'_n + \varepsilon_n$$

Thus $\displaystyle\lim_{n\to\infty} \frac{\log a_n}{\log a'_n} = \lim_{n\to\infty} \frac{\log a'_n + \varepsilon_n}{\log a'_n} = \lim_{n\to\infty} \Bigg(1 + \frac{\varepsilon_n}{\underbrace{\log a'_n}_{\to \log a \neq 0}}\Bigg) = 1.$

Example.

1. $\ln(n^4 + 2n^2) \sim \ln n^4 = 4 \ln n.$
2. $\ln\left(\frac{1}{\sqrt{n}} + \frac{1}{n}\right) \sim \ln \frac{1}{\sqrt{n}} = -\ln\sqrt{n} = -\frac{1}{2}\ln n.$

In both cases we have obtained an infinite equivalent to the initial one, with a simpler expression.

8.3 Potential-exponential

Given $a_n \sim a'_n,\ b_n \sim b'_n$, in general **we cannot assure** that $a_n^{b_n} \sim {a'_n}^{b'_n}$.

Example. The infinite $n+1+1/n$ is equivalent to n. But $e^{n+1+\frac{1}{n}} \not\sim e^n$, because the quotient of both expressions has limit $e \neq 1$.

These indeterminate cases are generally resolved by doing

$$\boxed{a_n^{b_n} = e^{\ln a_n^{b_n}} = e^{b_n \ln a_n}}$$

provided that $a_n > 0, \forall n \in \mathbb{N}$.

8.4 Addition and subtraction

Given $a_n \sim a'_n,\ b_n \sim b'_n$, in general **we cannot assure** that $a_n \pm b_n \sim a'_n \pm b'_n$

Example 1. The second order infinites n^2+n+1 and n^2-n+1 are equivalent to each other and to their principal part n^2. But, when subtracting them, we cannot replace them by n^2:

$$n^2 + n + 1 - (n^2 - n + 1) \not\sim (n^2 - n^2) = 0$$

because the difference would be 0. In fact, when subtracting we obtain the infinite $2n$ (order 1).

Example 2. The first order infinitesimals $\frac{1}{n} + \frac{1}{n^2}$ and $\frac{1}{n} - \frac{1}{n^2}$ are equivalent to each other and to their principal part $1/n$. But, when subtracting them, we cannot replace them by $1/n$.

$$\frac{1}{n} + \frac{1}{n^2} - \left(\frac{1}{n} - \frac{1}{n^2}\right) \not\sim \frac{1}{n} - \frac{1}{n} = 0,$$

Instead, its difference is the infinitesimal of second order $2/n^2$.

In both cases we are dealing with two equivalent infinites (infinitesimals), so they have the same principal part. When subtracting them, the principal parts disappear and infinites of lower order (infinitesimals of higher order) become important. But these are not taken into account if the substitution by the principal parts is made, thus the method is not valid.

F. Granero (p. 153) gives a **practical rule** for functions, valid also for sequences:

a) *Consider a limit in which a factor or divisor is formed by sums or differences of **infinites**. Let m be the highest of their orders. If replacing the infinites by their principal parts we obtain an infinite of order m, the substitution is correct and the limit does not vary.*

b) *Consider a limit in which a factor or divisor is formed by sums or differences of **infinitesimals**. Let m be the lowest of their orders. If replacing the infinitesimals by their principal parts we obtain an infinitesimal of order m, the substitution is correct and the limit does not vary.*

In summary, in sums and differences of infinites or infinitesimals we can use equivalences if the principal parts do not disappear, so the order of the infinites (infinitesimals) does not change.

Example 3.

a) Let the infinites $a_n = n^3 - n^2,\ b_n = n^3 + n^2$, so $a_n, b_n \sim n^3$. Then

$$\lim_{n\to\infty} \frac{a_n + b_n}{3a_n - 2b_n} = \lim_{n\to\infty} \frac{n^3 + n^3}{3n^3 - 2n^3} = 2$$

Before and after replacing them by their principal parts, the infinites are of order 3.

b) Let $a_n = \dfrac{1}{n} + \dfrac{2}{n^2},\ b_n = \dfrac{1}{n} - \dfrac{3}{n^2} \Longrightarrow a_n, b_n \sim \dfrac{1}{n}$

$$\lim_{n\to\infty} 4n(a_n + b_n) = \lim_{n\to\infty} 4n\left(\frac{1}{n} + \frac{1}{n}\right) = 8$$

Before and after replacing them by their principal parts, the infinitesimals are of order 1.

9 Some practical methods

9.1 Using number e

a Definition

Number e is defined as: $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$. But it can also be obtained as:

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{\alpha_n}\right)^{\alpha_n} = e} \quad \{\alpha_n\} \text{ divergent} \quad (*)$$

P: We will prove it in the case where $\{\alpha_n\}$ is monotone increasing of positive terms.

The floor function $\lfloor x \rfloor$ (also called the integer part of x) gives us the maximum integer less than or equal to $x \in \mathbb{R}$. Let a_n be the integer part of α_n, $a_n = \lfloor \alpha_n \rfloor$. Then

$$a_n \le \alpha_n < a_n + 1 \Longrightarrow \frac{1}{a_n} \ge \frac{1}{\alpha_n} > \frac{1}{a_n+1} \Longrightarrow$$

$$\left(1+\frac{1}{a_n}\right)^{a_n+1} > \left(1+\frac{1}{\alpha_n}\right)^{\alpha_n} > \left(1+\frac{1}{a_n+1}\right)^{a_n} \Longrightarrow$$

$$\underbrace{\left(1+\frac{1}{a_n}\right)^{a_n}}_{(1)}\underbrace{\left(1+\frac{1}{a_n}\right)}_{(2)} > \left(1+\frac{1}{\alpha_n}\right)^{\alpha_n} > \underbrace{\left(1+\frac{1}{a_n+1}\right)^{a_n+1}}_{(3)} : \underbrace{\left(1+\frac{1}{a_n+1}\right)}_{(4)}$$

Factors (1) and (3) correspond to subsequences of $\left(1+\dfrac{1}{n}\right)^n$, so they have limit e.

Factors (2) and (4) have limit 1. Then the sequence tends to e (prop. 6 of the limits).

b Generalization

The expression $(*)$ can also be proved (J. Burgos, p. 40) when sequence $\{\alpha_n\}$:

1. Is monotone decreasing of negative terms.
2. Has a finite number of negative terms and the rest are positive.
3. Has a finite number of positive terms and the rest are negative.
4. Has infinitely many positive terms and infinitely many negative terms.

c Application

From the definition $(*)$ of e using $\{\alpha_n\}$, different limits can be solved:

1. $$\lim_{n\to\infty}\left(1+\frac{1}{\alpha_n}\right)^{\alpha_n+a}=\lim_{n\to\infty}\left[\left(1+\frac{1}{\alpha_n}\right)^{\alpha_n}\left(1+\frac{1}{\alpha_n}\right)^{a}\right]\overset{\text{If }\exists}{=}\lim_{n\to\infty}\left(1+\frac{1}{\alpha_n}\right)^{\alpha_n}\lim_{n\to\infty}\left(1+\frac{1}{\alpha_n}\right)^{a}=e$$

2. $$\lim_{n\to\infty}\left(1+\frac{1}{\alpha_n+a}\right)^{\alpha_n}=\lim_{n\to\infty}\left(1+\frac{1}{\alpha_n+a}\right)^{\alpha_n+a-a}=\lim_{n\to\infty}\left[\left(1+\frac{1}{\alpha_n+a}\right)^{\alpha_n+a}:\right.$$
$$\left.\left(1+\frac{1}{\alpha_n+a}\right)^{a}\right]\overset{\text{If }\exists}{=}\lim_{n\to\infty}\left(1+\frac{1}{\alpha_n+a}\right)^{\alpha_n+a}:\lim_{n\to\infty}\left(1+\frac{1}{\alpha_n+a}\right)^{a}=e$$

3. $$\lim_{n\to\infty}\left(1+\frac{1}{\alpha_n}\right)^{\alpha_n a}=\lim_{n\to\infty}\left[\left(1+\frac{1}{\alpha_n}\right)^{\alpha_n}\right]^{a}\overset{\text{If }\exists}{=}\left[\lim_{n\to\infty}\left(1+\frac{1}{\alpha_n}\right)^{\alpha_n}\right]^{a}=e^a$$

4. $$\lim_{n\to\infty}\left(1+\frac{a}{\alpha_n}\right)^{\alpha_n}=\lim_{n\to\infty}\left[\left(1+\frac{1}{\alpha_n/a}\right)^{\frac{\alpha_n}{a}}\right]^{a}\overset{\text{If }\exists}{=}\left[\lim_{n\to\infty}\left(1+\frac{1}{\alpha_n/a}\right)^{\frac{\alpha_n}{a}}\right]^{a}=e^a$$

d Equivalences

From the definition $(*)$ of number e, different equivalences are also justified, as we will see below (in each one the place it occupies in the table of equivalences is indicated).
First we prove the relation between natural logarithms -in base e- and logarithms in base a.

1. Base change: $\boxed{\log_a x=\dfrac{\ln x}{\ln a}}$ $(a>0,\ a\neq 1)$.

   P: $\forall x>0,\ x=a^{\log_a x}\Longrightarrow \ln x=\ln\left(a^{\log_a x}\right)=\log_a x\ln a\Longrightarrow \log_a x=\dfrac{\ln x}{\ln a}$

2. $\boxed{\ln(1+\theta_n)\sim\theta_n}$ (B.1)

   P: We calculate the limit of the quotient of both expressions. Since $\{1/\theta_n\}$ is divergent,

$$\lim_{n\to\infty}\frac{\ln(1+\theta_n)}{\theta_n}=\lim_{n\to\infty}\frac{1}{\theta_n}\ln\left(1+\frac{1}{1/\theta_n}\right)=\lim_{n\to\infty}\ln\left(1+\frac{1}{1/\theta_n}\right)^{\frac{1}{\theta_n}}\overset{\text{If }\exists}{=}$$
$$\ln\lim_{n\to\infty}\left(1+\frac{1}{1/\theta_n}\right)^{\frac{1}{\theta_n}}=\ln e=1$$

   If the logarithm is not natural: $\log_a(1+\theta_n)=\dfrac{\ln(1+\theta_n)}{\ln a}\sim\dfrac{\theta_n}{\ln a}$

3. $\boxed{\ln u_n \sim u_n - 1}$ (B.2)

P: $u_n - 1 = \theta_n \Longrightarrow u_n = 1 + \theta_n \Longrightarrow \ln u_n = \ln(1+\theta_n) \sim \theta_n = u_n - 1$

If the logarithm is not natural: $\log_a u_n = \dfrac{\ln u_n}{\ln a} \sim \dfrac{u_n - 1}{\ln a}$

4. $\boxed{e^{\theta_n} - 1 \sim \theta_n}$ (B.3)

P: $e^{\theta_n} - 1 \to 0 \Longrightarrow e^{\theta_n} - 1 = \theta'_n \sim \ln(1+\theta'_n) = \ln\left(1 + e^{\theta_n} - 1\right) = \ln e^{\theta_n} = \theta_n$

If the base of the exponential is not e: $a^{\theta_n} - 1 = e^{\theta_n \ln a} - 1 \sim \theta_n \ln a$

Examples.

1. If $a = e^2$, $\log_a x = \dfrac{\ln x}{\ln a} = \dfrac{1}{2}\ln x$
2. $\ln\left(1 + \dfrac{1}{\sqrt{n}}\right) \sim \dfrac{1}{\sqrt{n}}$
3. $\ln\left(\dfrac{n+3}{n-2}\right) \sim \dfrac{n+3}{n-2} - 1 = \dfrac{n+3-n+2}{n-2} = \dfrac{5}{n-2} \sim \dfrac{5}{n}$
4. $2^{1/n} - 1 \sim \dfrac{1}{n}\ln 2 \Longrightarrow \lim\limits_{n\to\infty} n\,(2^{1/n} - 1) = \lim\limits_{n\to\infty} n\,\dfrac{1}{n}\ln 2 = \ln 2$

9.2 Polynomial expressions

a Division

A polynomial in the indeterminate n is an infinite equivalent to its highest degree term, because:

$$\lim_{n\to\infty} \frac{a_0 + \cdots + a_k n^k}{a_k n^k} = \lim_{n\to\infty} \frac{1}{a_k}\left(\frac{a_0}{n^k} + \frac{a_1}{n^{k-1}} + \cdots + a_k\right) = 1 \Longrightarrow \boxed{P_k(n) \sim a_k\, n^k} \quad \text{(C.1)}$$

$$\text{Then } \lim_{n\to\infty} \frac{P_k(n)}{Q_l(n)} = \lim_{n\to\infty} \frac{a_k\, n^k}{b_l\, n^l} = \begin{cases} \dfrac{a_k}{b_l} & k = l \\ 0 & k < l \\ \infty \cdot \operatorname{sign}(a_k/b_l) & k > l \end{cases}$$

b Logarithm

If $a_k > 0$, the natural logarithm of $P_k(n)$ is equivalent to the infinite $k \ln n$, because

$$\ln(a_0 + \cdots + a_k n^k) \sim \ln(a_k n^k) = \ln a_k + k \ln n \sim k \ln n \Longrightarrow \boxed{\ln P_k(n) \sim k \ln n} \quad \text{(C.2)}$$

c *nth* root

The *nth* root of $P_k(n)$ has limit 1. Indeed, from the root law (see **5.4**),

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l \Longrightarrow \lim_{n\to\infty} \sqrt[n]{a_n} = l$$

Therefore, considering the sequence with a general term $P_k(n)$:

$$\lim_{n\to\infty} \frac{P_k(n+1)}{P_k(n)} \overset{\mathbf{a}}{=} \lim_{n\to\infty} \frac{a_k (n+1)^k}{a_k n^k} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^k = 1 \Longrightarrow \boxed{\lim_{n\to\infty} \sqrt[n]{P_k(n)} = 1}$$

d Difference of roots of polynomials

Sometimes, replacing each of the polynomials by its higher degree term does not solve the problem. In this case we can take as common factor the root of the highest degree term and use the equivalence:

$$\boxed{\sqrt[p]{1+\theta_n}-1 \sim \frac{\theta_n}{p}} \quad \text{(D.1)}$$

P: $\underbrace{\sqrt[p]{1+\theta_n}-1}_{\theta'_n} \sim \ln(1+\theta'_n) = \ln\left(1+\sqrt[p]{1+\theta_n}-1\right) = \ln(1+\theta_n)^{1/p} = \dfrac{1}{p}\ln(1+\theta_n) \sim \dfrac{\theta_n}{p}$

Exercises. Justify the following results, using equivalences **a-d**:

1. $\displaystyle\lim_{n\to\infty} \frac{n^3-3n^2}{2n^3+4n} = \frac{1}{2}$.
2. $\ln(2n^3-n+1) \sim 3\ln n$.
3. $\displaystyle\lim_{n\to\infty} \sqrt[n]{\ln n} = 1$.
4. $\displaystyle\lim_{n\to\infty} n\left(\sqrt[3]{\frac{n+1}{n-1}}-1\right) = \frac{2}{3}$.
5. $\displaystyle\lim_{n\to\infty} \left(\sqrt{n^2+a}-\sqrt{n^2+b}\right) = 0$.

9.3 Recurrent sequences

In these sequences each term is deduced from the previous ones:

$$\boxed{a_{n+1} = f(a_n, a_{n-1}, \dots)}$$

A classic example is the Fibonacci sequence, in which any term is obtained by adding the two previous terms, the first two being 0 and 1:

$$x_n = x_{n-1} + x_{n-2}, \quad x_1 = 0,\ x_2 = 1 \Longrightarrow \{x_n\} = 0, 1, 1, 2, 3, 5, 8, 13\dots$$

To calculate the limit of a recurrent sequence, we usually proceed in two steps. First it must be proved that the limit exists (for example, because it is a monotone and bounded sequence). Then it is obtained from the recurrence relation.

Exercise. Prove the convergence and calculate the limit of the sequence defined by:

$$x_{n+1} = \frac{2x_n-1}{3}, \quad x_1 = 1 \ (\textbf{Sol: } l = -1)$$

9.4 Stirling and trigonometric equivalences

a Stirling formula

$$\boxed{n! \sim n^n e^{-n}\sqrt{2\pi n}} \quad \text{(E.1)}$$

Exercise. Using Stirling formula, it is proposed to justify the place of the factorial infinity in the table of infinites of different type (see **6.3**).

$$\boxed{\underset{p>1,\ a>0}{\left(\log_p n\right)^a} \ll \underset{b>0}{n^b} \ll \underset{c>1}{c^n} \ll \underset{0<d<1}{n^{dn}} \ll n! \ll \underset{f\geq 1}{n^{fn}}}$$

b Trigonometric equivalences

1. The tangent, the sine and the angle in radians are equivalent infinitesimals (prove it as an exercise).

$$\boxed{\tan\theta_n \sim \sin\theta_n \sim \theta_n} \quad \text{(F.1)}$$

2. From the above equivalence: $\boxed{1-\cos\theta_n \sim \frac{1}{2}\theta_n^2}$ (F.2)

P: $1-\cos\theta_n = 2\sin^2\frac{\theta_n}{2} \sim 2\left(\frac{\theta_n}{2}\right)^2 = \frac{1}{2}\theta_n^2$

Example. We solve the following limit using trigonometric equivalences:

$$\lim_{n\to\infty} n^3\left(\tan\frac{2}{n}-\sin\frac{2}{n}\right) = \lim_{n\to\infty} n^3\tan\frac{2}{n}\left(1-\cos\frac{2}{n}\right) = \lim_{n\to\infty} n^3\,\frac{2}{n}\,\frac{(2/n)^2}{2} = 4$$

9.5 Change of the type of indetermination

Indeterminations of the type 1^∞, 0^0 and ∞^0 are usually solved using the relation

$$a^b = e^{b\ln a} \quad (a>0)$$

thus taking the indetermination to the exponent. In the case $\infty-\infty$, one of the two infinites is used as common factor.

If θ_n represents a null sequence; α_n a sequence divergent to $+\infty$; and u_n a sequence of limit 1, we obtain the following cases.

a) Change of type 1^∞ to type $\infty\cdot 0$: $\boxed{u_n^{\alpha_n} = e^{\alpha_n\ln u_n}}$ (G.1)

b) Change of type 0^0 to type $0\cdot(-\infty)$: $\boxed{\theta_n^{\theta'_n} = e^{\theta'_n\ln\theta_n}}$ (G.2)

c) Change of type ∞^0 to type $0\cdot\infty$: $\boxed{\alpha_n^{\theta_n} = e^{\theta_n\ln\alpha_n}}$ (G.3)

d) Indetermination type $\infty-\infty$: $\boxed{\alpha_n-\alpha'_n = \alpha_n\left(1-\frac{\alpha'_n}{\alpha_n}\right) = \alpha'_n\left(\frac{\alpha_n}{\alpha'_n}-1\right)}$ (G.4)

Examples.

1. $L = \lim_{n\to\infty}\left(\cos\frac{1}{n}\right)^{n^2} = \lim_{n\to\infty} e^{n^2\ln\cos\frac{1}{n}} \overset{\text{If }\exists}{=} e^\lambda;\ \lambda = \lim_{n\to\infty} n^2\ln\cos\frac{1}{n} = \lim_{n\to\infty} n^2\left(\cos\frac{1}{n}-1\right) =$
$-\lim_{n\to\infty} n^2\left(1-\cos\frac{1}{n}\right) = -\lim_{n\to\infty} n^2\frac{1}{2n^2} = -\frac{1}{2} \Longrightarrow L = e^{-1/2} = \frac{1}{\sqrt{e}}$

2. $L = \lim_{n\to\infty}\left(\tan\frac{1}{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln\tan\frac{1}{n}} \overset{\text{If }\exists}{=} e^\lambda.$

   $\lambda = \lim_{n\to\infty}\frac{1}{n}\ln\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n}(-\ln n) = 0 \Longrightarrow L = e^0 = 1$

3. $\lim_{n\to\infty}(3^n - n^5\,2^n) = \lim_{n\to\infty} 3^n\left(1-\frac{2^n}{3^n}n^5\right) = \lim_{n\to\infty} 3^n\left(1-\underbrace{\frac{n^5}{(3/2)^n}}_{\to 0}\right) = \infty$

EQUIVALENCES TABLE (SEQUENCES)

$$\{\alpha_n\} \to +\infty; \qquad \{\theta_n\} \to 0; \qquad \{u_n\} \to 1; \qquad a_n \sim a'_n; \qquad b_n \sim b'_n$$

A. GENERAL EQUIVALENCES

1.	$a_n b_n$	$\sim$	$a'_n b'_n$	$\left(\text{Si } \exists \lim\limits_{n\to\infty} a'_n b'_n\right)$
2.	$\dfrac{a_n}{b_n}$	$\sim$	$\dfrac{a'_n}{b'_n}$	$\left(\text{Si } \exists \lim\limits_{n\to\infty} \dfrac{a'_n}{b'_n}\right)$
3.	$\log_p(a_n)$	$\sim$	$\log_p(a'_n)$	$\left(\text{Si } \exists \lim\limits_{n\to\infty} a_n \neq 1\right)$

B. FROM NUMBER e

1.	$\ln(1+\theta_n)$	$\sim$	θ_n
2.	$\ln u_n$	$\sim$	$u_n - 1$
3.	$e^{\theta_n} - 1$	$\sim$	θ_n

Remark: For logarithms of base p, the next relation is used: $\log_p x = \dfrac{\ln x}{\ln p}$

C. POLYNOMIAL EXPRESSIONS

1.	$a_0 + a_1\alpha_n + \ldots + a_p\alpha_n^p$	$\sim$	$a_p\alpha_n^p$
2.	$\ln(a_0 + a_1\alpha_n + \ldots + a_p\alpha_n^p)$	$\sim$	$p \ln \alpha_n$

D. ROOTS

1.	$\sqrt[p]{1+\theta_n} - 1$	$\sim$	$\dfrac{\theta_n}{p}$

E. STIRLING

1.	$n!$	$\sim$	$n^n e^{-n}\sqrt{2\pi n}$

F. TRIGONOMETRIC

1.	θ_n	$\sim$	$\sin\theta_n$	$\sim$	$\tan\theta_n$
2.	$1 - \cos\theta_n$	$\sim$	$\dfrac{1}{2}\theta_n^2$		

G. CHANGE OF INDETERMINATION

1.	$u_n^{\alpha_n}$	$\sim$	$e^{\alpha_n \ln u_n}$	$[1^\infty \to e^{\infty\, 0}]$
2.	$\theta_n^{\theta'_n}$	$\sim$	$e^{\theta'_n \ln \theta_n}$	$\left[0^0 \to e^{0\,(-\infty)}\right]$
3.	$\alpha_n^{\theta_n}$	$\sim$	$e^{\theta_n \ln \alpha_n}$	$[\infty^0 \to e^{0\,\infty}]$
4.	$\alpha_n - \alpha'_n$	$\sim$	$\alpha_n\left(1 - \dfrac{\alpha'_n}{\alpha_n}\right)$	$\left[\infty - \infty \to \infty\left(1 - \dfrac{\infty}{\infty}\right)\right]$

10 Self-assesment exercises

10.1 True/False exercise

Decide whether the following statements are true or false.

