2. Quadrics.

Throughout this chapter we will work on the Euclidean affine space E_3 with respect to a rectangular affine coordinate system $\{O; \bar{e}_1, \bar{e}_2, \bar{e}_3\}$. We will denote by (x, y, z) the affine coordinates with respect to this reference and by (x, y, z, t) the homogeneous coordinates.

1 Definition and equations.

Definition 1.1 *A* **quadric** *is a surface in E_3 determined, in affine coordinates, by a quadratic equation.*

In this way the **general equation of a quadric** will be:

$$\boxed{a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0}$$

with $(a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, a_{23}) \neq (0, 0, 0, 0, 0, 0)$ (to guarantee that the degree of the equation is 2.)

Other equivalent expressions of the equation of a quadric are:

1. In terms of the **matrix A associated to the quadric** (every symmetric matrix 4×4 determines a quadric):

$$\begin{pmatrix} x & y & z & 1 \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0, \quad \text{with } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}.$$

2. In terms of the **matrix T of quadratic terms**:

$$\begin{pmatrix} x & y & z \end{pmatrix} T \begin{pmatrix} x \\ y \\ z \end{pmatrix} + 2 \begin{pmatrix} a_{14} & a_{24} & a_{34} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + a_{44} = 0,$$

with

$$T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \neq \Omega.$$

3. In homogeneous coordinates:

$$\begin{pmatrix} x & y & z & t \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = 0, \quad \text{with } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}.$$

From the last equation we deduce that, in homogeneous coordinates, the points