1. Any convergent sequence is bounded and vice versa.
2. Any Cauchy sequence is convergent on $\mathbb{R}$.
3. If $\{a_n\} \to a \in \mathbb{R} \Longrightarrow \left\{\dfrac{1}{a_n}\right\} \to \dfrac{1}{a}$.
4. If $\{a_n\} \to a \in \mathbb{R},\ \{b_n\} \to b \in \mathbb{R} \Longrightarrow \left\{a_n^{b_n}\right\} \to a^b$.
5. If $\lim\limits_{n\to\infty} \dfrac{x_1 + \cdots + x_n}{n} = x \Longrightarrow \lim\limits_{n\to\infty} x_n = x$.
6. $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$, if the latter exists.
7. The difference of two equivalent infinites can be also an infinite, but it is negligible compared to both.
8. Let $a_n, a'_n > 0, \forall n \in \mathbb{N} \big/ a_n \sim a'_n$. Then it holds $\lim\limits_{n\to\infty} \ln a_n = \lim\limits_{n\to\infty} \ln a'_n$.
9. Let be a sum or difference of infinitesimals and m be the highest of its orders. If substituting the infinitesimals for their principal parts results in an infinitesimal of order m, the limit does not vary.
10. $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{\sqrt{n}}\right)^{\sqrt{4n}} = e^2$.
11. $n \ln\left(1 + \dfrac{1}{n^2}\right) \sim \dfrac{1}{n}$.
12. $\ln\left(3n^4 + 4n^2 + 4\right) \sim 4 \ln n$.
13. $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$.
14. The value of the expression $n^{\frac{1}{\ln n}}$ is e, $\forall n > 1$.
15. $\lim\limits_{n\to\infty} \left(n^2 \ln n - n \ln^2 n\right) = +\infty$.

10.2 Questions

Question 1. Reason the truth or falsity of this statement:

"Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\{a_n\} \to a \in \mathbb{R}$ and $\{b_n\}$ has no limit, finite or infinite. If the (product) sequence $\{a_n b_n\}$ has a limit $c \in \mathbb{R}$, then the limit $a = 0$."

Question 2. Give an example of a divergent sequence obtained as a sum of:

a) One divergent and one convergent. b) One divergent and one oscillating. c) Two divergent. d) Two oscillating. e) Two convergent.

Question 3. Let $\{a_n\} + \{b_n\} = \{a_n + b_n\}$. Is it sufficient that one of the two sequences is divergent for the sum to be divergent? Is it necessary?

Question 4. Reason what we can say about the sum of $\{a_n\}$ and $\{b_n\}$ if both are:
a) Convergent. b) Bounded. c) Oscillating. d) Divergent.

10.3 Solution to the True/False exercise

1. **F**. Every convergent sequence is bounded (prop. 8 of the limits), but not vice versa. For example, the sequence $\left\{1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{6}, \ldots\right\}$ is bounded but does not converge.

2. **T**. It is one of the ways to express the completeness of $\mathbb{R}$. It does not hold on $\mathbb{Q}$.

3. **F**. It is true only if $a \neq 0$.

4. **F**. The power of base a and exponent b is defined in general for $a > 0$. If $a > 0$, the implication is true.

5. **F**. Counterexample: the sequence $\{a, -a, a, -a, \ldots\}$. It holds from right to left.

6. **T**. It is a consequence of the Stolz theorem.

7. **T**. It is proved in section **7** of the unit (Equivalent sequences).

8. **T**. Being $a_n \sim a'_n$, both have the same limit a. We distinguish two cases:

   a) If $a \neq 1$, $\ln a_n \sim \ln a'_n$, so they have the same limit (section 8.2. of the unit).
   b) If $a = 1$, both $\ln a_n$ and $\ln a'_n$ tend to $\ln 1 = 0$.

9. **F**. …and m the **lowest** of their orders…(practical rule for sums and differences of infinitesimals).

10. **T**. Doing $\sqrt{4n} = 2\sqrt{n}$, we have one of the studied cases using number e (section 9.1).

11. **T**. $n \ln\left(1 + \frac{1}{n^2}\right) \sim n\,\frac{1}{n^2} = \frac{1}{n}$ (see table of equivalences).

12. **T**. $\ln(3n^4 + 4n^2 + 4) \sim \ln 3n^4 = \ln 3 + 4\ln n \sim 4\ln n$ (see table of equivalences).

13. **T**. Using Stirling's formula and comparing potential and exponential infinitesimals, we get:
$$\lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{n^n\, e^{-n}\sqrt{2\pi n}}{n^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}}{e^n} = 0$$

14. **T**. $n^{\frac{1}{\ln n}} = e^{\ln\left(n^{\frac{1}{\ln n}}\right)} = e^{\left(\frac{1}{\ln n}\ln n\right)} = e^1 = e,\ \forall n > 1$ (section **9.5** of the unit).

15. **T**. $\lim_{n\to\infty}\left(n^2\ln n - n\ln^2 n\right) = \lim_{n\to\infty} n^2 \ln n\left(1 - \frac{\ln n}{n}\right) = +\infty$ (section **9.5** of the unit).

10.4 Solution to the questions

Question 1 *Truth or falsity of: "Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\{a_n\} \to a \in \mathbb{R}$ and $\{b_n\}$ has no limit, finite or infinite. If the (product) sequence $\{a_n b_n\}$ has a limit $c \in \mathbb{R}$, then the limit $a = 0$."*

Since $\{b_n\}$ has no limit, it has to be either divergent or oscillating. In the first case, its terms become as large as we want in absolute value, starting from a given one. In the second case, neither the above is fulfilled nor $\{b_n\}$ has a finite limit.

It seems then logical that $\{a_n\} \to 0$ is a necessary condition for the convergence of $\{a_n b_n\}$. We will prove it by *reductio ad absurdum*.

Suppose $a \neq 0$. Then the terms of $\{a_n\}$ are different from 0 starting from a certain index and it will be satisfied:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{a_n b_n}{a_n} = \frac{\lim\limits_{n\to\infty} a_n b_n}{\lim\limits_{n\to\infty} a_n} = \frac{c}{a} \in \mathbb{R}$$

But this result contradicts the hypothesis that the sequence $\{b_n\}$ has no limit. Thus it is proved that, if $\{a_n b_n\}$ has a limit $c \in \mathbb{R}$, then the limit $a = 0$.

Question 2 *Examples of a divergent sequence obtained as a sum of:*

1. *One divergent and one convergent:* $\left\{n + \frac{1}{n}\right\} = \{n\} + \left\{\frac{1}{n}\right\}$
2. *One divergent and one oscillating:* $\{n + (-1)^n\} = \{n\} + \{(-1)^n\}$
3. *Two divergent:* $\{2n\} = \{n\} + \{n\}$
4. *Two oscillating:* $\{1, 2, 3, 4, 5, 6, \dots\} = \{1, 0, 3, 0, 5, 0, \dots\} + \{0, 2, 0, 4, 0, 6, \dots\}$
5. *Two convergent:* It does not exist. The sum of two convergent sequences converges to the sum of the limits (see section **4**).

Question 3 *Let* $\{a_n\} + \{b_n\} = \{a_n + b_n\}$*. Is it sufficient that one of the two sequences is divergent for the sum to be divergent? Is it necessary?*

- It is not sufficient: e.g. $\{1, 2, 3, 4, \dots\} + \{0, -2, 0, -4, \dots\} = \{1, 0, 3, 0, \dots\}$, oscillating.
- It is not necessary: in question 2, case d) a divergent sequence is obtained by adding two oscillating ones.

Question 4 *Reason what we can say about the sum of* $\{a_n\}$ *and* $\{b_n\}$ *if both are:*

a) *Convergent:* The sum is convergent (see question 2, case e)).

b) *Bounded:* The sum is bounded since, being both bounded,
$\exists\, k_1, k_2 > 0 \,/\, |a_n| < k_1,\ \ |b_n| < k_2\ \forall n$. So, $|a_n + b_n| \leq |a_n| + |b_n| < k_1 + k_2\ \forall n$.
As a consequence, the sum cannot be divergent.

c) *Oscillating:* We cannot assure anything.

Example 1: $\{0, 1, 0, 1, \dots\} + \{0, -1, 0, -1, \dots\} = \{0, 0, 0, 0, \dots\}$, convergent.

Example 2: $\{1, 0, 3, 0, \dots\} + \{0, 2, 0, 4, \dots\} = \{1, 2, 3, 4, \dots\}$, divergent.

Example 3: $\{0, 1, 0, 1, \dots\} + \{0, 1, 0, 1, \dots\} = \{0, 2, 0, 2, \dots\}$, oscillating.

d) *Divergent:* We cannot assure anything.

Example 1: $\{1, 2, 3, 4, \dots\} + \{-1, -2, -3, -4, \dots\} = \{0, 0, 0, 0, \dots\}$, convergent.

Example 2: $\{1, 2, 3, 4, \dots\} + \{1, 2, 3, 4, \dots\} = \{2, 4, 6, 8, \dots\}$, divergent.

Example 3: $\{1, 2, 3, 4, \dots\} + \{1, -2, 3, -4, \dots\} = \{2, 0, 6, 0, \dots\}$, oscillating.

Unit IV. Real functions (14.12.2023)

A. General notions

1 Concept of function

Definition. Let A and B be two sets and let $D \subset A$. A function f, from D to B, is any correspondence that associates, to each $x \in D$, a single element $f(x) \in B$.

$$f : D \to B; \quad x \to y = f(x)$$

x is called the independent variable; y is the dependent variable.
If $A = B = \mathbb{R}$, then it is a real function of a real variable, e.g. the function $y = \sqrt{1 - x^2}$.

Domain. It is the set D of values of the independent variable for which the function exists. In the example, the radicand must be greater than or equal to 0, so

$$1 - x^2 \geq 0 \Longrightarrow x^2 \leq 1 \Longrightarrow |x| \leq 1$$

thus the domain D is the interval $[-1, 1]$.

Image of the function. It is the set of images of the elements of D.

$$\boxed{f(D) = \big\{ f(x) \in B \,/\, x \in D \big\}}$$

In the example, $f(D) = [0, 1]$. The set B of which $f(D)$ is a subset, is called the codomain.

Graph. The graph of a function is the set of points

$$\boxed{G_f = \big\{ \big(x, f(x)\big) \in \mathbb{R}^2 \,/\, x \in D \big\}}$$

In the example, the graph of the function is the semicircle of equation $x^2 + y^2 = 1$, $y \geq 0$.

2 Operations with funcions

Functions defined on D. The set of all real functions of a real variable, defined on a certain set $D \subset \mathbb{R}$, is denoted $\mathcal{F}(D, \mathbb{R})$. D is usually an interval.

Operations between funtions. Between the elements of $\mathcal{F}(D, \mathbb{R})$ (functions defined on D), the operations addition, product and product by a real number are defined, as

$$\boxed{(f + g)(x) = f(x) + g(x)}, \quad \boxed{(f \cdot g)(x) = f(x) \cdot g(x)}, \quad \boxed{(\lambda \cdot f)(x) = \lambda \cdot f(x)}, \quad \forall x \in D$$

Order relation. We define the following order relation between functions:

$$\boxed{f \leq g \Longleftrightarrow f(x) \leq g(x)}, \forall x \in D$$

Bounded function. f is said to be bounded on $S \subset D$ if the set $f(S)$ (image of f) is bounded. In this case, the corresponding elements of $f(S)$ are called the supremum, infimum, maximum or minimum of f on S.

Composition of functions. Consider the functions $f : A \to B$ and $g : E \to F$, such that $f(A) \subset E$. We define the composite function $g \circ f$ as

$$g \circ f : A \to F \Big/ \boxed{(g \circ f)(x) = g\big(f(x)\big)}, \forall x \in A$$

The composition of functions satisfies the associative property, but not the commutative.

3 Types of functions

Even and odd functions. We say that a function f, defined on D, is:

- Odd, if and only if: $\boxed{f(-x) = -f(x)}$, $\forall x \in D$. Example: $\sin x$, x^3, $\tanh x$.
- Even, if and only if: $\boxed{f(-x) = f(x)}$, $\forall x \in D$. Example: $\cos x$, x^2, $\cosh x$.

Exercise. Prove that the only function which is at the same time even and odd is the null function.

Periodic functions. A function f is said to be periodic of period T if and only if

$$\exists T \in \mathbb{R} \Big/ \boxed{f(x) = f(x+T)}, \forall x \in D$$

Example. The sine and cosine functions. They are usually studied on the interval $[0, T]$.

Monotone functions. There are two cases:

- Monotone increasing: $\boxed{x_1 < x_2 \Longrightarrow f(x_1) \leq f(x_2)}$, $x_1, x_2 \in D$. Example: $y = \sqrt{x}$.
- Monotone decreasing: $\boxed{x_1 < x_2 \Longrightarrow f(x_1) \geq f(x_2)}$, $x_1, x_2 \in D$. Example: $1/x$ $(x > 0)$.

If $f(x_1)$ is never equal to $f(x_2)$, then f is strictly monotone.

The main elementary functions. They are the power, the exponential, the logarithm, the sine, the cosine and the tangent functions.

$$\boxed{x^a,\, a \in \mathbb{R}; \quad b^x,\, b \in \mathbb{R}^+; \quad \log_c x,\, c \in \mathbb{R}^+,\, c \neq 1; \quad \sin x; \quad \cos x; \quad \tan x}$$

The addition, the product, the division and the composition of a finite number of these main elementary functions give rise to the elementary functions.

Some particular functions.

1. Floor function (or integer part of x). Its value is the maximum integer less than or equal to $x \in \mathbb{R}$. It is represented as $\lfloor x \rfloor$ or floor(x): $\lfloor x \rfloor = p \in \mathbb{Z} \big/ p \leq x < p + 1$.

2. Fractional (or decimal) part function. We define it, only for $x > 0$, from the floor function. It is represented as $\{x\}$ or $frac(x)$: $\{x\} = x - \lfloor x \rfloor$.

3. Sign function: $sgn(x) = \begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

4. Dirichlet function: $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

B. Limits of functions

1 Limit of a function

Definition. Let f be a function in $\mathcal{F}(D, \mathbb{R})$. Let a be a point, such that f is defined on a punctured neighborhood of a, U_a^*. We say that f has limit φ at a if :

$$\boxed{\lim_{x \to a} f(x) = \varphi \Longleftrightarrow \forall \varepsilon > 0 \, \exists \, \delta > 0 \big/ 0 < |x-a| < \delta \Longrightarrow |f(x) - \varphi| < \varepsilon}$$

If x is close enough to a, its image $f(x)$ will be as close to φ as we want.

Note that we are not interested in the value of $f(a)$ (which does not even have to exist), but in the behavior of $f(x)$ when x approaches a $(x \to a)$.

Example. $f(x) = \begin{cases} x^2+1, & x \neq 0 \\ 2, & x = 0 \end{cases}$

The limit l of $f(x)$, when $x \to 0$, is equal to 1 $\big($note that $l \neq f(0)\big)$.

Remark. Let U_a be a neighborhood of a. If we remove the point a from it, we get a punctured neighborhood U_a^*. For example, if U_a is the interval $(a-\delta, a+\delta)$, U_a^* will be $(a-\delta, a) \cup (a, a+\delta)$. (A punctured ball centered at a is a punctured neighborhood of a: see Unit II, section 2).

2 One-sided limits

Definition. If, when calculating the limit of f at $x = a$, we approach the point only from the right or only from the left, we call the limit obtained a **one-sided limit**.
In this case, f must be defined on at least a punctured half-neighborhood of a: $(a-\delta, a)$ (limit from the left) or $(a, a+\delta)$ (limit from the right).

We denote the existence of limits of f at a from the left and from the right, respectively, as

$$\boxed{\lim_{x \to a^-} f(x) = \varphi^-} \quad ; \quad \boxed{\lim_{x \to a^+} f(x) = \varphi^+}$$

Example. Let $sgn(x)$ be the sign function. Its one-sided limits at $x = 0$ are:

$$\lim_{x \to 0^-} sgn(x) = -1; \quad \lim_{x \to 0^+} sgn(x) = 1$$

Relation between the limit of a function and the one-sided limits. If f exists on a neighborhood of a, we can approach the point from both sides. In that case the following theorem holds:

"It is a necessary and sufficient condition for f to have a limit at a that the one-sided limits exist at a and they coincide".

$$\boxed{\lim_{x \to a} f(x) = \varphi \Longleftrightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \varphi}$$

Exercise. Find the one-sided limits of the following functions:

a) Floor function of x, at points $x_p = p \in \mathbb{Z}$.

b) $f(x) = \begin{cases} x, & x \geq 0 \\ x-1, & x < 0 \end{cases}$, at $x = 0$.

3 Extension of the concept of limit

We have defined finite limit at a point. If we now look at the function $f(x) = 1/x^2$ (figure 1, left), we see that $f(x) \to \infty$ when $x \to 0$ and $f(x) \to 0$ when $x \to \infty$. This leads us to extend the concept of a limit, defining **limit infinity** and **limit at infinity**.

a) Limit infinity at a point: if x is sufficiently close to a, $f(x)$ becomes as large as we want.

$$\boxed{\lim_{x \to a} f(x) = \infty \iff \forall A > 0, \exists\, \delta > 0 \,/\, 0 < |x-a| < \delta \Longrightarrow f(x) > A}$$

b) Finite limit at infinity. If x grows sufficiently, $f(x)$ becomes as close to φ as we want.

$$\boxed{\lim_{x \to \infty} f(x) = \varphi \iff \forall \varepsilon > 0, \exists B > 0 \,/\, x > B \Longrightarrow |f(x) - \varphi| < \varepsilon}$$

c) Limit infinity at infinity. If x grows sufficiently, $f(x)$ becomes as large as we want.

$$\boxed{\lim_{x \to \infty} f(x) = \infty \iff \forall A > 0, \exists B > 0 \,/\, x > B \Longrightarrow f(x) > A}$$

In the figure on the left ($f(x) = 1/x^2$), examples of cases a) (where $a = 0$) and b) (where $\varphi = 0$) are shown.

In the figure on the right ($f(x) = x^2$), an example of case c) is shown.

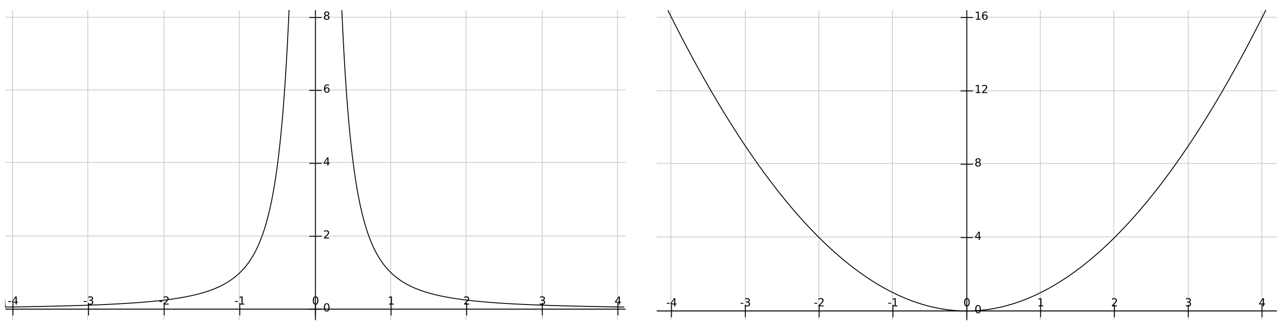

Figure 1: Cases a) $a = 0$; b) $\varphi = 0$; c) $a = \varphi = +\infty$.

We have defined the cases corresponding to $+\infty$. The definitions for $-\infty$ are very similar and we propose to obtain them as an exercise.

Exercise. Study the limits at $+\infty$ and at 0^+ of the following two functions and draw their graphs approximately:

a) $f(x) = e^{1/x}$

b) $f(x) = \begin{cases} \dfrac{1}{x}, & x \in \mathbb{Q} \\ -\dfrac{1}{x}, & x \notin \mathbb{Q} \end{cases}$

4 Sequential limit. Relation with the limit of a function

Let $f(x)$ be defined on a punctured neighborhood U_a^*, $a \in \mathbb{R}$. The following theorem holds:

"It is a necessary and sufficient condition for the function f to have limit φ at a that, for any sequence $\{x_n\}$ of limit a, its transformed sequence $\{f(x_n)\}$ has limit φ.".

$$\boxed{\lim_{x\to a} f(x) = \varphi \iff \forall \{x_n\} \,/\, \lim_{n\to\infty} x_n = a,\; \lim_{n\to\infty} f(x_n) = \varphi}$$

That is, if $f(x) \to \varphi$ (as $x \to a$), any sequence that tends to a is transformed by f into another that tends to φ.

And if there exists a sequence $\{x_n\}$ with limit a, whose transformed sequence does not have limit φ, the function will not have limit φ at $x = a$.

Application. This theorem is very useful for justifying the operations with limits of functions from the corresponding operations with limits of sequences (section **B.6**). It is also useful for proving that a certain function has no limit at a point a: we do it by finding a sequence of limit a, whose transformed sequence does not have a limit, as we will see in the following example.

Example. We study if the function $f(x) = \cos\dfrac{\pi}{x}$ has a limit at the origin.

When $x \to 0$, $\dfrac{\pi}{x} \to +\infty$. Then, for very small variations of x, the angle takes all the values in the successive intervals $[2\pi(n-1), 2\pi n]$, $n \in \mathbb{N}$, and the corresponding values of the function go over the interval $[-1, 1]$ again and again. Thus it seems clear that there is no limit.

We can demonstrate this by taking the sequence $x_n = 1/n$ whose limit is 0. Its transformed sequence by the function f is

$$\left\{\cos\frac{\pi}{x_n}\right\} = \left\{\cos\frac{\pi}{1/n}\right\} = \{\cos n\pi\}$$

which, when $n \in \mathbb{N}$, takes the values $\{-1, 1, -1, 1, \dots\}$, so it has no limit.

Since there is a sequence of limit 0 whose transformed sequence has no limit, it is proved that $f(x)$ has no limit at $x = 0$.

In the following figure, the behavior of the function near the origin is observed.

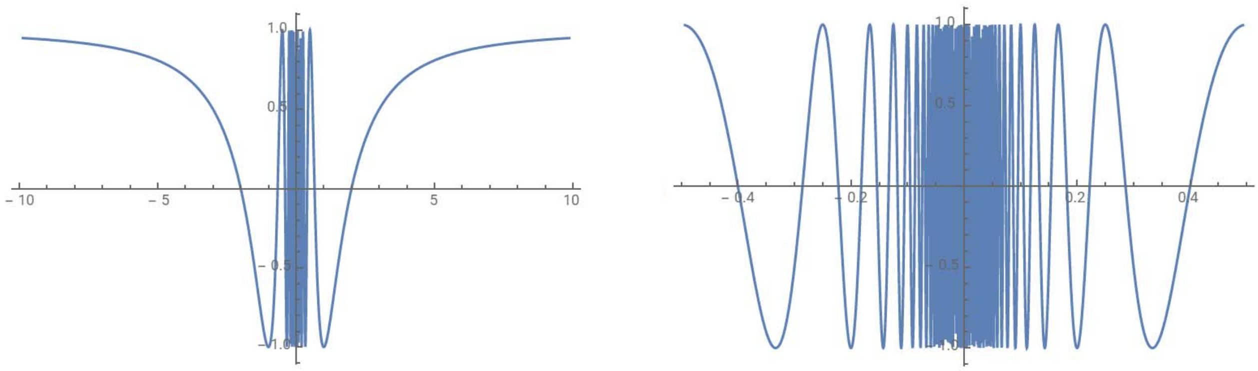

Figure 2: $f(x) = \cos\dfrac{\pi}{x}$. a) Between $x = -10$ and $x = 10$; b) Between $x = -0.5$ and $x = 0.5$.

Exercise. Study if the function $\sin\dfrac{\pi}{x}$ has a limit at the origin.

5 Properties of the limits

1) *If a function has a limit at a point, it is unique.*

D: By *reductio ad absurdum.* Let us suppose there are two limits, φ_1 and φ_2.

We take $\varepsilon > 0 \big/ 2\varepsilon < |\varphi_1 - \varphi_2|$ (ε less than the semidistance). Since φ_1 and φ_2 are limits,

$$\exists\delta_1 > 0 \ / \ 0 < |x-a| < \delta_1 \Longrightarrow |f(x) - \varphi_1| < \varepsilon \quad (1)$$
$$\exists\delta_2 > 0 \ / \ 0 < |x-a| < \delta_2 \Longrightarrow |f(x) - \varphi_2| < \varepsilon \quad (2)$$

Be $\delta = \min(\delta_1, \delta_2)$. If $0 < |x-a| < \delta$, conditions (1) and (2) are satisfied and

$$|\varphi_1 - \varphi_2| = |\varphi_1 - f(x) + f(x) - \varphi_2| \leq |\varphi_1 - f(x)| + |f(x) - \varphi_2| < 2\varepsilon$$

That is, $2\varepsilon > |\varphi_1 - \varphi_2|$, against the hypothesis.

2) *If f has a finite limit at a, the function is bounded on a punctured neighborhood of a, V_a^*.*

D: Since φ is the limit,

$$\forall\varepsilon > 0 \ \exists\delta > 0 \ / \ 0 < |x-a| < \delta \Longrightarrow |f(x) - \varphi| < \varepsilon$$

Then

$$-\varepsilon < f(x) - \varphi < \varepsilon \Longrightarrow \boxed{\varphi - \varepsilon < f(x) < \varphi + \varepsilon} \text{ for } x \in V_a^*$$

V_a^* is formed by the points that satisfy $0 < |x-a| < \delta$, i.e. $V_a^* = (a-\delta, a) \cup (a, a+\delta)$.

3) *If f has a finite non-zero limit φ at a, there exists a punctured neighborhood of a on which f has the sign of the limit.*

D: From property 2,

- If $\boxed{\varphi > 0}$ we take $\varepsilon < \varphi \Longrightarrow 0 < \varphi - \varepsilon < f(x) \Longrightarrow \boxed{f(x) > 0}$ for $x \in U_a^*$.

- If $\boxed{\varphi < 0}$ we take $\varepsilon < -\varphi \Longrightarrow f(x) < \varphi + \varepsilon < 0 \Longrightarrow \boxed{f(x) < 0}$ for $x \in V_a^*$.

4) *If $f(x) \leq g(x) \leq h(x)$ on a punctured neighborhood of a and functions f and h have a limit φ at a, then g has a limit φ at a.*

$$\left.\begin{array}{c} f(x) \leq g(x) \leq h(x) \text{ in } V_a^* \\ \\ \lim_{x\to a} f(x) = \lim_{x\to a} h(x) = \varphi \end{array}\right\} \Longrightarrow \lim_{x\to a} g(x) = \varphi$$

D: Since φ is the limit of f and h,

$$\forall\varepsilon > 0 \begin{cases} \exists\delta_1 > 0 \ / \ 0 < |x-a| < \delta_1 \Longrightarrow \varphi - \varepsilon < f(x) < \varphi + \varepsilon \\ \exists\delta_2 > 0 \ / \ 0 < |x-a| < \delta_2 \Longrightarrow \varphi - \varepsilon < h(x) < \varphi + \varepsilon \end{cases}$$

Also $f(x) \leq g(x) \leq h(x)$ for $0 < |x-a| < \delta$. Thus, if $\delta^* = \min(\delta, \delta_1, \delta_2)$, then

$$\forall\varepsilon > 0 \ \exists\delta^* > 0 \ / \ 0 < |x-a| < \delta^* \Longrightarrow \boxed{\varphi - \varepsilon < f(x) \leq g(x) \leq h(x) < \varphi + \varepsilon}$$

Therefore g has a limit φ at a.

6 Operations with limits

Operations with limits of functions are analogous to operations with limits of sequences (unit III). Proofs for functions can be deduced from the corresponding operations for sequences and from the relation between the limit of a function and the sequential limit (section **B.4**). The first of them is proved as an example at the end of this section.

Let f and g be two functions defined at least on a punctured neighborhood V_a^* of $a \in \mathbb{R}$, (a can be $\pm\infty$). In the first 6 paragraphs the limits are finite. In paragraph 7 we study operations when infinite limits appear, which sometimes give rise to indeterminate cases.

1. $\lim\limits_{x\to a} f(x) = \varphi,\ \lim\limits_{x\to a} g(x) = \gamma \Longrightarrow \boxed{\lim\limits_{x\to a} (\lambda f(x) + \mu g(x)) = \lambda\varphi + \mu\gamma,\ \forall\lambda,\mu \in \mathbb{R}}$

2. $\lim\limits_{x\to a} f(x) = \varphi,\ \lim\limits_{x\to a} g(x) = \gamma \Longrightarrow \boxed{\lim\limits_{x\to a} f(x)\, g(x) = \varphi\,\gamma}$

3. $\lim\limits_{x\to a} f(x) = \varphi \neq 0 \Longrightarrow \boxed{\lim\limits_{x\to a} \dfrac{1}{f(x)} = \dfrac{1}{\varphi}}$

4. $\lim\limits_{x\to a} f(x) = \varphi \Longrightarrow \boxed{\lim\limits_{x\to a} |f(x)| = |\varphi|}$

5. $\lim\limits_{x\to a} f(x) = \varphi > 0,\ p > 0\ (p \neq 1) \Longrightarrow \boxed{\lim\limits_{x\to a} \log_p(f(x)) = \log_p \varphi}$

6. $\lim\limits_{x\to a} f(x) = \varphi > 0,\ \lim\limits_{x\to a} g(x) = \gamma \Longrightarrow \boxed{\lim\limits_{x\to a} (f(x))^{g(x)} = \varphi^\gamma}$

7. $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = \pm\infty \Longrightarrow$

   a) $\boxed{\lim\limits_{x\to a} f(x) + g(x) = \pm\infty}$

   b) $\boxed{\lim\limits_{x\to a} f(x)\, g(x) = +\infty}$

   c) $\boxed{\lim\limits_{x\to a} \alpha f(x) = \pm\infty,\ \forall\alpha > 0}$

   d) $\boxed{\lim\limits_{x\to a} \dfrac{1}{f(x)} = 0}$

   e) $\boxed{\lim\limits_{x\to a} f(x) - g(x) = ?}$

   f) $\boxed{\lim\limits_{x\to a} f(x)/g(x) = ?}$

Types of indetermination. Paragraphs 6 (in certain cases of limit zero or infinity in the base or exponent) and 7 (cases e and f) give rise to the following types of indetermination:

$$\boxed{\frac{0}{0};\ \ \frac{\infty}{\infty};\ \ 0\cdot\infty;\ \ \infty-\infty;\ \ 1^\infty;\ \ 0^0;\ \ \infty^0}$$

Proof of 1: Since both functions have a limit at $x = a$, every sequence $\{x_n\}$, with limit a, is transformed -by means of function f- into the sequence $\{f(x_n)\}$, with limit φ; and -by means of function g- in the sequence $\{g(x_n)\}$, with limit γ.

Then, by means of $\lambda f + \mu g$, any sequence $\{x_n\}$, with limit a, is transformed in the sequence $\{\lambda f(x_n) + \mu g(x_n)\}$, which has limit $\lambda\varphi + \mu\gamma$ (see the operations with limits of sequences).

7 Infinites and infinitesimals

7.1 Definitions

Let $f \in \mathcal{F}(D, \mathbb{R})$ and $a \in \mathbb{R}$ or $a = \pm\infty$. We say that:

a) f is an infinite at a if $\boxed{\lim_{x \to a} f(x) = \pm\infty.}$

b) f es un infinitesimal at a if $\boxed{\lim_{x \to a} f(x) = 0.}$

Example. If $x \to \infty$, x^3 is an infinite and $1/x^3$ an infinitesimal. If $x \to 1$, $(x-1)^2$ is an infinitesimal and $1/(x-1)^2$ an infinite.

7.2 Comparison

Let $f, g \in \mathcal{F}(D, \mathbb{R})$ be infinites (infinitesimals) at a, such that they are not zero on a punctured neighborhood of a.

a) If f/g is an infinite (infinitesimal) at a, f **is of higher order than** g at a.

Example. If $x \to \infty$, x^3 is an infinite of higher order than x^2, because their quotient x^3/x^2 tends to infinity; and $1/x^4$ is an infinitesimal of higher order than $1/x^2$, because their quotient tends to zero.

b) If $\lim_{x \to a} \dfrac{f(x)}{g(x)} = k \in \mathbb{R},\ k \neq 0$, f **and** g **are of the same order** at a. Then:

b.1. Let f be an infinite for $x \to \infty$. If

$$\lim_{x \to \infty} \frac{f(x)}{x^p} = k \in \mathbb{R},\ k \neq 0,$$

$f(x)$ **is of order** p **and** $\boxed{kx^p}$ **is its principal part**.

b.2. Let f be an infinitesimal for $x \to \infty$. If

$$\lim_{x \to \infty} \frac{f(x)}{1/x^p} = k \in \mathbb{R},\ k \neq 0,$$

$f(x)$ **is of order** p **and** $\boxed{\dfrac{k}{x^p}}$ **is its principal part**.

b.3. Let f be an infinitesimal at $a \in \mathbb{R}$. If

$$\lim_{x \to a} \frac{f(x)}{(x-a)^p} = k \in \mathbb{R},\ k \neq 0,$$

$f(x)$ **is of order** p **and** $\boxed{k(x-a)^p}$ **is its principal part**.

Example. $2x + 3x^2$ is an infinitesimal of order 1 at $x = 0$ (principal part: $2x$). It is also an infinite of order 2 for $x \to \infty$ (principal part: $3x^2$).

Exercise. Obtain the order and the p.p. of $\sqrt{9x^2 - 2x}$ and of $\left(\dfrac{1}{x} + \dfrac{1}{x^2}\right)^2$ $(x \to \infty)$.

c) If $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$, f is negligible compared to g at a.

Example. $\sqrt{x}$ is negligible compared to x^2 and $\frac{1}{x^3}$ is negligible compared to $\frac{1}{x}$ $(x \to \infty)$.

Remark. From the equivalences between functions (see table of equivalences at the end of section **9**), we can determine the order and the principal part of non-polynomial infinites and infinitesimals. For example, if $x \to a$, it results:

$$f(x) = \sin^2(x-a) \sim (x-a)^2$$

thus $f(x)$ is an infinitesimal of order 2 and its principal part is $(x-a)^2$.

Exercise. If $x \to 2$, obtain the order and the principal part of $g(x) = e^{2x-4} - 1$.

7.3 Relation between types of infinite

It can be shown that every logarithmic infinite is negligible compared to any potential infinite; every potential compared to any exponential; and every exponential compared to any potential-exponential. We represent it with the symbol $\ll$:

$$\boxed{\underset{p>1,\ a>0}{\left(\log_p x\right)^a} \ll \underset{b>0}{x^b} \ll \underset{c>1}{c^x} \ll \underset{d>0}{x^{dx}}}$$

8 Equivalent functions at a point

Let $f, g \in \mathcal{F}(D, \mathbb{R})$ be functions with the same limit, finite or infinite, at a ($a \in \mathbb{R}$ or $a = \pm\infty$). We say that f g are equivalent at a if the following condition is satisfied:

$$\boxed{f \sim g \text{ at } a \Longleftrightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = 1}$$

The equivalence of functions is a somewhat broader topic than that of sequences, since the equivalence of functions is defined for $x \to a$ or for $x \to \infty$, while for sequences it is defined only when $n \to \infty$. But the reasoning made for sequences is also valid here, which allows us to abbreviate the study of the equivalence of functions and their substitution by equivalent functions.

Thus, if $a \in \mathbb{R}$ or $a = \pm\infty$, then:

- If $f \sim g$ at a, then both functions have the same limit at a.

- If f and g have the same finite and not null limit at a, then $f \sim g$ at a.

- If f and g are equivalent infinitesimals at a (so of the same order), then $f - g$ is an infinitesimal of higher order at a.

- If f and g are equivalent infinites at a, then $f - g$ is negligible compared to any of them at a (we cannot assure it is an infinite).

- If f is an infinite at a, then it is equivalent to its principal part at a.

- If f is an infinitesimal at a, then it is equivalent to its principal part at a.

9 Substitution by equivalent functions

As in section **B.6** here we are using the results obtained for the equivalence of sequences (unit III) as well as the relation between the limit of a function and the sequential limit (section **B.4**).

9.1 Product and division

It can be shown that *when substituting in products (quotients) a function by an equivalent one, we obtain an expression equivalent to the first.* A necessary condition for this is that the limit of the product (quotient) of the equivalent functions exists.

That is, if $f_1 \sim f_2$ and $g_1 \sim g_2$ at a, then

$$\boxed{f_1 \cdot g_1 \sim f_2 \cdot g_2 \text{ at } a} \quad \left(\text{if } \exists \lim_{x \to a} f_2 \cdot g_2\right)$$

$$\boxed{f_1/g_1 \sim f_2/g_2 \text{ at } a} \quad \left(\text{if } \exists \lim_{x \to a} f_2/g_2\right)$$

9.2 Logarithm

It can be shown that *when substituting the argument of a logarithm for an equivalent function, the result is an expression equivalent to the first*, provided that the function does not have limit 1 at a.

That is, if the function f_1 takes only positive values on a neighborhood of a and its limit at a is $\varphi \geq 0,\ \varphi \neq 1$ (φ can be $+\infty$), it is satisfied

$$\boxed{f_1 \sim f_2 \text{ at } a \Longrightarrow \ln f_1 \sim \ln f_2 \text{ at } a}$$

9.3 Potential-exponential

Given $f_1 \sim f_2$ and $g_1 \sim g_2$ at a, in general **we cannot assure** the equivalence between $(f_1)^{g_1}$ and $(f_2)^{g_2}$.

$$(f_1)^{g_1} \not\sim (f_2)^{g_2}$$

For example, if $x \to \infty,\ x+1 \sim x$. But $e^{x+1} \not\sim e^x$ (their quotient has limit $e \neq 1$).

These indeterminations are generally resolved by doing the following

$$\boxed{f^g = e^{g \ln f}} \quad (\text{if } f > 0)$$

9.4 Addition and subtraction

As we saw in sequences, in a sum (difference) we cannot, in general, replace functions by their equivalents, if the sum (difference) of limits is zero. Nor can we do it in a difference of infinites (indetermination of the type $\infty - \infty$). F. Granero (p. 153) gives the following practical rule.

a) *Consider a limit in which a factor or divisor is formed by sums or differences of* ***infinites****. Let m be the highest of their orders. If replacing the infinites by their principal parts results in an infinite of order m, the substitution is correct and the limit does not vary.*

b) *Consider a limit in which a factor or divisor is formed by sums or differences of* ***infinitesimals****. Let m be the lowest of their orders. If replacing the infinitesimals by their principal parts results in an infinitesimal of order m, the substitution is correct and the limit does not vary.*

In summary, in sums and differences of infinites or infinitesimals we can use equivalences if the principal parts do not disappear, so the order of the infinites (infinitesimals) does not change.

EQUIVALENCES TABLE (FUNCTIONS)

$$\lim_{x\to x_0}\alpha(x)=\infty;\quad \lim_{x\to x_0}\theta(x)=0;\quad \lim_{x\to x_0}u(x)=1;\quad f_1(x)\sim f_2(x);\quad g_1(x)\sim g_2(x)$$

These equivalences are at $x_0 \begin{cases} x_0\in\mathbb{R} \\ x_0=\pm\infty \end{cases}$; $f_1(x)\sim f_2(x)$ at $x_0 \Leftrightarrow \lim_{x\to x_0}\frac{f_1(x)}{f_2(x)}=1$

A. GENERAL EQUIVALENCES

1.	$f_1(x)\cdot g_1(x)$	$\sim$	$f_2(x)\cdot g_2(x)$	$\left(\text{If } \exists \lim_{n\to\infty} f_2(x)g_2(x)\right)$
2.	$\dfrac{f_1(x)}{g_1(x)}$	$\sim$	$\dfrac{f_2(x)}{g_2(x)}$	$\left(\text{If } \exists \lim_{n\to\infty} \dfrac{f_2(x)}{g_2(x)}\right)$
3.	$\log_p(f_1(x))$	$\sim$	$\log_p(f_2(x))$	$\left(\text{If } \exists \lim_{n\to\infty} f_1(x)\neq 1\right)$

B. FROM NUMBER e

1.	$\ln(1+\theta(x))$	$\sim$	$\theta(x)$
2.	$\ln u(x)$	$\sim$	$u(x)-1$
3.	$e^{\theta(x)}-1$	$\sim$	$\theta(x)$

Remark: For logarithms of base p, we use the relation: $\log_p x=\dfrac{\ln x}{\ln p}$

C. POLYNOMIALS

1.	$a_0+a_1\alpha(x)+\ldots+a_p\alpha^p(x)$	$\sim$	$a_p\alpha^p(x)$
2.	$\ln(a_0+a_1\alpha(x)+\ldots+a_p\alpha^p(x))$	$\sim$	$p\ln\alpha(x)$

D. ROOTS

1.	$\sqrt[p]{1+\theta(x)}-1$	$\sim$	$\dfrac{\theta(x)}{p}$

E. TRIGONOMETRIC

1.	$\theta(x)$	$\sim$	$\sin\theta(x)$	$\sim$	$\tan\theta(x)$
2.	$1-\cos\theta(x)$	$\sim$	$\dfrac{1}{2}\theta(x)^2$		

F. CHANGE OF INDETERMINATION

1.	$f(x)^{g(x)}$	$\sim$	$e^{g(x)\ln f(x)}$	$[\text{for } 1^\infty, 0^0, \infty^0]$
2.	$\alpha_p(x)-\alpha_q(x)$	$\sim$	$\alpha_p(x)\left(1-\frac{\alpha_q(x)}{\alpha_p(x)}\right)$	$\left[\infty-\infty\to\infty\left(1-\frac{\infty}{\infty}\right)\right]$
3.	$\alpha_p(x)-\alpha_q(x)$	$\sim$	$\dfrac{\frac{1}{\alpha_p(x)}-\frac{1}{\alpha_q(x)}}{\frac{1}{\alpha_p(x)\alpha_q(x)}}$	$\left[\infty-\infty\to\dfrac{0}{0}\right]$

C. Continuity of functions

1 Continuous function

Definition. Let $f \in \mathcal{F}(D, \mathbb{R})$ be a function defined on a neighborhood of a, U_a. We say that f is continuous at a if:

$$\boxed{\forall \varepsilon > 0 \; \exists \delta > 0 \big/ \; |x-a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon}$$

Since f is defined on U_a, we can take values of x as close to a as we want and find the limit. So the above definition is equivalent to:

$$\boxed{f \text{ is continuous at } a \Longleftrightarrow \lim_{x \to a} f(x) = f(a)}$$

Example. Let f be defined as $f(x) = \begin{cases} x^2 \cos \dfrac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

We define the function in this way because the expression outside the origin is not valid at $x = 0$. Studying the limit when $x \to 0$, we observe that the first factor tends to 0 and the second is bounded, so the limit is zero and the function is continuous at the origin.

Exercise. Study the continuity at the origin of $f(x) = \begin{cases} \dfrac{1}{x} \sin x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

2 One-sided continuity

Definition. If f is defined on at least a half-neighborhood of a, that is, on $[a, a+\delta)$ or $(a-\delta, a]$, we say that:

- f is continuous at a^+ (a from the right) if and only if $\lim_{x \to a^+} f(x) = f(a)$
- f is continuous at a^- (a from the left) if and only if $\lim_{x \to a^-} f(x) = f(a)$

Let f be defined on $U_a = (a-\delta, a+\delta)$. From the relation between one-sided and functional limits (**B.2**), we have that *"f is continuous at a if and only if it is continuous at a^+ and at a^-"*.

Continuity on an interval. f is continuous on $I = (a, b)$, if it is continuous $\forall x \in I$.

Closed interval. We say that f is continuous on $I = [a, b]$ if it is continuous on (a, b), at a^+ and at b^-.

Example. We study the continuity at $x = 0$ of $f(x) = \begin{cases} e^{-x}, & x \leq 0 \\ x, & x > 0 \end{cases}$

We obtain the one-sided limits:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} e^{-x} = e^0 = f(0) \, ; \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0 \neq f(0)$$

so we see that f is continuous at 0^- and discontinuous at 0^+.

Exercises.

1. Graph and study the function $f(x) = e^{1/x}$, when $x \to 0^-$.
2. Graph and study the function $f(x) = |x+1|$.
3. Study the continuity at the origin of $f(x) = \begin{cases} |x|, & x < 0 \\ |1+x|, & x \geq 0 \end{cases}$

3 Discontinuities

If f is not continuous at a point a, we say that there is a discontinuity at that point. We distinguish the following types:

a) Removable discontinuity. It occurs when the limit of f at a exists, but it does not coincide with $f(a)$ (either because $f(a)$ does not exist or because both values exist and are different).

$$\boxed{\exists \lim_{x\to a} f(x) \neq f(a)}$$

To remove this discontinuity, we can take the value of the limit as $f(a)$.

$$f(a) = \lim_{x\to a} f(x)$$

Example. If $f(x) = \dfrac{1-x^2}{2+2x}$, then $f(-1)$ does not exist. We obtain the limit at $x = -1$:

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{(1-x)(1+x)}{2(1+x)} = \lim_{x\to -1} \frac{(1-x)}{2} = 1$$

By giving the value 1 to $f(-1)$, the function becomes continuous at that point.

b) Jump discontinuity. It occurs when the one-sided limits exist and are different .

Example. The floor function at all points $x \in \mathbb{Z}$. The sign function at $x = 0$.

c) Essential discontinuity. In this case one of the one-sided limits either does not exist or is infinite.

Example. The functions $f(x) = \sin 1/x$ or $g(x) = 1/x^2$ at the origin.

Exercises. Study the continuity of the following functions at the indicated points, defining the value $f(a)$ if the discontinuity is removable.

1. $f(x) = \dfrac{\sin(x+2)}{2x+4}$, $a = -2$.
2. $g(x) = \dfrac{f(x)}{x}$, $a = 2$, where $f(x)$ is the floor function.
3. $h(x) = \dfrac{1}{x-1}$, $a = 1$.
4. $r(x) = e^{-1/x^2}$, $a = 0$.

4 Operations with continuous functions

Let $f, g \in \mathcal{F}(I, \mathbb{R})$ be continuous on I. From the properties of the operations with limits, it is easy to prove the continuity on I of:

1. $\lambda f + \mu g$, $\forall \lambda, \mu \in \mathbb{R}$
2. $f \cdot g$
3. $1/g$, except at the points where g is zero.

As a consequence of the above, the polynomial function $f(x) = P_k(x)$ is continuous, as is the quotient $g(x) = \dfrac{P_k(x)}{Q_l(x)}$, except at the points where $Q_l(x)$ is zero.

5 Continuity of the elementary functions

From the definition of elementary functions and the properties of the operations with limits, the continuity on I of the following functions on their corresponding intervals can be proved:

1. x^a, $a \in \mathbb{R}$, on $(0, \infty)$ (for certain values of a, e.g. $a \in \mathbb{N}$, the domain can be larger).
2. b^x, $b > 0$, on $\mathbb{R}$.
3. $\log_c x$, $c > 0$, $c \neq 1$, on $(0, \infty)$.
4. x^x, on $(0, \infty)$.
5. $\sinh x, \cosh x, \tanh x$, on $\mathbb{R}$.
6. $\sin x, \cos x$, on $\mathbb{R}$.
7. $\tan x$, on $\mathbb{R} \setminus \left\{(2k-1)\dfrac{\pi}{2}\right\}_{k \in \mathbb{Z}}$.

6 Composition of continuous functions

Consider the functions f and g such that f is continuous on I, g is continuous on J and $f(I) \subset J$. Under these conditions, it can be proved that *"the composite function $g \circ f$ is continuous on I"*.

Example. The function $f(x) = 1/x$ is continuous on $\mathbb{R} \setminus \{0\}$ and the function $g(x) = e^x$ is continuous on $\mathbb{R}$. By composing them in the two possible ways, we obtain:

- Function $(g \circ f)(x) = e^y\big|_{y=\frac{1}{x}} = e^{\frac{1}{x}}$ is continuous on $\mathbb{R} \setminus \{0\}$.
- Function $(f \circ g)(x) = \left.\dfrac{1}{y}\right|_{y=e^x} = \dfrac{1}{e^x} = e^{-x}$ is continuous on $\mathbb{R}$.

Remark. Observe that the domain of the composite function $g \circ f$ is formed by those points x in the domain of f that satisfy that the values $f(x)$ are part of the domain of g.

$$D_{g \circ f} = \left\{x \in D_f \,/\, f(x) \in D_g\right\}$$

By applying the composition of continuous functions to the elementary functions introduced in the previous section, we can ensure the continuity, on the corresponding intervals, of functions such as $\sin(\ln x)$, $\arctan \sqrt{x}$, $e^{\cosh x}$, etc.

Exercises.

1. Consider the functions $f(x) = \sqrt{x}$, continuous on $[0, \infty)$ and $g(x) = \sin x$, continuous on $\mathbb{R}$. Find the functions $g \circ f$ and $f \circ g$. Check that:

   - The function $g \circ f$ is continuous on $[0, \infty)$.

   - The function $f \circ g$ is continuous on $\bigcup_{k\in\mathbb{Z}} [2k\pi, (2k+1)\pi]$.

2. Consider the functions $f(x) = |x|$, continuous on $\mathbb{R}$ and $g(x) = \ln x$, continuous on $(0, \infty)$. Find the functions $g \circ f$ and $f \circ g$. Check that:

   - The function $g \circ f$ is continuous on $\mathbb{R} \setminus \{0\}$.

   - The function $f \circ g$ is continuous on $(0, \infty)$.

7 Theorems on continuous functions

7.1 Bolzano's theorem

If a continuous function on $[a, b]$ has values of different sign at the endpoints of the interval, there is at least one intermediate point where the value of f is zero.

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, such that $f(a) \cdot f(b) < 0$. It is satisfied:

$$\boxed{\exists c \in (a, b) \ / \ f(c) = 0}$$

Proof. We divide $[a, b]$ in half. If $f = 0$ at the midpoint, the theorem has been proved.
Otherwise, we choose the semi-interval $[a_1, b_1]$ on whose endpoints f have values of different sign. We now divide $[a_1, b_1]$ in half, choosing the semi-interval $[a_2, b_2]$ on whose endpoints f have different signs. By repeating the operation over and over again, we obtain a sequence of nested intervals $[a_n, b_n]$, of lengths $l_n = (b - a)/2^n$.
When $n \to \infty$, this sequence defines a point c (see unit III, section **3.3**), at which f must be zero. Indeed, if $f(c) \neq 0$, since f is continuous, it will be satisfied

$$\lim_{x\to c} f(x) = f(c) \neq 0$$

so there will exist a neighborhood of c in which f takes the sign of the limit (prop. 3 of the limits). But, due to the way we have found c, in every neighborhood of it there are points where $f > 0$ and points where $f < 0$. Then, by *reductio ad absurdum*, $f(c) = 0$.

7.2 Darboux property (intermediate value property)

If a continuous function on $[a, b]$ has different values at the endpoints of the interval, f takes all intermediate values at least once.

Let us assume $f(a) < f(b)$ (if $f(a) > f(b)$ the proof is analogous). It is satisfied that:

$$\boxed{\forall y \ / \ f(a) < y < f(b), \ \exists c \in (a, b) \ / \ f(c) = y}$$

Proof. We define the function $g(x) = f(x) - y$. As $f(a) < y < f(b)$,

$$g(a) = f(a) - y < 0; \quad g(b) = f(b) - y > 0$$

Since g has different signs at a and at b, there is an intermediate point where its value is zero (see **7.1**). Therefore

$$\exists c \in (a, b) \ / \ g(c) = f(c) - y = 0 \Longrightarrow f(c) = y$$

so the value y is taken by the function at $x = c$.

7.3 Weierstrass theorem

If a function is continuous on a closed interval, it attains its maximum and its minimum on it.

Proof. We do it in two parts (by *reductio ad absurdum*):

a) Let us see that **a continuous function** on $[a, b]$ **is bounded** on $[a, b]$. We assume that it is not bounded and divide $[a, b]$ in half, choosing the semi-interval $[a_1, b_1]$ on which f is not bounded (at least it will not be bounded on one of them). Now we divide $[a_1, b_1]$ in half, again choosing the semi-interval $[a_2, b_2]$ on which f is not bounded.

Repeating the operation over and over again, we obtain a sequence of nested intervals $[a_n, b_n]$, of length $(b-a)/2^n$. When $n \to \infty$, this sequence defines a point c (unit III, section **3.3**), such that f is not bounded on any neighborhood of it.

But, since f is continuous, it must have a limit at c, so it must be bounded on a neighborhood of this point (prop. 2 of the limits), with which we have reached a contradiction.

b) Since f is bounded on $I = [a, b]$, the set $f(I)$ has a supremum M and an infimum m (property of the supremum, unit I, section 6). But these values are reached by f, so it **has a maximum and a minimum** (we will see it for M by *reductio ad absurdum*):

Indeed, if f does not reach the value M, $f(x) - M$ cannot be zero and the function

$$g(x) = \frac{1}{M - f(x)}$$

is continuous on I, then it is bounded as we have just seen. So, $\forall x \in I$,

$$\exists k > 0 \; / \; \frac{1}{M - f(x)} < k \Longrightarrow \frac{1}{k} < M - f(x) \Longrightarrow f(x) < M - \frac{1}{k},$$

Therefore $M - 1/k$ (a value less than M) is an upper bound of f, so M is not the supremum, against the hypothesis.

7.4 Image of a closed interval

A continuous function transforms a closed interval into a closed interval.

Proof. From Weierstrass's theorem, every f, continuous on a closed interval $[a, b]$, reaches in it a maximum M and a minimum m.

Furthermore, due to the Darboux property, f will reach all values between m and M. Thus, the interval $[a, b]$ is transformed, by means of f, into the closed interval $[m, M]$.

If, in addition, the function is monotone, $[a, b]$ will become $[f(a), f(b)]$ (if f is increasing) or $[f(b), f(a)]$ (if it is decreasing).

7.5 Image of an interval

A continuous function transforms an interval into an interval (J. Burgos, pg. 152).

It is proved that a continuous function f transforms an interval I into another interval $f(I)$. It will take one of the following four forms:

$$(\alpha, \beta), \; [\alpha, \beta], \; [\alpha, \beta), \; (\alpha, \beta]$$

where α and β are, respectively, the infimum and the supremum of the function f on I; that is, the infimum and the supremum of the set $f(I)$.

$$\alpha = \inf f(I), \quad \beta = \sup f(I)$$

8 Uniform continuity

8.1 Definition

As seen in **C.2**, a function f is continuous on I if it is continuous at all its points (if I contains its endpoints, the continuity is one-sided at them). To obtain the continuity condition on I, we use the condition at a point $a \in I$, writing x instead of a. And instead of the point x ("close enough" to a) we use an $x' \in I$. Thus, the continuity condition on I will be

$$\boxed{\forall \varepsilon > 0,\ \exists\, \delta > 0 \,\big/\, |x' - x| < \delta \Longrightarrow |f(x') - f(x)| < \varepsilon \ \ (x' \in I)}$$

This value δ depends in general on ε and also on the point x of the interval, so

$$\delta = \delta(\varepsilon, x)$$

When we can express the continuity condition independently of the point x, so that the value of δ depends only on ε we say that the **continuity** is **uniform**. Formally,

$$f \text{ uniformly cont. on } I \Longleftrightarrow \boxed{\forall \varepsilon > 0,\ \exists\, \delta(\varepsilon) > 0 \,\big/\, |x_1 - x_2| < \delta \Longrightarrow |f(x_1) - f(x_2)| < \varepsilon}$$

That is, continuity is uniform if the difference between the values of f at two points depends exclusively on the distance between these points, regardless of their position in the interval.

Example 1. Consider the function $f(x) = ax + b$, $a > 0$. We study the difference between the values of f at two points and look for the value of δ that ensures that this difference is less than ε.

$$|f(x_1) - f(x_2)| = |ax_1 + b - (ax_2 + b)| = a\,|x_1 - x_2| < \varepsilon \Longleftrightarrow |x_1 - x_2| < \frac{\varepsilon}{a}$$

In this case, if the distance between two values of the variable is less than ε/a, the difference between the corresponding values of f will be less than ε. Then, having chosen a value for ε, we can obtain the necessary δ by dividing ε by a, with which we have removed the dependence on x and the continuity is uniform.

This result suggests that if, from the condition $|f(x_1) - f(x_2)| < \varepsilon$, we get a relation

$$|x_1 - x_2| < h(\varepsilon)$$

where $h(\varepsilon)$ can be made independent of the point x, then we can ensure that the continuity is uniform by taking $\delta = h(\varepsilon)$. This is immediate in the case we have just seen: a function whose graph is a straight line for which the relation between ε and δ is constant.

Example 2 . We study the function $y = 1/x$, which is continuous (and unbounded) on $(0, \infty)$:

$$|f(x_1) - f(x_2)| = \left|\frac{1}{x_1} - \frac{1}{x_2}\right| = \left|\frac{x_2 - x_1}{x_1\,x_2}\right| = \frac{|x_1 - x_2|}{|x_1|\,|x_2|} < \varepsilon \Longleftrightarrow |x_1 - x_2| < \varepsilon\,|x_1|\,|x_2|$$

The product $\varepsilon\,|x_1|\,|x_2|$, can be made as close to 0 as we want by taking x_1, x_2, smaller and smaller. This means that we **cannot** find a value $\delta > 0$ that allows us to ensure that

$$|x_1 - x_2| < \delta \Longrightarrow |f(x_1) - f(x_2)| < \varepsilon$$

Therefore, this function is not uniformly continuous on $(0, \infty)$.

The following two theorems allow us to justify in a simple way that certain functions are uniformly continuous.

8.2 Heine's theorem

"If a function f is continuous on a closed interval I, then it is uniformly continuous on I".

Example 1. As we saw in the previous section, the function $f(x) = 1/x$ is not uniformly continuous on $(0, \infty)$. However, if we consider a closed interval $[a, b]$, $a, b > 0$, it turns out that there is a minimum value a of the variable x on the interval, so that the expression $\varepsilon\,|x_1|\,|x_2|$ is bounded from below,

$$\varepsilon\,|x_1|\,|x_2| \geq \varepsilon\, a\, a = \varepsilon\, a^2$$

and we can take this value for δ.

Indeed, if $|x_1 - x_2| < \delta$,

$$|f(x_1) - f(x_2)| = \frac{|x_1 - x_2|}{|x_1|\,|x_2|} < \frac{\varepsilon\, a^2}{|x_1|\,|x_2|} = \varepsilon\,\frac{a}{|x_1|}\,\frac{a}{|x_2|} \leq \varepsilon$$

with which the continuity condition is satisfied, with δ depending only on ε, therefore the continuity is uniform.

Example 2. As a consequence of the theorem, the function $y = \sqrt{x}$, continuous on $[0, \infty)$, is uniformly continuous on any interval $[0, k]$, $k \in \mathbb{R}$ (despite having vertical tangent at the origin).

8.3 Composition of uniformly continuous functions

Let the functions f and g be such that

$$f : I \to J; \quad g : U \to V; \quad f(I) \subset U$$

The composite function $g \circ f$ exists, being I its domain (see section **A.2**).

It is satisfied that: "*If both functions are uniformly continuous on their respective domains, then the composite function $g \circ f$ is uniformly continuous on its domain*".

D. Differentiability of functions

1 Derivability and differentiability

Thanks to the concept of derivative of a function we can obtain the equation of the tangent to a curve, calculate the speed of a moving object or find the maxima and minima of a function. From the higher order derivatives of a function we will obtain its Taylor expansion, which will allow us to approximate the value of the function in the surroundings of a point. The concept of differential is fundamental for finding primitives and for the different applications of integral calculus: to obtain areas, volumes and lengths of curves, among many others.

1.1 Derivable function

a) Definition. Let $f \in \mathcal{F}(I, \mathbb{R})$. We say that f is derivable at $a \in I$ if and only if the following limit exists and is finite, in which case we call it the derivative of f at a.

$$\boxed{\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)} \tag{1}$$

Otherwise, we say that f is not derivable at a.

Alternative form. If $x - a = h$, then $x = a + h$ and the limit (1) takes the following form:

$$\boxed{\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)}$$

Example. The derivative of $f(x) = x^3$ at $a = 1$ is

$$\lim_{x \to 1} \frac{x^3 - 1^3}{x - 1} = \lim_{x \to 1}(x^2 + x + 1) = 3$$

Exercise. Obtain the derivative of $f(x) = x^3$ at $a = 1$, in the alternative form.

b) One-sided derivatives. The derivative condition (1) consists in obtaining a limit at a. If the one-sided limits at a^+ and at a^- exist, we call them the right derivative and the left derivative at a, respectively $f'(a^+)$ and $f'(a^-)$.
It can be proved that f *is derivable at a if and only if its lateral derivatives exist and coincide.*

c) Vertical tangent. If the limit (1) is $\pm\infty$ and the function is continuous at a, we say that f has a vertical tangent at a. For example, if $f(x) = \sqrt{x}$, then

$$\lim_{x \to 0^+} \frac{\sqrt{x} - 0}{x - 0} = +\infty$$

d) Singular point. If $f'(a^+)$ and $f'(a^-)$ exist and are different, f has a singular point at $x = a$. For example, $f(x) = |x|$ has a singular point at the origin.

$$f'(0^+) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x - 0}{x - 0} = 1; \quad f'(0^-) = \lim_{x \to 0^-} \frac{-x - 0}{x - 0} = -1$$

e) Derivative function. If $f'(x)$ exists $\forall x \in I$, we say that f is derivable on I. The function that maps every point x to the value of the derivative of f at x is called the derivative function f'. If the interval is closed, $I = [a, b]$, f must be derivable on (a, b), at a^+ and at b^-.

To obtain the derivative at a generic point x, we use the alternative form, with x and $x+h$ instead of a and $a+h$. For example, to derive $f(x) = x^2$, we do the following:

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = 2x$$

Exercise. Obtain the derivative of $f(x) = \sin x$.

1.2 Differenciable function

When studying a function near a point a, we analyze the increment in its value between the points a and $a + \Delta x$, when Δx is very small. In many functions this increment is "almost" proportional to Δx, that is

$$\Delta f = f(a + \Delta x) - f(a) \approx c\,\Delta x$$

being $c \in \mathbb{R}$. In these cases, the graph of f is well approximated by a straight line in a neighborhood of a.

To express this more formally, we say that a function $f \in \mathcal{F}(I, \mathbb{R})$ is differentiable at $a \in I$ if and only if

$$\Delta f = c\,\Delta x + \varepsilon(\Delta x)\,\Delta x,\ c \in \mathbb{R} \quad \left(\text{being } \lim_{\Delta x\to 0} \varepsilon(\Delta x) = 0\right) \tag{2}$$

In the above condition, $\varepsilon(\Delta x)$ represents an expression that tends to 0 when $\Delta x \to 0$, for example $(\Delta x)^{1/2}$. Thus the second addend of Δf tends to 0 faster than the first.

When Δx is very small, $\varepsilon(\Delta x)\Delta x$ is negligible compared to $c\,\Delta x$, which means that the increment of f is expressed very accurately by the value of $c\,\Delta x$.

Writing Δx as $x - a$ in (2), the **differentiability condition** of f at a becomes

$$\boxed{f(x) - f(a) = c\,(x-a) + \varepsilon(x-a)(x-a)} \tag{3}$$

Definition of differential. If a function f is differentiable at a, the product $c\,(x-a)$ is the differential of f at a.

1.3 Relation between both concepts

"A function is differentiable at a point if and only if it is derivable at that point.".

Proof. We will prove it only from left to right. We will see that, if f is differentiable, then it is derivable and we will obtain the proportionality coefficient c.

Indeed, if f satisfies the differentiability condition (3),

$$f(x) - f(a) = \big[c + \varepsilon(x-a)\big](x-a) \Longrightarrow \lim_{x\to a} \frac{f(x) - f(a)}{x-a} = \lim_{x\to a} \big[c + \varepsilon(x-a)\big] = c$$

so c is the derivative $f'(a)$ and the differential of f at a becomes $\boxed{df(a) = f'(a)(x-a)}$.

The **differenciability condition** (3) takes then the form we will more frequently use:

$$\boxed{f(x) - f(a) = f'(a)(x-a) + \varepsilon(x-a)(x-a)} \tag{4}$$

We observe that the first addend on the right is $df(a)$, proportional to $(x-a)$, so it is an infinitesimal of $(x-a)$. The second addend is the product of $(x-a)$ and $\varepsilon(x-a)$, that tends to 0 with $(x-a)$. This product is, then, an infinitesimal of higher order than $(x-a)$, which is denoted

$$\varepsilon(x-a)(x-a) = \circ(x-a)$$

The **differenciability condition** takes the following useful expression

$$\boxed{\Delta f = df(a) + \circ(x-a)} \quad \left(\text{being } \lim_{x\to a} \frac{\circ(x-a)}{x-a} = 0\right) \tag{5}$$

Differential of the variable. We can now obtain the differential of x, that is, the differential of the function $f(x) = x$.

$$dx = df|_{f(x)=x} \Longrightarrow dx = x'(x-a) = x-a$$

and we see that **the differential of the variable** x **is the increment of** x.
Replacing the above in the expression of the differential, we get

$$\boxed{df(a) = f'(a)\,dx} \Longrightarrow \boxed{f'(a) = \frac{df(a)}{dx}}$$

which tells us that the differential of a function f is the product of its derivative times the differential of x. And we can also express the derivative of f as a quotient of differentials.

1.4 Graphic interpretation

As we have just seen, the increment of the function between a and x is decomposed into two addends, $\Delta f = df(a) + \circ(x-a)$, which we interpret graphically in the following figure.

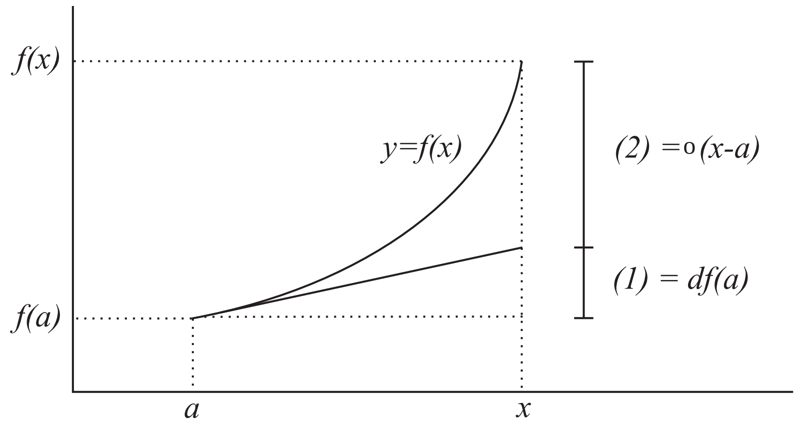

Figure 1: Graphic interpretation of the differential.

It is known that $f'(a)$ is the value of the slope of the tangent to the curve $y = f(x)$ at $x = a$.
Thus the tangent passes through the point $\big(a, f(a)\big)$ with a slope $f'(a)$ and its equation is:

$$y - f(a) = f'(a)(x-a)$$

Hence the product of $f'(a)\,(x-a)$, which equals $df(a)$, is the length of the segment *(1)*.
Therefore, the length of the segment *(2)*, that is $\Delta f - df(a)$, represents the second addend $\circ(x-a)$ in the differentiability condition (4). As it can be seen in the figure, this term can be much more important than $df(a)$ when we are at a certain distance from a, but as we get closer to the point, it becomes negligible compared to $df(a)$.
Thus Δf and df tend to merge when $x \to a$, since they are equivalent infinitesimals, while their difference $\circ(x-a)$ is an infinitesimal of higher order.
Geometrically, the differential of f is the increment of the ordinate of the line tangent to the curve $y = f(x)$, when the variable is increased by a value $x - a$.
We can see that the behavior of the curve near the point, is well described by the tangent line.

1.5 Relation between continuity and differentiability

"If a function is differentiable at a point, then it is continuous at that point".

Indeed, if f is differentiable at a, from the differentiability condition (4) (section **1.3**) it results

$$f(x) = f(a) + [f'(a) + \varepsilon(x-a)](x-a) \Longrightarrow \lim_{x\to a} f(x) = f(a)$$

then f is continuous at a.

The reciprocal is not true. For example, $f(x) = |x|$ is continuous throughout $\mathbb{R}$, but has two different one-sided derivatives at $x = 0$ (*corner point*), so it is not differentiable at the origin.

1.6 Operations with differenciable functions

The method to obtain the derivatives of the sum, the product and the quotient of differentiable functions is assumed to be known. It is easy to verify that the differentiation follows the same rules of the derivation.

a) $d(f+g) = (f+g)'\,dx = (f'+g')\,dx = f'dx + g'dx = df + dg$

b) $d(f\cdot g) = (f\cdot g)'\,dx = (f'g + fg')\,dx = f'g\,dx + fg'dx = g\,df + f dg$

Exercise. Obtain the differential of the quotient f/g.

2 Chain rule. Applications

2.1 Derivative of the composite function

Let $f \in \mathcal{F}(I,\mathbb{R})$ and $g \in \mathcal{F}(J,\mathbb{R})$ be functions such that $f(I) \subset J$. It holds that: *If f is differentiable at $a \in I$ and g is differentiable at $b = f(a) \in J$, then the composite function $\phi(x) = g\big(f(x)\big)$ is differentiable at a.* Its derivative is obtained as

$$\boxed{\phi'(a) = g'\big(f(a)\big)\cdot f'(a)} \quad \text{or} \quad \boxed{\left.\frac{d\phi}{dx}\right|_{x=a} = \left.\frac{dg}{dy}\right|_{y=f(a)} \left.\frac{df}{dx}\right|_{x=a}}$$

Proof. We start from the differentiability conditions of f at $x = a$ and g at $y = b$:

$$f(x) - f(a) = [f'(a) + \varepsilon_f(x-a)]\,(x-a), \quad \text{being } \lim_{x\to a}\varepsilon_f = 0 \tag{6}$$

$$g(y) - g(b) = [g'(b) + \varepsilon_g(y-b)]\,(y-b), \quad \text{being } \lim_{y\to b}\varepsilon_g = 0 \tag{7}$$

Every point $y \in f(I)$ is the image by f of a certain $x \in I$, and $b = f(a)$ by hypothesis; so, in condition (7), we write $f(x)$ and $f(a)$ instead of y and b:

$$g\big(f(x)\big) - g\big(f(a)\big) = \Big[g'\big(f(a)\big) + \varepsilon_g\big(f(x) - f(a)\big)\Big]\big(f(x) - f(a)\big) \tag{8}$$

Now we replace $f(x) - f(a)$ in (8) by its value in condition (6). In the resulting product of brackets, we omit the dependencies of ε_f and ε_g for simplicity. Operating, it turns out

$$g\big(f(x)\big) - g\big(f(a)\big) = \Big[g'\big(f(a)\big) + \varepsilon_g\Big]\Big[f'(a) + \varepsilon_f\Big](x-a) =$$

$$\Big[g'\big(f(a)\big)\cdot f'(a) + g'\big(f(a)\big)\cdot\varepsilon_f + \varepsilon_g\cdot f'(a) + \varepsilon_g\cdot\varepsilon_f\Big](x-a)$$

If $x \to a$, then $f(x) \to f(a)$ (continuity of f), so both ε_f and ε_g tend to 0. Thus the 2nd., 3rd. and 4th addends of the first factor are expressions of the type $\varepsilon(x-a)$, which we group together. It results

$$g\big(f(x)\big) - g\big(f(a)\big) = \big[g'\big(f(a)\big) \cdot f'(a) + \varepsilon(x-a)\big]\,(x-a), \quad \text{being } \lim_{x\to a} \varepsilon(x-a) = 0$$

The term $g'\big(f(a)\big) \cdot f'(a)$ is the derivative of the composite function, formed by the product of the derivatives of g with respect to y, at $y = f(a)$, and of f with respect to x, at a.
If $\phi(x)$ is the composite function $g\big(f(x)\big)$, the differentiability condition of ϕ at a results:

$$\phi(x) - \phi(a) = \left[\left.\frac{d\phi}{dx}\right|_{x=a} + \varepsilon(x-a) \right](x-a),$$

where the derivative of ϕ is the product of the corresponding derivatives of g and f

$$\left.\frac{d\phi}{dx}\right|_{x=a} = \left.\frac{dg}{dy}\right|_{y=f(a)} \left.\frac{df}{dx}\right|_{x=a}$$

If the above is true $\forall x \in I$, then the composite function $\phi(x) = g(f(x))$ is differentiable at x, $\forall x \in I$, being its derivative function

$$\boxed{\frac{d\phi}{dx} = \left.\frac{dg}{dy}\right|_{y=f(x)} \cdot \frac{df}{dx}}$$

what is known as the **chain rule**.

2.2 Applications of the chain rule

The main elementary functions are differentiable in their domain. We assume that the derivatives of the logarithm, power and exponential functions of x are known, as well as the sine, cosine and tangent, both trigonometric and hyperbolic. From them we are going to find the derivatives of some composite functions.

a) $\phi(x) = \ln u(x)$. Since $(\ln x)' = 1/x$,

$$\phi' = \frac{1}{u}\,u' \Longrightarrow \boxed{\Big(\ln u(x)\Big)' = \frac{u'(x)}{u(x)}}$$

b) $\phi(x) = u^m(x)$. Since $(x^m)' = m\,x^{m-1}$,

$$\phi' = m\,u^{m-1}u' \Longrightarrow \boxed{\Big(u^m(x)\Big)' = m\Big(u(x)\Big)^{m-1} u'(x)}$$

c) $\phi(x) = a^{u(x)}$, $a > 0$. Since $(a^x)' = a^x \ln a$,

$$\phi' = a^u \ln a\, u' \Longrightarrow \boxed{\Big(a^{u(x)}\Big)' = a^{u(x)} \ln a\, u'(x)}$$

d) $\phi(x) = \Big(u(x)\Big)^{v(x)}$. Taking logarithms, we get $\ln \phi(x) = v(x) \ln u(x)$. Now we find the derivatives of a logarithm (left) and of a product (right), and solve for the derivative of ϕ:

$$\frac{\phi'}{\phi} = v' \ln u + v\,\frac{u'}{u} \Longrightarrow \phi' = u^v \left(v' \ln u + v\,\frac{u'}{u}\right) = u^v \ln u\, v' + v\, u^{v-1} u'$$

This result can be interpreted as the sum of two derivatives of u^v: in the first one, we treat u as a constant and find the derivative of an exponential function; in the second one, we treat v as a constant and find the derivative of a power function.

Examples.

1. $\left(x^x\right)' = x\,x^{x-1} + x^x \ln x = x^x(1+\ln x)$.

2. $\left(\sin\sqrt{\ln\cos x}\right)' = \cos\sqrt{\ln\cos x}\,\dfrac{1}{2\sqrt{\ln\cos x}}\,\dfrac{1}{\cos x}\,(-\sin x) = -\dfrac{1}{2}\,\dfrac{\cos\sqrt{\ln\cos x}\,\tan x}{\sqrt{\ln\cos x}}$.

Exercises. Obtain the derivatives of $f(x) = (\sin x)^{\cos x}$ and $g(x) = \arctan\left(\tan^3 1/x\right)$.

3 Derivative of the inverse function

3.1 Existence of the inverse function

Theorem. *If f is continuous and bijective, then its inverse function f^{-1} exists and is also continuous and bijective.*

Example. The function $f(x) = x^2$ (square of a number) is continuous on $\mathbb{R}$ and bijective in $[0,k]$, $k \in \mathbb{R}$, therefore it has an inverse function. To obtain it, we solve for x in terms of y:

$$y = f(x) = x^2 \Longrightarrow x = f^{-1}(y) = \sqrt{y}$$

Thus the inverse function of the "square of a number" is the "square root" function. We express this function in the usual way, using x as a variable:

$$f(x) = x^2 \Longrightarrow f^{-1}(x) = \sqrt{x}$$

That is, to obtain the inverse function of a given one $y = f(x)$, we solve for x in terms of y, and then we permute x and y in the resulting formula.

3.2 Derivative of the inverse function

Let f^{-1} be the inverse function of f on an interval I. Let us assume that f' exists and is a non-null function on I, $f' \neq 0$ on I. We will prove that *function f^{-1} is derivable on I and its derivative is the inverse of the derivative of function f.*

$$\boxed{\left(f^{-1}(x)\right)' = \frac{1}{f'(y)|_{y=f^{-1}(x)}}}$$

Proof. Let f be a function and $y = f^{-1}(x)$ the inverse of f. To obtain the derivative of f^{-1}: 1) we solve for x in terms of y; 2) we differentiate on both sides with respect to x, using the chain rule; 3) we solve for $y'(x)$, writing y as a function of x in the right-hand side term $f'(y)$:

$$y = f^{-1}(x) \Longrightarrow x = f(y) \overset{\prime}{\Longrightarrow} 1 = f'(y)\,y'(x) \Longrightarrow y'(x) = \frac{1}{f'(y)|_{y=f^{-1}(x)}}$$

Example. Let f be the sine function and f^{-1} its inverse function, $y = \arcsin x$. We find $\left(f^{-1}\right)'$:

$$y = f^{-1}(x) = \arcsin x \Longrightarrow x = f(y) = \sin y \overset{\prime}{\Longrightarrow} 1 = \cos y \cdot y'$$

Solving for y' and writing the result in terms of x:

$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}} \Longrightarrow \boxed{(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}}$$

Exercise. Find the derivative of $f(x) = \arctan x$.

4 Mean value theorems

4.1 Rolle's theorem

Let $f \in \mathcal{F}(I, \mathbb{R})$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, it is satisfied:

$$\boxed{\exists \xi \in (a, b) \;/\; f'(\xi) = 0}$$

Proof. By Weierstrass theorem we know that, since f is continuous on $I = [a, b]$, it reaches a maximum and a minimum on I, that is

$$\exists\, x_M \in [a, b] \,/\, M = f(x_M) \geq f(x) \;\forall x \in [a, b]$$
$$\exists\, x_m \in [a, b] \,/\, m = f(x_m) \leq f(x) \;\forall x \in [a, b]$$

If the maximum and the minimum coincide, since all the values of f are between them, the function is constant. Then its derivative is zero and the theorem is satisfied.

In a general case, M will be greater than m and they will be reached at different points. We distinguish two cases:

a) The maximum is reached at an interior point of $[a, b]$.

Since f is differentiable on (a, b), we study the derivative at x_M^+. We observe that the numerator is less than or equal to 0 and the denominator is positive, thus the quotient is less than or equal to 0 and so is its limit.

$$f'(x_M^+) = \lim_{h \to 0^+} \frac{f(x_M + h) - f(x_M)}{h} \leq 0$$

We now study the derivative at x_M^-: the numerator is less than or equal to 0, but now the denominator is negative, so the quotient is greater than or equal to 0 and so is its limit.

$$f'(x_M^-) = \lim_{h \to 0^-} \frac{f(x_M + h) - f(x_M)}{h} \geq 0$$

Since f is differentiable on x_M, both derivatives must coincide, so

$$f'(x_M^+) = f'(x_M^-) = 0$$

b) The maximum is reached at the endpoints.

Since the maximum and the minimum are reached at different points, the minimum will be reached inside the interval. As before, we study the one-sided derivatives at x_m, noting that now the numerators are greater than or equal to 0. We obtain:

$$f'(x_m^+) = \lim_{h \to 0^+} \frac{f(x_m + h) - f(x_m)}{h} \geq 0$$
$$f'(x_m^-) = \lim_{h \to 0^-} \frac{f(x_m + h) - f(x_m)}{h} \leq 0$$

Both derivatives must coincide, hence

$$f'(x_m^+) = f'(x_m^-) = 0$$

We conclude that at least one extremum is reached at an interior point and, at it, $f' = 0$.

Exercise. The function $f(x) = 1 - x^{2/3}$ is equal to zero at $x = \pm 1$. Reason if it is possible to apply the theorem on the interval $[-1, 1]$.

4.2 Cauchy's theorem

Let $f, g \in \mathcal{F}(I, \mathbb{R})$ be continuous on $[a, b]$ and differentiable on (a, b). It is satisfied:

$$\boxed{\exists \xi \in (a, b) \ / \ f'(\xi)[g(b) - g(a)] = g'(\xi)[f(b) - f(a)]}$$

If, in addition, $g(b) \neq g(a)$ and g' and f' are not zero at the same time, then:

$$\boxed{\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}}$$

Proof. From f and g we define the function F, which will also be continuous and differentiable.

$$F(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$$

$F(a) = F(b)$ (check it as an exercise), so F satisfies the conditions of Rolle's theorem and it holds

$$\exists \xi \in (a, b) \ / \ F'(\xi) = 0 \Longrightarrow f'(\xi)[g(b) - g(a)] = g'(\xi)[f(b) - f(a)]$$

which proves the first expression of the theorem.

If the factor $g(b) - g(a)$ is not null, we can move it to the other side, which allows us to ensure that $g'(\xi) \neq 0$. Indeed, if $g'(\xi)$ were zero, $f'(\xi)$ would also be zero and f' and g' would be null at the same time, against the hypothesis.

So, taking $g'(\xi)$ to the left, we get the expression we were looking for

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

4.3 Lagrange's theorem (mean value theorem)

Let $f \in \mathcal{F}(I, \mathbb{R})$ be continuous on $[a, b]$ and differentiable on (a, b). It is satisfied:

$$\boxed{\exists \xi \in (a, b)/f'(\xi) = \frac{f(b) - f(a)}{b - a}}$$

Alternative form. If $b - a = h$ then $b = a + h$ and we can write the above expression as

$$\boxed{f(a + h) = f(a) + hf'(a + \theta h), \ 0 < \theta < 1}$$

Proof. This is a particular case of Cauchy's theorem, doing $g(x) = x$.

Geometric interpretation. The theorem states that, if f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point c in (a, b) such that the tangent line at c is parallel to the secant line through the points $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$.

4.4 Differentiable monotone functions

Let $f \in \mathcal{F}(I, \mathbb{R})$ be differentiable on I. It is satisfied that:

a) $f'(x) \geq 0, \ \forall x \in I \Longleftrightarrow f$ *is monotone increasing on* I.

b) $f'(x) \leq 0, \ \forall x \in I \Longleftrightarrow f$ *is monotone decreasing on* I.

c) $f'(x) > 0, \ \forall x \in I \Longrightarrow f$ *is strictly increasing on* I.

d) $f'(x) < 0, \ \forall x \in I \Longrightarrow f$ *is strictly decreasing on* I.

Proof of case a) (the others are proved analogously).

(→) Given any two points of I, we apply Lagrange's theorem to the interval $[x_1, x_2]$:

$$\forall x_1, x_2 \in I \;/\; x_1 < x_2 \Longrightarrow \exists \xi \in (x_1, x_2) \;/\; f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)$$

If $f'(x) \geq 0$, $\forall x \in I$, then $f'(\xi) \geq 0$ so $f(x_2) \geq f(x_1)$ and function f is monotone increasing.

(←) Given two points x, $x + h$, with $h > 0$, since f is increasing, it holds:

$$\frac{f(x+h) - f(x)}{h} \geq 0 \Longrightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \geq 0$$

In cases c) and d) there is no equivalence since the implication not always holds from right to left. Indeed, although the quotient of Δf and h is strictly greater than 0 (or less than 0), the limit can be equal to 0 and so the derivative. For example, the functions $y = \pm x^3$ are strictly monotone, but their derivative at the origin is 0.

4.5 Constant functions

Let $f \in \mathcal{F}(I, \mathbb{R})$ be differentiable on I. It is satisfied:

$$\boxed{f'(x) = 0,\ \forall x \in I \Longleftrightarrow f(x) = K,\ \forall x \in I}$$

That is, a function is constant if and only if its derivative is null.

Proof.

(→) Again we apply Lagrange's theorem to any two points on the interval:

$$\forall x_1, x_2 \in I \,/\, x_1 < x_2 \Longrightarrow \exists \xi \in (x_1, x_2) \;/\; f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)$$

If $f'(x) = 0\ \forall x \in I$, then $f'(\xi) = 0$ will hold, so $f(x_2) = f(x_1)$. Hence f is constant.

(←) If f is constant, its derivative is zero.

Corollary. If $f' = g' \Longrightarrow f = g + K$. That is, *if f and g are primitives of the same function, their difference is a constant.*

Proof. Let f and g be differentiable on I. We define $\phi = f - g$ and apply the the property we have just seen:

$$\phi' = f' - g' = 0 \Longrightarrow \phi = K \Longrightarrow f - g = K$$

5 The derivative as a limit of derivatives

Let $f : I \to \mathbb{R}$ be a continuous function, from which we obtain the derivative f' at a generic point. If f' exists $\forall x \in I$, we say that f is differentiable on I and f' is its derivative function.

In the event that function f' does not exist at a point $a \in I$ but its limit when $x \to a$ does exist, function f will also be differentiable at a.

We will prove it by seing that the one-sided derivatives of f at a coincide with the one-sided limits of f' at a, from which the derivative of f exists at a if f' has a limit at a.

Derivative at a^+. Let f be defined on $I = [a, a+\delta)$. If f is continuous on I, differentiable on $I \setminus \{a\}$ and such that $\exists \lim_{x \to a^+} f'(x)$, then f is differentiable at a^+ and it holds

$$\boxed{f'(a^+) = \lim_{x \to a^+} f'(x)}$$

Proof. We consider the interval $[a, a+h],\ 0 < h < \delta$. Applying the Lagrange theorem to f,

$$\exists \xi \in (a, a+h) \, / \, \frac{f(a+h) - f(a)}{h} = f'(\xi)$$

If $h \to 0^+$, then $a + h \to a^+$, therefore the intermediate point $\xi \to a^+$. Taking limits,

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} = \lim_{\xi \to a^+} f'(\xi)$$

But if $\lim_{x \to a^+} f'(x)$ exists and is equal to l, then $\lim_{\xi \to a^+} f'(\xi)$ will also exist and be equal to l, with what

$$\lim_{x \to a^+} f'(x) = l \Longrightarrow \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} = l \Longrightarrow f'(a^+) = l$$

So, if the limit of f' exists at a^+, it coincides with the derivative of f at a^+.

Derivative at a^-. Studying by the same method the limit of f' at a^-, it is shown that -if it exists- it coincides with the derivative of f at a^-. Then we can conclude what follows:

Derivative at a. If f is defined on a neighborhood of a and the limit of f' exists at a, the limits of f' at a^+ and at a^- exist and coincide. Since each limit is equal to the corresponding one-sided derivative, the values of $f'(a^+)$ and $f'(a^-)$ exist and coincide. Thus the function is differentiable at a.

$$f'(a^+) = \lim_{x \to a^+} f'(x) = \lim_{x \to a^-} f'(x) = f'(a^-), \Longrightarrow \boxed{f'(a) = \lim_{x \to a} f'(x)}$$

This result is useful for calculating the derivative of f at a point a, when f is defined at a independently of its expression on the rest of the interval (for example, because the expression does not makes sense at a).

Example. We study, at $x = 0$, the derivative of $f(x) = \begin{cases} x^3 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

The derivative at $x \neq 0$ is $f'(x) = 3x^2 \sin(1/x) - x \cos(1/x)$, which is not defined at $x = 0$. However, since f' has a limit at that point,

$$f'(0) = \lim_{x \to 0} f'(x) = 0$$

Remark. The condition (existence of a limit of f') is sufficient for the existence of the derivative of f, but is not a necessary one. If $f'(x)$ has no limit at a, it is still possible that f is differentiable at a, as it can be seen in the following exercise.

Exercise. Verify that the function $f(x) = x^2 \sin(1/x)$, $x \neq 0$; $f(0) = 0$, is differentiable at the origin (obtaining $f'(0)$ by the definition), but its derivative function has no limit as $x \to 0$.

6 L'hôpital rules

The theorems stated below allow us to solve limits in the form of quotients of the type $0/0$ and ∞/∞. We state them for one-sided limits at $a, b \in \mathbb{R}$, but **they are also valid when** $x \to \pm\infty$.

Let f and g be differentiable on (a, b), being $g'(x) \neq 0\ \forall x \in (a, b)$. Let $\gamma \in \mathbb{R}$ or $\gamma = \pm\infty$.

a) If f and g are null at a and are continuous at a^+, that is

$$\lim_{x\to a^+} f(x) = f(a) = 0; \quad \lim_{x\to a^+} g(x) = g(a) = 0,$$

it holds

$$\boxed{\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = \gamma \Longrightarrow \lim_{x\to a^+} \frac{f(x)}{g(x)} = \gamma}$$

b) If f and g are null at b and are continuous at b^-, that is

$$\lim_{x\to b^-} f(x) = f(b) = 0; \quad \lim_{x\to b^-} g(x) = g(b) = 0,$$

it holds

$$\boxed{\lim_{x\to b^-} \frac{f'(x)}{g'(x)} = \gamma \Longrightarrow \lim_{x\to b^-} \frac{f(x)}{g(x)} = \gamma}$$

c) If $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = \pm\infty$, it holds

$$\boxed{\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = \gamma \Longrightarrow \lim_{x\to a^+} \frac{f(x)}{g(x)} = \gamma}$$

d) If $\lim_{x\to b^-} f(x) = \lim_{x\to b^-} g(x) = \pm\infty$, it holds

$$\boxed{\lim_{x\to b^-} \frac{f'(x)}{g'(x)} = \gamma \Longrightarrow \lim_{x\to b^-} \frac{f(x)}{g(x)} = \gamma}$$

In all four cases, if the limit of the quotient of derivatives exists, the limit of the quotient of functions exists and takes the same value.

The theorem has been stated for one-sided limits. In the (usual) case that we want to find the limit at an certain point $c \in \mathbb{R}$, we will consider one interval to the right and another to the left of c, $(c, c+\delta)$ and $(c-\delta, c)$. If the limit of the quotient of derivatives exists at c, there will exist one-sided limits at c^+ and at c^-. Applying the theorem on both intervals, it results:

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = \gamma \Longrightarrow \left\{ \begin{array}{c} \displaystyle\lim_{x\to c^+} \frac{f'(x)}{g'(x)} = \gamma \Longrightarrow \lim_{x\to c^+} \frac{f(x)}{g(x)} = \gamma \\ \\ \displaystyle\lim_{x\to c^-} \frac{f'(x)}{g'(x)} = \gamma \Longrightarrow \lim_{x\to c^-} \frac{f(x)}{g(x)} = \gamma \end{array} \right\} \Longrightarrow \lim_{x\to c} \frac{f(x)}{g(x)} = \gamma$$

Remarks. (See examples of both in supplementary documents).

1) That f'/g' has a limit is a sufficient condition for the existence of the limit of f/g, but not a necessary one; that is, although the limit of the quotient of derivatives does not exist, the limit of the quotient of functions could exist.

2) If the limit of f'/g' is still indeterminate, the L'hôpital rule can be applied again, if the conditions of the theorem (for the quotient f'/g') are still satisfied.

Proof for case a). (Indetermination type 0/0, with limit at a^+).

Given (a,b), we choose any $x \in (a,b)$. It is satisfied:

1. $g' \neq 0$ on (a,x) since it is satisfied on (a,b) by hypothesis.
2. $g(x) \neq g(a)$ since, if $g(x) = g(a)$, by Rolle's theorem there would exist $\xi \in (a,x)$ such that $g'(\xi)$ would be null, which cannot occur by hypothesis.

Then, by Cauchy's Theorem, there will exist $\xi \in (a,x)$ such that

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(\xi)}{g'(\xi)} \Longrightarrow \frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}$$

because f and g are null at a. That is, the quotient of functions at the point x has the same value as uotient of derivatives at a point $\xi \in (a,x)$.

Since the equality holds $\forall x \in (a,b)$, we take limits as $x \to a^+$. As $\xi \in (a,x)$, if $x \to a^+$, then $\xi \to a^+$ too, so we write $\xi \to a^+$ in the right-hand side limit:

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{\xi\to a^+} \frac{f'(\xi)}{g'(\xi)}$$

Now, if the quotient of $f'(x)$ and $g'(x)$ has a limit (when $x \to a^+$), the quotient of $f'(\xi)$ and $g'(\xi)$ (when $\xi \to a^+$) will have the same, from which

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = \gamma \Longrightarrow \lim_{\xi\to a^+} \frac{f'(\xi)}{g'(\xi)} = \gamma \Longrightarrow \lim_{x\to a^+} \frac{f(x)}{g(x)} = \gamma$$

Example. We find the limit of the quotient of $\ln x$ and $\sin(x-1)$ when $x \to 1$:

$$\lim_{x\to 1} \frac{\ln x}{\sin(x-1)} \overset{\text{If } \exists}{=} \lim_{x\to 1} \frac{(\ln x)'}{\sin'(x-1)} = \lim_{x\to 1} \frac{1/x}{\cos(x-1)} = \frac{1}{\cos 0} = 1$$

7 Higher-order derivatives

If a function f is differentiable on the interval I, we can obtain the expression f' of its derivative at a generic point. In this case we say that f' is the **derivative function of f on I**.

If the derivative of function f' exists at $a \in I$, it is called the second derivative of f at a, $f''(a)$. And if this occurs at x, $\forall x \in I$, $f''(x)$ will be the **second derivative function of f at I**.

Generalizing the above, we obtain the **n-th derivative function of** f (or derivative of order n), which we express as $f^{(n}$ or in the alternative form $\dfrac{d^n f}{dx^n}$. Thus

$$\begin{aligned} f' &= \frac{df}{dx} \\ f'' &= \frac{d}{dx}(f') = \frac{d^2 f}{dx^2} \\ f''' &= \frac{d}{dx}(f'') = \frac{d^3 f}{dx^3} \\ &\vdots \\ f^{(n} &= \frac{d}{dx}\left(f^{(n-1)}\right) = \frac{d^n f}{dx^n} \end{aligned}$$

Example 1. Exponential function $y = a^{px}$, $a > 0$, $p \in \mathbb{R}$.

$$\begin{aligned} y' &= a^{px}\, p\, \ln a \\ y'' &= a^{px}\, p^2\, (\ln a)^2 \\ &\vdots \\ y^{(n} &= a^{px}\, p^n\, (\ln a)^n \end{aligned}$$

Example 2. Function $y = \sin x$.

$$\begin{aligned} y' &= \cos x = \sin\left(x + \frac{\pi}{2}\right) \\ y'' &= \cos(x + \frac{\pi}{2}) = \sin\left(x + 2\,\frac{\pi}{2}\right) \\ &\vdots \\ y^{(n} &= \sin\left(x + n\,\frac{\pi}{2}\right) \end{aligned}$$

Example 3. Power of a binomial $y = (ax + b)^m$, $m \in \mathbb{Z}$.

We distinguish three cases: m greater than, less than or equal to 0.

a) $m > 0$, that is, power of natural exponent.

$$\begin{aligned} y' &= m\,(ax + b)^{m-1}\, a \\ y'' &= m\,(m-1)\,(ax + b)^{m-2}\, a^2 \\ &\vdots \\ y^{(n} &= \begin{cases} m\,(m-1)\cdots(m-n+1)\,(ax+b)^{m-n}\,a^n & n \le m \\ 0 & n > m \end{cases} \end{aligned}$$

We can see that the m-th derivative is constant and the higher-order ones are null.

b) $m < 0$. In this case, the decreasing exponent $m - n$ does not become zero, so the derivative does not become constant, as in the previous case. Then

$$y^{(n} = m\,(m-1)\cdots(m-n+1)\,(ax+b)^{m-n}\,a^n,\ \forall n \in \mathbb{N} \tag{9}$$

c) $m = 0$. The function is constant and its derivatives are zero.

Particular case. From the power of a binomial for $m \in \mathbb{Z}^-$, we obtain the n-th derivative of $y = 1/(x - c)$. To do it, we use the formula (9) with $m = -1$, $a = 1$, $b = -c$. It turns out:

$$\left(\frac{1}{x-c}\right)^{(n} = (-1)\,(-2)\cdots(-1-n+1)\,(x-c)^{-1-n} = \frac{(-1)^n\, n!}{(x-c)^{n+1}}$$

Exercise. Find the n-th derivative of function f, given by a quotient of polynomials:

$$f(x) = \frac{x^2+1}{x^2-3x+2}$$

Solution:

$$f^{(n}(x) = (-1)^n\, n!\left[\frac{5}{(x-2)^{n+1}} - \frac{2}{(x-1)^{n+1}}\right]$$

8 Limited Taylor and MacLaurin expansions

Our objective is to approximate a function f near a point a by means of a polynomial, so that the more terms the polynomial has, the better the approximation. To do this, we will obtain the polynomial whose first and higher-order derivatives at a coincide with those of f.

8.1 Limited Taylor expansion of order n

Let f be a function n times differentiable at a. This is equivalent to say that f is $n-1$ times differentiable in a neighborhood U_a of a and n times at a.

a Taylor polynomial of degree n at a

We take a polynomial of degree n and unknown coefficientes α_i, in powers of $(x-a)$.

$$P_n(x) = \alpha_0 + \alpha_1(x-a) + \alpha_2(x-a)^2 + \dots \alpha_n(x-a)^n$$

To identify the coefficients, we find its successive derivatives, equating them to the corresponding derivatives of $f(x)$ at a. That is

$$P_n(a) = f(a), \quad P_n'(a) = f'(a), \quad P_n''(a) = f''(a), \quad \dots \quad P_n^{(n}(a) = f^{(n}(a) \tag{10}$$

Thus

$$\begin{array}{rcll}
P_n(x) & = & \alpha_0 + \alpha_1(x-a) + \alpha_2(x-a)^2 + \cdots + \alpha_n(x-a)^n & \Longrightarrow P_n(a) = \alpha_0 = f(a) \\
P_n'(x) & = & \alpha_1 + 2\alpha_2(x-a) + \cdots + n\,\alpha_n(x-a)^{n-1} & \Longrightarrow P_n'(a) = \alpha_1 = f'(a) \\
P_n''(x) & = & 2\alpha_2 + \cdots + n(n-1)\,\alpha_n(x-a)^{n-2} \Longrightarrow P_n''(a) = 2\alpha_2 = f''(a) \Longrightarrow & \alpha_2 = 1/2 f''(a) \\
 & \vdots & & \\
P_n^{(n}(x) & = & n(n-1)\cdots 1\cdot\alpha_n = n!\,\alpha_n \Longrightarrow P_n^{(n}(a) = n!\,\alpha_n = f^{(n}(a) & \Longrightarrow \boxed{\alpha_n = \frac{1}{n!} f^{(n}(a)}
\end{array}$$

with which we obtain the Taylor polynomial of degree n at a:

$$\boxed{P_n(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \dots \frac{f^{(n}(a)(x-a)^n}{n!}}$$

b Remainder term

It is the difference between $f(x)$ and $P_n(x)$. It will be differentiable as many times as f is differentiable.

$$\boxed{T_n(x) = f(x) - P_n(x)} \tag{11}$$

c Approximation order

The remainder is the error made when taking $P_n(x)$ as the value of $f(x)$. It is an infinitesimal **of higher order than** $(x-a)^n$, that is, the quotient of both tend to 0, as $x \to a$.

Proof. Taking into account the equality (10) of the derivatives of f and $P_n(x)$ at a and the definition (11) of $T_n(x)$, the first n derivatives of T_n will be null at a:

$$\left.\begin{array}{r}
P_n(a) - f(a) = P_n'(a) - f'(a) = \dots = P_n^{(n}(a) - f^{(n}(a) = 0 \\[2ex]
T_n^{(k}(x) = f^{(k}(x) - P_n^{(k}(x), \; k = 0, 1, \dots, n
\end{array}\right\} \Rightarrow T_n(a) = T_n'(a) = \cdots = T_n^{(n}(a) = 0$$

Then we study the limit

$$\lim_{x\to a} \frac{T_n(x)}{(x-a)^n}$$

which is an indetermination of type 0/0. Applying L'hôpital's rule, the derivatives of both the numerator and the denominator also have a null limit and we obtain another indetermination of the same type. Repeating the process $n-1$ times, it results in:

$$\lim_{x\to a} \frac{T_n(x)}{(x-a)^n} \overset{\text{if } \exists}{=} \lim_{x\to a} \frac{T_n'(x)}{n(x-a)^{n-1}} \overset{\text{if } \exists}{=} \lim_{x\to a} \frac{T_n''(x)}{n(n-1)(x-a)^{n-2}} \overset{\text{if } \exists}{=} \cdots \overset{\text{if } \exists}{=} \lim_{x\to a} \frac{T_n^{n-1}(x)}{n!(x-a)} \tag{12}$$

We have assumed that f is only $n-1$ times differentiable on U_a, so we cannot differentiate $T_n^{n-1}(x)$ on U_a. However, since $T_n^{(n-1}(a)$ is null, we can subtract it from the numerator, obtaining the expression of the derivative of $T_n^{n-1}(x)$ at a (which is also null):

$$\lim_{x\to a} \frac{T_n^{n-1}(x)}{n!(x-a)} = \frac{1}{n!} \lim_{x\to a} \frac{T_n^{n-1}(x) - T_n^{n-1}(a)}{(x-a)} = \frac{T^{(n}(a)}{n!} = 0 \tag{13}$$

Therefore, taking into account (12) and (13),

$$\boxed{\lim_{x\to a} \frac{T_n(x)}{(x-a)^n} = 0}$$

This result can be expressed as

$$\boxed{T_n(x) = \circ(x-a)^n}$$

and we can write $f(x)$ as the sum of the polynomial that approximates it plus the error made in the approximation, which is known as the **limited Taylor expansion of order** n **at** a.

$$\boxed{f(x) = P_n(x) + T_n(x) = \sum_{i=0}^{n} f^{(i}(a)\frac{(x-a)^i}{i!} + T_n(x)} \quad \text{being} \quad \lim_{n\to\infty} \frac{T_n(x)}{(x-a)^n} = 0$$

It can be proved that if $f(x)$ admits a limited expansion of order n at a, then it is unique.

d Geometric meaning of the first three terms of the expansion

1. $y = f(a)$ is the equation of the horizontal line passing through the point $P\big(a, f(a)\big)$.
2. $y = f(a) + f'(a)(x-a)$ is the equation of the tangent to the curve $y = f(x)$ at point P.
3. $y = f(a) + f'(a)(x-a) + \dfrac{1}{2!}f''(a)(x-a)^2$ is the equation of the parabola tangent to the curve $y = f(x)$ at point P.

Remark. There are infinitely many parabolas tangent to $y = f(x)$ at a point, but the Taylor parabola is the one with the best approximation of all of them. In fact, if $y = f(x)$ corresponds to the equation of a parabola, the one obtained with the Taylor formula is the same.

Example. If $y = x^2 + x + 1$, calculating $f(1)$, $f'(1)$ y $f''(1)$, the Taylor parabola for $a = 1$ is $y = (x-1)^2 + 3(x-1) + 3$. Operating, it becomes $y = x^2 + x + 1$.

Exercise. Obtain the Taylor parabola, for $a = 1$, of $y = 2x^2 - 1$.

8.2 Limited MacLaurin expansion of order n

This is the particular case of Taylor expansion for $a = 0$.

$$P_n(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \ldots \frac{f^{(n}(0)x^n}{n!} + T_n(x)$$

The following figures show the curve $y = e^x$ and the approximation obtained by increasing the number of terms of the MacLaurin expansion. On the left the function is represented. The central figure shows the polynomials P_1 (tangent line) and P_2 (tangent parabola). On the right we can see the almost total coincidence of $f(x)$ with P_5, on the interval $[-3, 3]$.

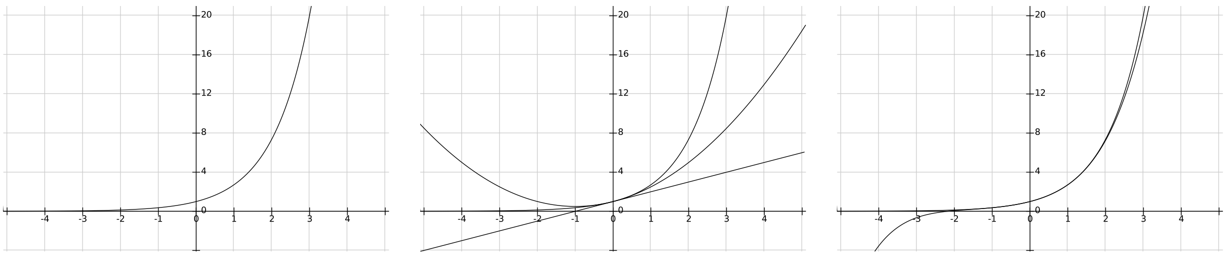

Figure 2: a) Function $f(x) = e^x$; b) Approximation with P_1, P_2; c) Approximation with P_5.

Below are the first terms of the expansions that we will use most often.

1. $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$
2. $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \ldots$
3. $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$
4. $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$
5. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots$
6. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$

8.3 Remainder of Lagrange

There are different expressions for the remainder term. One of the most frequently used is due to Lagrange. To prove it, f is assumed to be $n+1$ times differentiable on a neighborhood of a and Rolle's theorem is repeatedly applied. The resulting expression for $T_n(x)$ is

$$T_n(x) = \frac{f^{(n+1}(\xi)(x-a)^{n+1}}{(n+1)!} \qquad \xi \in (a, x)$$

ξ represents a certain intermediate value between a and x, if x is to the right of a. If x is to the left of a, then $\xi \in (x, a)$. For this reason, the expression of the remainder term does not give us the exact value of the error made in the approximation. However we can obtain a upper bound of the error, as we will see in the following example.

Example. Let $f(x) = e^x$ and $a = 0$. We write the Lagrange formula of the remainder considering $x > 0$, so that $0 < \xi < x$. Since e^x is strictly increasing, we have

$$T_n(x) = \frac{f^{(n+1}(\xi)\, x^{n+1}}{(n+1)!} = \frac{e^\xi\, x^{n+1}}{(n+1)!} < \frac{e^x\, x^{n+1}}{(n+1)!}$$

what allows us to find an error bound. Taking $n = 9$ (i.e. using the first 10 terms of the expansion), we can obtain an upper bound of $T_n(x)$ for different values of x:

$$T_n(x) < \frac{e^x\, x^{10}}{10!} \Longrightarrow \begin{cases} \text{if } x = 1, & T_n(x) < 8 \cdot 10^{-7} \\ \text{if } x = 2, & T_n(x) < 2 \cdot 10^{-3} \\ \text{if } x = 5, & T_n(x) < 400 \end{cases}$$

This result tells us that the approximation is very good for $x = 1$, acceptable for $x = 2$ and useless for $x = 5$, since the error made is greater than the value of the function ($e^5 \approx 148$). In this last case, to get an acceptable error (e.g. less than 10^{-3}), we need to take more terms.

Writing the condition for $x = 5$ and proceeding by trial and error, we find the value of n

$$\frac{e^5\, 5^{n+1}}{(n+1)!} < 10^{-3} \Longrightarrow n = 21$$

what means that we need to use 22 terms. In this case, the upper bound of $T_n(x)$ is $3.15 \cdot 10^{-4}$

Exercise. Repeat the process with the function $y = \sin x$.

8.4 Local extrema property

a Definition of local extremum

A function $f : I \to \mathbb{R}$ has a local (or relative) maximum at $a \in I$ if and only if there exists a neighborhood of a in which the values of the function are less than or equal to $f(a)$, that is

$$\boxed{\exists\, U_a \,/\, \forall x \in U_a \cap I, \quad f(x) \le f(a)}$$

For a minimum, the condition is the same, using $\ge$. For strict maximum and minimum we use the symbols $<$ and $>$ respectively, as well as a punctured neighborhood U_a^* (does not include a).

(The conditions for a function to have global –or absolute– extrema are analogous, considering the whole domain instead of the neighborhood of a point).

Remark. The local extrema are not always located at points where f has a null derivative; in fact, f does not even need to be differentiable at those points. For example, on $I = [-1, 1]$, the function $y = x$ has its extrema at $x = \pm 1$, and its derivative is $f' = 1$ throughout I; while the function $y = |x|$ has a local minimum at $x = 0$, where it is not differentiable.

The following property shows the relation between the local extrema of a differentiable function and the points at which the derivative is zero.

b Property

Let be the function $f : I \to \mathbb{R}$, differentiable at a (a is an interior point of I). If f has a local extremum at a, then $f'(a) = 0$.

Proof for a local maximum. (For a minimum the proof is analogous).

The function f is differentiable at an interior point a, so it has one-sided derivatives. The one at a^+ is:

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Since there exists a local maximum at a, in a certain neighborhood U_a of a the numerator must be less than or equal to 0, while the denominator h is positive. Then the quotient is less than or equal to 0 and so is its limit for $h \to 0^+$, that is $f'(a^+)$. Then

$$f'(a^+) \leq 0$$

On the other hand, the derivative at a^- is

$$f'(a^-) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

In this case, the numerator must be again less than or equal to 0 in the neighborhood U_a of a, while the denominator h is negative. Therefore, the quotient is greater than or equal to 0 and so is its limit for $h \to 0^+$, that is $f'(a^-)$. Then

$$f'(a^-) \geq 0$$

Since f is differentiable at a, the one-sided derivatives must coincide. Therefore

$$f'(a^+) = f'(a^-) = 0 \Longrightarrow \boxed{f'(a) = 0}$$

c Location of local extrema.

We have seen that, if the function has an extremum at an interior point where it is differentiable, then $f'(a) = 0$. We will also have to analyze the points that we previously discarded. As a conclusion, the possible local extrema of f are found in:

- Interior points where $f' = 0$ (**stationary** or **critical points**).
- The endpoints of the interval.
- Points where f is not differentiable.

8.5 Applications of the Taylor expansion

a Study of a function in the neighborhood of a point. Maxima and minima

- If a function is sufficiently differentiable, using the Taylor expansion we can study its behavior on the neighborhood of a point. In particular, we can obtain its extrema at interior points of the domain. Let us consider the expansion of $f(x)$:

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \ldots$$

- If $f'(a) = 0$ and x is very close to a, the behavior of $y = f(x)$ is approximately given by that of $y = f(a) + \frac{1}{2} f''(a)(x-a)^2$ (the next terms are infinitesimal of higher order).

  Its graph is a parabola with the vertex at $x = a$. If $f''(a) > 0$, we have a minimum at $x = a$ and, if $f''(a) < 0$, we have a maximum.

- If, in addition, $f''(a) = 0$, near a the function behaves as $y = f(a) + \frac{1}{6} f'''(a)(x-a)^3$, which is a cubic function with an inflection point at $x = a$.

  This means that, if $f'''(a) > 0$, at $x = a$ the first derivative (null at a) stops decreasing and starts increasing. If $f'''(a) < 0$, the opposite happens.

  Geometric consequence: the center of curvature changes to the other side of the curve.

Example 1. $y = x^2 - x + 1$. We obtain $f'(\frac{1}{2}) = 0,\ f''(\frac{1}{2}) = 2 \Rightarrow$ mínimum $\big(f(\frac{1}{2}) = 3/4\big)$.

Example 2. $y = 8x^3 - 12x^2 + 6x - 1$. Now $f'(\frac{1}{2}) = f''(\frac{1}{2}) = 0,\ f''' \neq 0 \Rightarrow$ point of inflection.

Remark. The existence of an inflection point, does not imply that the tangent to the curve is horizontal. For example, we study the curve $y = f(x) = x^3 + kx$ at $x = 0$.
Since $f''(0) = 0, f'''(0) = 6 \neq 0$, the curve has an inflection point at $x = 0$. But, $f'(0) = k$, hence the slope of the tangent at the origin is $f'(0) = k$.

b Equivalences of infinitesimals

Some equivalences of infinitesimals can be deduced from Taylor expansions. To do it, we eliminate in the expansions infinitesimals of higher order:

1. $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \cdots \Longrightarrow e^x - 1 \sim x$
2. $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots \Longrightarrow \ln(1+x) \sim x$
3. $\sin x = x - \dfrac{x^3}{3!} + \cdots \Longrightarrow \sin x \sim x$
4. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots \Longrightarrow 1 - \cos x \sim \dfrac{x^2}{2}$

c Sum of series

Giving values to x, we can obtain the sum of numerical series. For example, if $x = 1$ in the expansions of e^x and $\arctan x$, we get:

1. $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \cdots \overset{x=1}{\Longrightarrow} \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots = e - 1$
2. $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots \overset{x=1}{\Longrightarrow} 1 - \dfrac{1}{3} + \dfrac{1}{5} - \cdots = \dfrac{\pi}{4}$

d Approximate value of definite integrals

We can obtain an approximate value of the following integral (error $\varepsilon < 10^{-4}$), taking the first 3 terms of the expansion of the sine.

$$\int_0^1 \frac{\sin x}{x}\,dx \approx \int_0^1 \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!}}{x}\,dx = \int_0^1 \left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right) dx = x - \frac{x^3}{18} + \frac{x^5}{600}\bigg|_0^1 = 0.9461...$$

8.6 Expansions deduced from others

Let $f, g \in \mathcal{F}(I, \mathbb{R})$, defined in a neighborhood of the origin, that admit limited Taylor expansions of order n at the origin. Let p and q be their polynomials. It can be proved (J. Burgos, p. 241):

a) The function $\boxed{\alpha f + \beta g,\ \alpha, \beta \in \mathbb{R}}$ admits limited expansion of order n at the origin, its polynomial being the linear combination of polynomials, $\alpha p + \beta q$.

b) The function $\boxed{f \cdot g}$ admits limited expansion of order n at the origin. Its polynomial is obtained by multiplying $p \cdot q$ and eliminating the terms of degree higher than n.

c) If $g(0) \neq 0$, then the function $\boxed{f/g}$ admits limited expansion of order n at the origin. Its polynomial is obtained by dividing p/q according to increasing powers up to degree n.

d) If $f(0) = 0$, the composite function $\boxed{g \circ f}$ admits limited expansion of order n at the origin. Its polynomial is obtained by eliminating the terms of degree higher than n in $q \circ p$. The polynomial $q \circ p$ is obtained by replacing the variable of q by the polynomial p.

e) If F is a primitive of f, then $\boxed{F}$ admits limited expansion of order $n+1$ at the origin, its polynomial P being the primitive of p. The constant of integration is determined by equating the function and the polynomial at the origin, $P(0) = F(0)$.

Example. Given the functions $f(x) = \sin x$ and $g(x) = \cos x$ whose polynomials are

$$p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}, \quad q_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

find the following polynomials:

1. Taylor polynomial of degree 5 of function $h(x) = \sin 2x = 2 \sin x \cos x$.

$$2p(x) \cdot q(x) = 2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right) = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{1}{45}x^7 + \frac{1}{1440}x^9 \Longrightarrow$$

$$\boxed{r_5(x) = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5}$$

2. Taylor polynomial of degree 5 of function $h(x) = \tan x = \sin x / \cos x$.

$$\frac{p(x)}{q(x)} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}} = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{19}{360}x^7 + \ldots \Longrightarrow \boxed{r_5(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5}$$

3. Taylor polynomial of degree 5 of function $h(x) = \sin y|_{y=2x}$. Replacing the variable y by $2x$ in the sine polynomial, we obtain the same result as in case 1 (obviously, the polynomial of function $y = 2x$ is $2x$).

$$\boxed{r_5(x) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5}$$

4. Taylor polynomial of degree 5 of function $f(x) = \sin x$.

$$\sin x = \int \cos x \, dx \Longrightarrow p(x) = \int \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right) dx = x - \frac{x^3}{3!} + \frac{x^5}{5!} + k$$

Since $p(0) = \sin(0) \Longrightarrow k = 0$. Then $\boxed{p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}}$.

9 Representation of curves

It is very useful to know how to graph functions expressed in explicit Cartesian coordinates, as well as curves in parametric or polar coordinates. In the supplementary document "Notes on drawing curves" (in Spanish), brief notions are given for graphing functions, accompanied by examples and proposed exercises.

10 Self-assesment exercises

10.1 True/False exercises

Exercise 1. Decide whether the following statements are true or false.

1. Let f, g be defined in I. The relation $f \leq g \Longleftrightarrow f(x) \leq g(x),\ \forall x \in I$ is a partial order.
2. Let $\mathrm{A} \subset \mathbb{R}$ and $f : \mathrm{A} \to \mathbb{R}$. If $m = \inf f(x)$ in A, then $\forall p > m$, $\exists a \in \mathrm{A} \big/ f(a) < p$.
3. The function $f(x) = e^{-x}$ can be decomposed into the sum of an even function and an odd function.
4. If $\exists \{x_n\}_{n\in\mathbb{N}} \big/ \lim\limits_{n\to\infty} x_n = a$, but $\lim\limits_{n\to\infty} f(x_n) \neq \varphi$, then the function f has no limit φ at $x = a$.
5. If $\lim\limits_{x\to a} f(x) = \varphi \in \mathbb{R},\ f$ is bounded in every neighborhood V_a^*.
6. If $\lim\limits_{x\to a} f(x) = \varphi \in \mathbb{R} \setminus \{0\}$ and $\lim\limits_{x\to a} g(x) = \gamma \in \mathbb{R}$, then $\lim\limits_{x\to a} f(x)^{g(x)} = \varphi^\gamma$.
7. The function $f(x) = 1/x^3 + x^3$ is an infinite, both for $x \to 0$ and for $x \to \infty$.
8. If $f_1 \sim f_2, g_1 \sim g_2$ at a, then $f_1^{g_1} \sim f_2^{g_2}$ at a.
9. The decimal part function, $f(x) = x - \mathrm{E}(x)$, has two different one-sided limits at the origin, so there is a removable discontinuity at this point.
10. Let be the functions power of positive rational exponent $\left(x^{m/n},\ m/n \in \mathbb{Q}^+\right)$ and power of positive irrational exponent $(x^a,\ a \notin \mathbb{Q},\ a > 0)$. Both are continuous on $\mathbb{R}$.
11. Every continuous function on a closed interval is bounded, so it has a supremum and an infimum. It does not always have a maximum and a minimum.
12. Every continuous function on a bounded interval I is uniformly continuous on I (Heine's theorem).

Exercise 2. Decide whether the following statements are true or false.

1. Let $a \in I = (\alpha, \beta) \subset \mathbb{R}$. If the function f is continuous on I, differentiable on $I \setminus \{a\}$, and $\exists \lim\limits_{x\to a} f'(x)$, then f is differentiable at a and it holds that $f'(a) = \lim\limits_{x\to a} f'(x)$.
2. The differential of the product of f and g is the product of the differentials of the functions.
3. $(x^x)' = x^x(1 + \ln x)$.
4. $f' > 0$ on I if and only if f is strictly increasing on I.
5. If f and g are infinites at a, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$.
6. The remainder term $T_n(x) = f(x) - P_n(x)$ is an infinitesimal of order n.
7. Let be $f : I \to \mathbb{R}$. The relative extremes of f on I can be only at the points of zero derivative or at the endpoints of the interval I.
8. If f and g admit a limited expansion of order n at the origin, then $f \cdot g$ admits a limited expansion of order n at the origin. Its polynomial is obtained by multiplying the polynomials of f and g and eliminating the terms of degree higher than n.
9. The function $x^2 + \dfrac{1}{x^2}$ is both an infinite (at the origin) and an infinitesimal (for $x \to \infty$).

10.2 Question

Obtain an approximate value for $\sqrt{627}$, using the concept of differential and knowing that $25^2 = 625$.

10.3 Solution to the True/False exercises

Exercise 1.

1. **T**. The relation thus defined satisfies the reflexive, antisymmetric and transitive properties. But, given any two functions, the values of one of them do not have to be less than or equal to those of the other for every point in the domain, so it does not have to be an order relation between them. See section **A.1** of the unit.

2. **T**. We prove it by *reductio ad absurdum*. We assume the opposite, that is $\exists\, p > m \,\big/\, \forall a \in$ A, $f(a) \geq p > m$. Then m is not the infimum (the largest of the lower bounds) since p is a lower bound greater than m.

3. **T**. Every function defined in a symmetric domain admits the decomposition $f(x) = \frac{1}{2}\Big(f(x) + f(-x)\Big) + \frac{1}{2}\Big(f(x) - f(-x)\Big)$, where the first addend is an even function and the second an odd one. For example, if $f(x) = e^x$, then $e^x = \cosh x + \sinh x,\ \forall x \in \mathbb{R}$.

4. **T**. It is a consequence of the relation between the limit of a function and the sequential limit. See section **B.4** of the unit.

5. **F**. Property 2 of limits states that f is bounded on some punctured neighborhood of a, which does not ensure that it is bounded on any. For example, $f(x) = 1/x$ has limit $1/2$ at $x = 2$ and is bounded on $U_2^* = (1, 2) \cup (2, 3)$ but it is not bounded on $V_1^* = (0, 2) \cup (2, 4)$.

6. **F**. To be true, $\varphi > 0$ must be true.

7. **F**. f is an infinite for $x \to \infty$. At the origin it is not an infinite since it does not have a constant sign when approaching 0. Indeed, if $x \to 0^+,\ f \to \infty$ and, if $x \to 0^-,\ f\ to - \infty$. We can only say that $|f| \to \infty$ when $x \to 0$. The function $g(x) = 1/x^4 + x^4$ would be an infinite at the origin (and also for $x \to \infty$).

8. **F**. We cannot assure it. For example, $f_1 = f_2 = e;\ g_1 = x^2 + x,\ g_2 = x^2$. When $x \to +\infty$ the quotient between $f_1^{g_1}$ and $f_2^{g_2}$ tends to ∞.

9. **F**. If the one-sided limits exist and do not coincide, as in this case, we say that there is a jump discontinuity.

10. **F**. The function $x^{m/n},\ m/n \in \mathbb{Q}^+$ exists and is continuous on $\mathbb{R}$, if n is odd; and only on $\mathbb{R}^+ \cup \{0\}$, if n is even. The function $x^a,\ a \notin \mathbb{Q},\ a > 0$ exists and is continuous only on $\mathbb{R}^+ \cup \{0\}$. A negative number raised to an irrational exponent makes no sense.

11. **F**. By Weierstrass's theorem (theorems of continuous functions), a continuous function on a closed interval attains a maximum and a minimum on it (they are, respectively the supremum and the infimum).

12. **F**. Heine's theorem only ensures the uniform continuity of a continuous function on a closed interval. For example, $f(x) = 1/x$ is not uniformly continuous on the bounded interval $(0, 1)$.

Exercise 2.

1. **T**. See section D.5. of the unit (The derivative as a limit of derivatives).
2. **F**. $d(f \cdot g) = f dg + g\, df$.
3. **T**. $(x^x)' = x x^{x-1} + x^x \ln x = x^x + x^x \ln x = x^x(1 + \ln x)$.
4. **F**. See section **D.4.4** of the unit (mean value theorems). Counterexample: the function $y = x^5$ is strictly increasing but has zero derivative at $x = 0$.
5. **F**. By L'Hopital's rule, if $\lim_{x \to a} \dfrac{f'(x)}{g'(x)} = \alpha$, then there exists $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \alpha$.

   But the first limit may not exist and the second may (see section **D.6** of the unit and a supporting document).
6. **F**. The remainder term $T_n(x)$ is an infinitesimal of order higher than n. See section **D.8.1** of the subject (Taylor limited expansion).
7. **F**. There may be extremes at points where the function is not differentiable. For example, the function $y = |x|$ at $x = 0$.
8. **T**. See section **D.8.6** of the unit (Expansions deduced from others).
9. **F**. It is an infinite, both at the origin and for $x \to \infty$.

10.4 Solution to the question

The condition that the function f must satisfy to be differentiable at a is:

$$f(x) - f(a) = [f'(a) + \varepsilon(x - a)]\,(x - a) = f'(a)\,(x - a) + \varepsilon(x - a)\,(x - a)$$

where $\varepsilon(x - a) \to 0$ when $x \to a$.

If a function is differentiable, we can approximate the increment of f between two points by means of the differential, neglecting infinitesimals of order higher than that of $(x - a)$:

$$f(x) - f(a) \approx df(a) = f'(a)\,(x - a)$$

So if $f(x) = \sqrt{x},\ f'(x) = \dfrac{1}{2\sqrt{x}},\ x = 627,\ a = 625$, we will have:

$$\sqrt{627} - \sqrt{625} \approx \frac{1}{2\sqrt{625}}\,2 = 0.04 \Longrightarrow \sqrt{627} \approx 25.04$$

In other words, we have substituted the value of f at the point x for that of the ordinate of the line tangent to the curve $y = f(x)$ at the point a:

$$f(x) \approx f(a) + f'(a)\,(x - a)$$

Note: The exact value of $\sqrt{627}$ is 25.039968... The relative error made when approximating the value of the increment by that of the differential is:

$$E_r = \frac{0.04 - 0.039968}{0.04}\,100 = 0.08\%$$

Unit V. Techniques of integration (14.12.2023)

1 Introduction and basic concepts

This unit is intended as a support for practical classes on the obtaining of primitives, recalling some concepts and introducing techniques of integration.

First, some basic ideas about trigonometric and logarithmic functions are briefly recalled. Then the hyperbolic functions and their inverses are defined, which are widely applied in integration (a somewhat more detailed study of these issues can be seen in the Mathematics precourse).

Next, the concepts of primitive and immediate integral are defined and the changes of variable are introduced. After describing the method of integration by parts, it is applied to the reduction formulas. Finally, the usual techniques to solve rational, trigonometric and irrational integrals are shown.

1.1 Trigonometric functions

a) Definitions. From the sine and cosine, we define the tangent, cotangent, secant and cosecant:

$$\tan x = \frac{\sin x}{\cos x}\,; \qquad \operatorname{cotan} x = \frac{\cos x}{\sin x}\,; \qquad \sec x = \frac{1}{\cos x}\,; \qquad \operatorname{cosec} x = \frac{1}{\sin x}$$

b) Sine, cosine and tangent of angle sum and double angle.

1. $\sin(x+y) = \sin x \cos y + \cos x \sin y \Longrightarrow \sin 2x = 2 \sin x \cos x$.
2. $\cos(x+y) = \cos x \cos y - \sin x \sin y \Longrightarrow \cos 2x = \cos^2 x - \sin^2 x$.
3. $\tan(x+y) = \dfrac{\sin(x+y)}{\cos(x+y)} = \cdots = \dfrac{\tan x + \tan y}{1 - \tan x \tan y} \Longrightarrow \tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$.

c) Derivatives of sine, cosine and tangent.

$$\sin' x = \cos x\,; \qquad \cos' x = -\sin x\,; \qquad \tan' x = \frac{1}{\cos^2 x} = 1 + \tan^2 x$$

d) Some relations between trigonometric functions.

$$\sin^2 x + \cos^2 x = 1\,; \qquad 2\sin^2 x = 1 - \cos 2x\,; \qquad 2\cos^2 x = 1 + \cos 2x$$

1.2 Natural logarithm

The formulas of the logarithms of the product, quotient and power are very important.

$$\ln xy = \ln x + \ln y\,; \qquad \ln(x/y) = \ln x - \ln y\,; \qquad \ln x^y = y \ln x$$

2 Hyperbolic functions

2.1 Definition

Hyperbolic functions are defined from the function e^x.

$$\sinh x = \frac{e^x - e^{-x}}{2}\,; \qquad \cosh x = \frac{e^x + e^{-x}}{2}\,; \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

2.2 Odd and even function

The following properties (easy to verify) show that the cosine is an even function, while the hyperbolic sine and tangent are odd (the same as in trigonometric functions).

$$\sinh(-x) = -\sinh(x)\,; \qquad \cosh(-x) = \cosh(x)\,; \qquad \tanh(-x) = -\tanh(x)$$

2.3 Sine, cosine and hyperbolic tangent of $x + y$

a) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y \Longrightarrow \sinh 2x = 2 \sinh x \cosh x$.

b) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y \Longrightarrow \cosh 2x = \cosh^2 x + \sinh^2 x$.

c) $\tanh(x+y) = \dfrac{\sinh(x+y)}{\cosh(x+y)} = \cdots = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \Longrightarrow \tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$.

2.4 Derivatives of the hyperbolic functions

From the definition of these functions, their derivatives are obtained (check it as an exercise).

$$\sinh' x = \cosh x\,; \qquad \cosh' x = \sinh x\,; \qquad \tanh' x = \cdots = \frac{1}{\cosh^2 x} = 1 - \tanh^2 x$$

2.5 Relation between hyperbolic sine and cosine

It is similar to that between trigonometric sine and cosine.

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \cdots = 1$$

2.6 Inverse functions

Intuitive notion. Every function f associates, to each value x of the independent variable, a value $y = f(x)$ of the dependent variable. The inverse function of a given one goes the opposite way, associating to each y the corresponding value $x = f^{-1}(y)$. For example, if f associates to any x the value $y = 2x$, the inverse function f^{-1} associates to any y the corresponding $x = y/2$.

A practical way to obtain the inverse function of a given one is to solve for the variable x in terms of y, and then interchange the names of the variables x and y. For example, if f associates to each x the value $y = x^2$,

$$y = f(x) = x^2 \Longrightarrow x = \sqrt{y} \Longrightarrow y = f^{-1}(x) = \sqrt{x}$$

The inverse of the "square" function is the "square root" function.

Example. Inverse trigonometric functions and their derivatives. The inverse functions of the three main trigonometric functions are, respectively, the arc sine, the arc cosine and the arc tangent, whose derivatives are shown below (in unit IV, the general way of obtaining the derivative of the inverse of any function is studied).

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}\,; \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}\,; \qquad (\arctan x)' = \frac{1}{1+x^2}$$

Relation between the graphs of a function and its inverse. They are symmetric with respect to the bisector of the first quadrant (reason it as an exercise).

2.7 Inverse hyperbolic functions

a) Inverse hyperbolic sine. The inverse function of $y = \sinh x$ will be $y = \operatorname{argsh} x$ (argument whose hyperbolic sine is x), which we are going to write in logarithmic form. To do this, we start from the inverse function $y = \operatorname{argsh} x$, solve for the variable x and use the definition of the hyperbolic sine function.

$$y = \operatorname{argsh} x \Longrightarrow x = \sinh y = \frac{e^y - e^{-y}}{2} \Longrightarrow 2x = e^y - e^{-y}$$

Now we multiply both sides by e^y and solve a quadratic equation of unknown e^y.

$$2x\, e^y = (e^y)^2 - 1 \Longrightarrow (e^y)^2 - 2x\, e^y - 1 = 0 \Longrightarrow e^y = x \pm \sqrt{x^2+1}$$

Finally we obtain y as the natural logarithm of e^y. For the sign $+$ before the root, the expression takes a positive value $\forall x$. On the other hand, for the sign $-$ its value is negative regardless of the value of x, so no value of y is a solution to the equation. Then it turns out

$$\boxed{y = \ln\left(x + \sqrt{x^2+1}\right), \quad x \in (-\infty, \infty)}$$

b) Inverse hyperbolic cosine. We repeat the steps of the previous section. Starting from the inverse hyperbolic cosine, we write x as a function of e^y.

$$y = \operatorname{argch} x \Longrightarrow x = \cosh y = \frac{e^y + e^{-y}}{2} \Longrightarrow 2x = e^y + e^{-y}$$

We multiply by e^y and solve.

$$2x\, e^y = (e^y)^2 + 1 \Longrightarrow (e^y)^2 - 2x\, e^y + 1 = 0 \Longrightarrow e^y = x \pm \sqrt{x^2-1}$$

The root exists only for $x^2 \geq 1$, that is, $x \geq 1$ or $x \leq -1$. For $x \leq -1$, $e^y < 0$, so there is no solution. For $x \geq 1$, e^y takes a positive value for both root signs and each results in a value of y that is a solution to the equation. We take only the positive sign because it is a function, so each x can only have one image. It results

$$\boxed{y = \ln\left(x + \sqrt{x^2-1}\right), \quad x \in [1, \infty)}$$

Exercise. Obtain the relation between the values of y, that correspond to the two signs of the root. Can you relate the result to the properties of the graph of the inverse function?

Inverse hyperbolic tangent. We start from the inverse hyperbolic tangent, writing x as a function of e^y and multiplying the numerator and denominator by e^y

$$y = \operatorname{argth} x \Longrightarrow x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1}$$

We solve the quadratic equation

$$x\left(e^{2y} + 1\right) = e^{2y} - 1 \Longrightarrow e^{2y} = \frac{1+x}{1-x}$$

and take logarithms.

$$\boxed{y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \ln\sqrt{\frac{1+x}{1-x}}, \quad x \in (-1, 1)}$$

Exercise. Reason the indicated fields of existence.

2.8 Derivatives of te inverse hyperbolic functions

To obtain the derivatives of these functions, we express them in logarithmic form and derive by the usual method.

a) $(\operatorname{argsh} x)' = \left[\ln\left(x+\sqrt{x^2+1}\right)\right]' = \dfrac{\left(x+\sqrt{x^2+1}\right)'}{x+\sqrt{x^2+1}} = \cdots = \dfrac{1}{\sqrt{x^2+1}}.$

b) $(\operatorname{argch} x)' = \left[\ln\left(x+\sqrt{x^2-1}\right)\right]' = \dfrac{\left(x+\sqrt{x^2-1}\right)'}{x+\sqrt{x^2-1}} = \cdots = \dfrac{1}{\sqrt{x^2-1}}.$

c) $(\operatorname{argth} x)' = \dfrac{1}{2}\left[\ln\left(\dfrac{1+x}{1-x}\right)\right]' = \dfrac{1}{2}\left(\dfrac{1+x}{1-x}\right)' : \left(\dfrac{1+x}{1-x}\right) = \cdots = \dfrac{1}{1-x^2}.$

3 Primitive function. Immediate integrals

The concepts of integral function and definite integral will be seen in detail later, after having studied functions and their derivatives. For the moment we will define the primitive of a function, also called indefinite integral or antiderivative.

3.1 Primitive of a function

Given a function f, defined on an interval I, we call primitive of f any function F, derivable in I, such that its derivative coincides with f $(F' = f)$. In that case we write

$$F(x) = \int f(x)\,dx$$

If C is any constant, then

$$\big(F(x)+C\big)' = F'(x)+0 = f(x)$$

so $F(x)+C$ is also a primitive of f, therefore $F(x)+C$, $\forall\, C \in \mathbb{R}$ is the set of primitives of $f(x)$. We express this as

$$\int f(x)\,dx = F(x)+C$$

3.2 Differential

The factor dx that appears in the above expression is called the differential of x. As will be seen in unit IV, the differential of a function is equal to the product of its derivative times the differential of the variable, that is to say

$$dg(x) = g'(x)\,dx$$

For example, the differential of the sine function is the cosine times the differential of x.

$$u = \sin x \Longrightarrow du = \cos x dx$$

From here it results

$$F'(x) = f(x) \Longrightarrow F(x) = \int f(x)\,dx = \int F'(x)\,dx = \int dF(x) \Longrightarrow F(x) = \int dF(x)$$

This means that the integral of the differential of a function is the function itself. As a consequence, the primitive we are looking for is the function whose differential is the integrand. This will be useful in the calculation of definite integrals.

3.3 Linearity of the integral

The integral of a linear combination is the linear combination of the integrals.

$$\int \big(\alpha f(x) + \beta g(x)\big)\, dx = \alpha \int f(x)\, dx + \beta \int g(x)\, dx$$

3.4 Immediate integrals

From the formulas for the derivation of the most common functions, a table of immediate integrals can be obtained that helps in the obtaining of primitives. The last section of this unit is a table with the 19 most frequently used integrals, most of which are recommended to be memorized.

The last three cases have two different solutions. In the last two, one of the solutions has a larger interval of existence than the other.

4 Semiinmediate integrals and changes of variable

Semi-immediate (or almost immediate) integrals are those that become immediate by means of some simple operation on the integrand, for example using the chain rule or doing a change of variable. For example, to integrate

$$\int \sin^2 x \cos x\, dx$$

we can replace $\sin x$ by u, so $du = \cos x\, dx$. Taking into account the first immediate integral of the table, it results

$$\int \sin^2 x \cos x\, dx = \int u^2\, du = \frac{u^3}{3} = \frac{\sin^3 x}{3} + C$$

The above is an implicit change of variable. An explicit one is shown below, which allows us to simplify the integral (it is not solved completely). Doing $x = \sin t$ and $dx = \cos t\, dt$, and taking into account one of the relations seen in section **1.1**, it results

$$\int \sqrt{1-x^2}\, dx = \int \sqrt{1-\sin^2 t}\, \cos t\, dt = \int \cos^2 t\, dt = \frac{1}{2}\int (1+\cos 2t)\, dt = \frac{1}{2}\int dt + \frac{1}{2}\int \cos 2t\, dt$$

The supplementary document “Introduction to changes of variable” describes these two types of changes in more detail and shows different examples.

5 Integration by parts

We start from the differential of the product of two functions of x, $u(x)$ and $v(x)$. As will be studied in unit IV,

$$d(uv) = (uv)'dx = (uv' + u'v)dx = uv'dx + vu'dx = udv + vdu$$

We solve for udv and integrate both members. Remembering that the integral of the differential of a function is the function itself, it results

$$udv = d(uv) - vdu \Longrightarrow \int udv = uv - \int vdu$$

To apply the method we have to obtain the factors u and dv, calculate from them du and v and replace. The method will be useful if the integral of vdu is easier to obtain than the initial one.

6 Reduction formulas

1) Integral depending on a parameter. Let us consider an integral, which we denote $I(n)$, whose integrand depends on a parameter $n \in \mathbb{R}$. Suppose that, applying some method (usually integration by parts), we can express it as a function of the same integral, with smaller values of the parameter, that is $I(n) = f\big(I(n-1),\, I(n-2)\dots\big)$.

In this case we have found a **reduction formula**, valid in principle $\forall n \in \mathbb{R}$. Most often, $I(n)$ will result in a function of $I(n-1)$ or $I(n-2)$ or both.

Example. Let $I(n) = \displaystyle\int \ln^n x\, dx$. If $n \neq 0$, we integrate by parts. It results:

$$\left.\begin{array}{l} u = \ln^n x \Rightarrow du = n \ln^{n-1} x \, \dfrac{1}{x}\, dx \\ \\ dv = dx \Rightarrow v = x \end{array}\right\} \Longrightarrow I(n) = \int \ln^n x\, dx = x \ln^n x - nI(n-1)$$

2) The parameter n is a natural number. In this case, applying the formula again and again, we would obtain $I(n-1)$ as a function of $I(n-2)$, this one as a function of $I(n-3)$, and so on. We would thus be reducing the degree, arriving at $I(0)$ or $I(1)$. These are integrated directly, since they are usually very simple. Sometimes it is observed that they are particular cases of the general formula.

3) Explícit formula. Sometimes, by iterating the method, we can arrive at an explicit formula that directly gives us the value of $I(n)$, although it is usually complicated. In the above case:

$$\int \ln^n x\, dx = x\big(\ln^n x - n \ln^{n-1} x + n(n-1) \ln^{n-2} x + \cdots + (-1)^n\, n!\big)$$

4) The parameter is a negative integer. Let us suppose that the parameter n is a negative integer and we apply again and again the reduction formula. This results in an endless process, in which n takes decreasing negative values. In these cases, we have to obtain $I(n-1)$ as a function of $I(n)$ or, what is the same, $I(n)$ as a function of $I(n+1)$. Thus, in each step the value of the parameter increases until it reaches $I(-1)$, $I(0)$, which are calculated directly.

5) Change of sign of the parameter. Since the reduction formula is valid $\forall n \in \mathbb{R}$, we can sometimes obtain the formula of one integral from that of another, changing the sign of n.

Example. Let $I(n) = \displaystyle\int x^n e^x\, dx$; $J(n) = \displaystyle\int \frac{e^x}{x^n}\, dx$.

We see that $J(n) = I(-n)$, so, if we know the reduction formula for $I(n)$, we can obtain the one for $J(n)$ without having to integrate. It suffices to change the sign of the parameter in the reduction formula of $I(n)$ and operate.

That is, since we know $I(n)$ as a function of $I(n-1)$, then $J(n)$ (which is $I(-n)$) can be written as a function of $I(-n-1)$, which is equal to $J(n+1)$. To finish, we have to solve for $J(n+1)$ as a function of $J(n)$ and, from there, to get the relation between $J(n)$ and $J(n-1)$.

In the above example is easy to obtain $I(n) = x^n e^x - nI(n-1)$. From this expression,

$$J(n) = I(-n) = x^{-n} e^x + nI(-n-1) = x^{-n} e^x + nJ(n+1) \Longrightarrow J(n+1) = \frac{1}{n} J(n) - \frac{e^x}{nx^n}$$

from which we get the expression, that would also be obtained directly solving $J(n)$,

$$J(n) = \frac{1}{n-1} J(n-1) - \frac{1}{n-1} \frac{e^x}{x^{n-1}}$$

7 Rational functions

The integrand is a rational function, that is, a quotient of polynomials.

$$I = \int \frac{P_k(x)}{Q_l(x)}\,dx,\ k < l$$

For the application of the method, the numerator and denominator must not have common factors and the degree of the numerator must be less than that of the denominator.

If $k \geq l$, we divide the polynomials and the problem is reduced to integrating the quotient between the remainder of the division and the polynomial $Q_l(x)$.

According to the Fundamental Theorem of Algebra, every polynomial with real coefficients and degree l has l roots. These roots will be simple or multiple, real or complex; and for each complex root its conjugate root is also a root. When factoring the polynomial, the real roots give rise to factors of the type $(x-a)$, while each pair of complex conjugate roots gives rise to a quadratic factor with no real roots, such as x^2+1, whose roots would be $\pm i$, or x^2+x+2 (roots $1 \pm i$).

Let us consider a polynomial $Q_l(x)$ with the following roots: two real, one simple and the other multiple and four complex: two simple (conjugates) and two multiple (also conjugates). If the coefficient of the highest degree term is l, the resulting factoring will be

$$Q_l(x) = (x-a)(x-b)^m\,(x^2+cx+d)(x^2+ex+f)^n$$

where the degree l equals the sum of the degrees of the factors, $l = 1+m+2+2n = 2n+m+3$.

We decompose the quotient into simple fractions, obtaining

$$\frac{P_k(x)}{Q_l(x)} = \frac{A}{x-a} + \frac{B_1}{x-b} + \cdots + \frac{B_m}{(x-b)^m} + \frac{Cx+D}{x^2+cx+d} + \frac{E_1x+F_1}{x^2+ex+f} + \cdots + \frac{E_nx+F_n}{(x^2+ex+f)^n}$$

(we see that the number of indetermined coefficients is equal to l).

Next we make a common denominator in the second member and identify the numerators, obtaining a system of l equations with l unknowns. Once the system is solved, it gives us the indetermined coefficients.

To finish we must integrate the different resulting fractions.

a) $\displaystyle\int \frac{dx}{x-a} = \ln|x-a| + C.$

b) $\displaystyle\int \frac{dx}{(x-b)^m} = \cdots = -\frac{1}{m-1}\frac{1}{(x-b)^{m-1}} + C.$

c) Fractions of the form $\dfrac{Cx+D}{x^2+cx+d}$ yield a natural logarithm plus an arc tangent, as in the following case.

$$\int \frac{x+3}{x^2+2x+3}\,dx = \frac{1}{2}\int \frac{2x+6}{x^2+2x+3}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+3}\,dx + 2\int \frac{dx}{(x+1)^2+2} =$$

$$\frac{1}{2}\ln|x^2+2x+3| + \sqrt{2}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + C.$$

d) Fractions of the type $\dfrac{E_nx+F_n}{(x^2+ex+f)^n}$ can be integrated by a fairly laborious process, resulting in a recurrrent expression that allows to reduce the degree of the denominator. So, when the denominator contains factors of the form $(x^2+ex+f)^n,\ n > 1$, the Hermite's method is much more practical.

Hermite's method (for quotients of polynomials in which the denominator has quadratic factors without real roots, raised to an exponent greater than 1).

As before, we assume that $Q_l(x)$ has two real roots, one simple and another multiple; four complex, two simple and two multiple; and that the coefficient of the term of highest degree is l. The expression of the factored polynomial is:

$$Q_l(x) = (x-a)(x-b)^m\,(x^2+cx+d)(x^2+ex+f)^n$$

The method consists in decomposing the integrand as follows:

$$\frac{P_k(x)}{Q_l(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{Cx+D}{x^2+cx+d} + \frac{Ex+F}{x^2+ex+f} + \frac{d}{dx}\left(\frac{R_s(x)}{(x-b)^{m-1}(x^2+ex+f)^{n-1}}\right)$$

That is, in the first four addends of the second member, the factors of $Q_l(x)$ appear with exponent equal to 1. In the denominator of the derived quotient, the exponent of the factors is reduced by 1, so those with exponent 1 do not appear. Finally, the degree of the polynomial $R_s(x)$ (with indetermined coefficients) is less than that of the denominator, that is, of degree $2n+m-4$ at most.

After deriving the quotient, we reduce the addends to a common denominator and equal the numerators, resulting in a system of $2n+m+3$ equations, with as many unknowns. Now we identify the indetermined coefficients and integrate: the first two fractions give rise to natural logarithms; the next two, to logarithm plus arctangent. The last fraction is the easiest to integrate:

$$\int \frac{d}{dx}\left(\frac{R_s(x)}{(x-b)^{m-1}(x^2+ex+f)^{n-1}}\right) dx = \frac{R_s(x)}{(x-b)^{m-1}(x^2+ex+f)^{n-1}}$$

Example. $\displaystyle\int \frac{x^2}{\left(x^2+1\right)^2}\,dx.$

a) We decompose into fractions, derive and reduce to a common denominator:

$$\frac{x^2}{\left(x^2+1\right)^2} = \frac{Ax+B}{x^2+1} + \frac{d}{dx}\left(\frac{Cx+D}{x^2+1}\right) = \frac{Ax+B}{x^2+1} + \frac{C(x^2+1)-(Cx+D)2x}{\left(x^2+1\right)^2} =$$
$$\frac{(Ax+B)(x^2+1)+Cx^2+C-2Cx^2-2Dx}{\left(x^2+1\right)^2} = \frac{Ax^3+(B-C)x^2+(A-2D)x+B+C}{\left(x^2+1\right)^2}$$

b) We set up the system, equaling the coefficients of the terms of the same degree:

$$A=0,\quad B-C=1,\quad A-2D=0,\quad B+C=0$$

c) We solve the system: $A=D=0,\quad B=-C=1/2.$

d) Integrating, we get:

$$I = \int \frac{1/2}{1+x^2}\,dx + \int \frac{d}{dx}\left(\frac{-x/2}{1+x^2}\right) dx = \frac{1}{2}\arctan x - \frac{x}{2\left(1+x^2\right)} + C$$

Remark. This example is only intended to show the application of the method in a simple case. The exercise would be more easily solved by parts (making $u=x$).

Exercise. Solve by Hermite's method: $\displaystyle\int \frac{x^2}{(x-1)\left(x^2+1\right)^2}\,dx.$

8 Trigonometric integrals. Changes of variable

1) Integrand "odd function of sine". If $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, $\boxed{\cos x = t}$.

$$\boxed{\sin x = \sqrt{1-t^2}}\ ;\ -\sin x\ dx = dt \Longrightarrow \boxed{dx = \frac{-dt}{\sqrt{1-t^2}}}$$

Example. $\displaystyle\int \sin^3 x \cos^2 x\ dx = \int \underbrace{\sin^2 x}_{1-t^2} \cos^2 x \underbrace{\sin x\ dx}_{-dt} = -\int (1-t^2)\, t^2\, dt.$

2) Integrand "odd function of cosine". If $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, $\boxed{\sin x = t}$.

$$\boxed{\cos x = \sqrt{1-t^2}}\ ;\ \cos x\ dx = dt \Longrightarrow \boxed{dx = \frac{dt}{\sqrt{1-t^2}}}$$

Example. $\displaystyle\int (\cos^3 x + \cos x) \sin^2 x\ dx = \int (\underbrace{\cos^2 x + 1}_{2-t^2}) \sin^2 x \underbrace{\cos x\ dx}_{dt} = \int (2-t^2) t^2\ dt.$

3) Integrand is an "even function of sine-cosine". If $R(-\sin x, -\cos x) = R(\sin x, \cos x)$, we may change $\boxed{\tan x = t}$, so

$$\boxed{\cos x = \frac{1}{\sqrt{1+t^2}}}\ ;\ \boxed{\sin x = \frac{t}{\sqrt{1+t^2}}}\ ;\ (1+\tan^2 x)\ dx = dt \Longrightarrow \boxed{dx = \frac{dt}{1+t^2}}$$

Example. $\displaystyle\int \frac{\cos^2 x}{\sin^4 x}\ dx = \int \frac{1}{1+t^2} \frac{(1+t^2)^2}{t^4} \frac{dt}{1+t^2} = \int \frac{dt}{t^4}.$

4) Rest of cases. If the above changes do not work, we may change $\boxed{\tan \frac{x}{2} = t}$, so

$$\cos \frac{x}{2} = \frac{1}{\sqrt{1+t^2}}\ ;\ \sin \frac{x}{2} = \frac{t}{\sqrt{1+t^2}} \Longrightarrow \boxed{\sin x = \frac{2t}{1+t^2}}\ ;\ \boxed{\cos x = \frac{1-t^2}{1+t^2}}$$

$$\left(1 + \tan^2 \frac{x}{2}\right)\ d\left(\frac{x}{2}\right) = dt \Longrightarrow \boxed{dx = \frac{2dt}{1+t^2}}$$

Example. $\displaystyle\int \frac{2+\sin x}{2+\cos x}\ dx = \cdots = 4 \int \frac{1+t+t^2}{(3+t^2)(1+t^2)}\ dt.$

5) Change of products into sums. We start from the sine and the cosine of $x+y$ and $x-y$. By adding or subtracting, we can obtain the needed expressions to replace the products.

$$\begin{aligned} \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \cos(x-y) &= \cos x \cos y + \sin x \sin y \\ \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \sin(x-y) &= \sin x \cos y - \cos x \sin y \end{aligned} \Longrightarrow \begin{aligned} &\boxed{\sin x \sin y = \frac{\cos(x-y) - \cos(x+y)}{2}} \\ &\boxed{\cos x \cos y = \frac{\cos(x-y) + \cos(x+y)}{2}} \\ &\boxed{\sin x \cos y = \frac{\sin(x-y) + \sin(x+y)}{2}} \end{aligned}$$

9 Irrational integrals. Changes of variable

1) Quotient of binomials with different rational exponents. That is, the integrand is of the type

$$R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{p}{q}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{r}{s}}, \dots\right]$$

We change: $\boxed{\dfrac{ax+b}{cx+d} = t^m}$ being $m = \text{m.c.m.}(q, s \dots)$.

Example. $I = \displaystyle\int \sqrt{\frac{x+1}{x+2}}\, dx$. Doing $\dfrac{x+1}{x+2} = t^2 \Longrightarrow x = \dfrac{2t^2-1}{1-t^2}$; $dx = \dfrac{2t\, dt}{(1-t^2)^2}$.

It results $I = \displaystyle\int \frac{2t^2}{(1-t^2)^2}\, dt$, that is solved by partial fraction decomposition.

2) The integrand contains the root of a sum or difference of squares.

a) $R\left[x, \sqrt{c^2 - a^2x^2}\right]$: $\boxed{ax = c\sin t} \Longrightarrow \sqrt{c^2-a^2x^2} = c\cos t;\ dx = \dfrac{c}{a}\cos t\, dt.$

Example. $\displaystyle\int \sqrt{5-x^2}\, dx : x = \sqrt{5}\sin t \Longrightarrow \sqrt{5-x^2} = \cdots = \sqrt{5}\cos t;\ dx = \sqrt{5}\cos t dt.$

$$I = 5\int \cos^2 t dt = \frac{5}{2}\int (1+\cos 2t)dt = \frac{5}{2}(t + \sin t\cos t) = \frac{5}{2}\arcsin\frac{x}{\sqrt{5}} + \frac{x}{2}\sqrt{5-x^2} + C.$$

Particular case. $\displaystyle\int \frac{dx}{\sqrt{5-x^2}} = \arcsin\frac{x}{\sqrt{5}} + C$ (immediate).

b) $R\left[x, \sqrt{a^2x^2 - c^2}\right]$: $\boxed{ax = \dfrac{c}{\cos t}} \Longrightarrow \sqrt{a^2x^2-c^2} = c\tan t;\ dx = \dfrac{c}{a}\dfrac{\sin t}{\cos^2 t}\, dt.$

Example. $\displaystyle\int \frac{dx}{(x^2-5)^{3/2}} : x = \frac{\sqrt{5}}{\cos t} \Longrightarrow (x^2-5)^{3/2} = \cdots = 5^{3/2}\tan^3 t;\ dx = \sqrt{5}\frac{\sin t}{\cos^2 t}\, dt.$

$$I = \int \frac{\cos^3 t}{5^{3/2}\sin^3 t}\frac{\sqrt{5}\sin t}{\cos^2 t}dt = \frac{1}{5}\int \frac{\cos t}{\sin^2 t}dt = \frac{1}{5}\frac{-1}{\sin t} = -\frac{1}{5}\frac{x}{\sqrt{x^2-5}} + C.$$

Particular case. $\displaystyle\int \frac{dx}{\sqrt{x^2-5}} = \text{argch}\frac{x}{\sqrt{5}} + C = \ln\left|x + \sqrt{x^2-5}\right| + C$ (immediate).

c) $R\left[x, \sqrt{a^2x^2 + c^2}\right]$: $\boxed{ax = c\tan t} \Longrightarrow \sqrt{a^2x^2+c^2} = \dfrac{c}{\cos t};\ dx = \dfrac{c}{a}\dfrac{1}{\cos^2 t}\, dt.$

Example. $\displaystyle\int \sqrt{x^2+5}\, dx : x = \sqrt{5}\tan t \Longrightarrow \sqrt{x^2+5} = \cdots = \frac{\sqrt{5}}{\cos t};\ dx = \frac{\sqrt{5}}{\cos^2 t}dt.$

$I = 5\displaystyle\int \frac{dt}{\cos^3 t} = 5\int \frac{\cos t\, dt}{\cos^4 t}$. Doing the change $\sin t = u$, it results $5\displaystyle\int \frac{du}{(1-u^2)^2}$.

Particular case. $\displaystyle\int \frac{dx}{\sqrt{x^2+5}} = \text{argsh}\frac{x}{\sqrt{5}} + C = \ln\left|x + \sqrt{x^2+5}\right| + C$ (immediate).

3) The integrand contains the root of a second degree polynomial. It can be reduced to the preceeding case.

a) $a > 0 \Longrightarrow ax^2 + bx + c = \left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 + \left(c - \frac{b^2}{4a}\right)$.

Example. $2x^2 + x + 1 = \left(\sqrt{2}x + \frac{1}{2\sqrt{2}}\right)^2 + 1 - \frac{1}{(2\sqrt{2})^2} = \left(\sqrt{2}x + \frac{1}{2\sqrt{2}}\right)^2 + \frac{7}{8}$.

b) $a < 0 \Longrightarrow ax^2 + bx + c = -\left(-ax^2 - bx - c\right) = \ldots$

Example. $-x^2 - x + 2 = -\left(x^2 + x - 2\right) = -\left[\left(x + \frac{1}{2}\right)^2 - 2 - \frac{1}{4}\right] = \frac{9}{4} - \left(x + \frac{1}{2}\right)^2$.

4) German method. We use it if the integrand is of the type $\dfrac{P_m(x)}{\sqrt{ax^2 + bx + c}}$.

$$\boxed{\frac{P_m(x)}{\sqrt{ax^2 + bx + c}} = \frac{d}{dx}\left[Q_{m-1}(x)\sqrt{ax^2 + bx + c}\right] + \frac{\lambda}{\sqrt{ax^2 + bx + c}}}$$

After identifying λ and the coefficients of $Q_{m-1}(x)$, we integrate, obtaining

$$I = \left[Q_{m-1}(x)\sqrt{ax^2 + bx + c}\right] + \int \frac{\lambda}{\sqrt{ax^2 + bx + c}}\, dx,$$

which we solve using paragraphs **2)** and **3)**.

Example. $I = \displaystyle\int \frac{x}{\sqrt{3x - x^2 - 2}}\, dx$. We do $\dfrac{x}{\sqrt{}} = \dfrac{d}{dx}\left(k\sqrt{}\right) + \dfrac{\lambda}{\sqrt{}}$. Then

$$\frac{x}{\sqrt{}} = k\,\frac{3 - 2x}{2\sqrt{}} + \frac{\lambda}{\sqrt{}} \overset{\cdot 2\sqrt{}}{\Longrightarrow} 2x = 3k - 2kx + 2\lambda \Longrightarrow k = -1,\ \lambda = \frac{3}{2}$$

and the integral becomes

$$I = \int \frac{d}{dx}\left(k\sqrt{}\right)\, dx + \int \frac{\lambda}{\sqrt{}}\, dx = -\sqrt{3x - x^2 - 2} + \frac{3}{2}\int \frac{dx}{\sqrt{3x - x^2 - 2}}$$

To solve the second term, we do $3x - x^2 - 2 = \cdots = \dfrac{1}{4} - \left(x - \dfrac{3}{2}\right)^2$. Thus

$$I_1 = \int \frac{dx}{\sqrt{}} = \arcsin(2x - 3) \Longrightarrow I = -\sqrt{3x - x^2 - 2} + \frac{3}{2}\arcsin(2x - 3) + C$$

5) The integrand is of the type $\dfrac{1}{(x - \alpha)^p\sqrt{ax^2 + bx + c}}$. Change $\boxed{x - \alpha = \dfrac{1}{t}}$. Then:

$$x = \frac{1}{t} + \alpha,\ dx = -\frac{dt}{t^2} \Longrightarrow \cdots\ I = -\int \frac{t^{p-1}}{\sqrt{Q(t)}} dt,$$

being $Q(t)$ a second degree polynomial (if $\alpha = 0$, then $Q(t) = a + bt + ct^2$).

In the case $p = 1$, we apply **3)**. If $p > 1$, the German method is used.

Table of inmediate integrals

1) $\int u^m\,du = \frac{u^{m+1}}{m+1} + C,\ m \neq -1$

2) $\int \frac{1}{u}\,du = \ln|u| + C$

3) $\int \frac{1}{2\sqrt{u}}\,du = \sqrt{u} + C$

4) $\int a^u\,du = \frac{a^u}{\ln a} + C, a > 0, a \neq 1$

5) $\int \sin u\,du = -\cos u + C$

6) $\int \cos u\,du = \sin u + C$

7) $\int \frac{1}{\cos^2 u}\,du = \int (1+\tan^2 u)du = \tan u + C$

8) $\int \frac{1}{\sin^2 u}du = \int (1+\operatorname{cotan}^2 u)du = -\operatorname{cotan} u + C$

9) $\int \operatorname{cotan} u\,du = \ln|\sin u| + C$

10) $\int \tan u\,du = -\ln|\cos u| + C$

11) $\int \frac{1}{\sqrt{\alpha^2 - u^2}}\,du = \arcsin\frac{u}{\alpha} + C$

12) $\int \frac{1}{\alpha^2 + u^2}\,du = \frac{1}{\alpha}\arctan\frac{u}{\alpha} + C$

13) $\int \sinh u\,du = \cosh u + C$

14) $\int \cosh u\,du = \sinh u + C$

15) $\int \frac{1}{\cosh^2 u}\,du = \tanh u + C$

16) $\int \frac{1}{\sinh^2 u}\,du = -\coth u + C$

17) $\int \frac{1}{\sqrt{u^2 + \alpha^2}}\,du = \operatorname{argsh}\frac{u}{\alpha} + C$

17′) $\int \frac{1}{\sqrt{u^2 + \alpha^2}}\,du = \ln\left(u + \sqrt{u^2 + \alpha^2}\right) + C$

18) $\int \frac{1}{\sqrt{u^2 - \alpha^2}}\,du = \operatorname{argch}\frac{u}{\alpha} + C$

18′) $\int \frac{1}{\sqrt{u^2 - \alpha^2}}\,du = \ln\left|u + \sqrt{u^2 - \alpha^2}\right| + C$

19) $\int \frac{1}{\alpha^2 - u^2}\,du = \frac{1}{\alpha}\operatorname{argth}\frac{u}{\alpha} + C$

19′) $\int \frac{1}{\alpha^2 - u^2}\,du = \frac{1}{2\alpha}\ln\left|\frac{\alpha + u}{\alpha - u}\right| + C$

REMARKS:

1: The primitive of **17'** is the logarithm form of the primitive of **17**. Both have the same field of existence $(-\infty, \infty)$.

2: The primitive of **18** exists only for $u > \alpha$. The one of **18'** exists $\forall u \neq \pm\alpha$.

3: The primitive of **19** exists only for $-\alpha < u < \alpha$. The one of **19'** exists $\forall u \neq \pm\alpha$.

4: In the cases **11, 12, 17-19'**, we denote by α the positive square root of α^2. We write α^2 to indicate that it is a positive number.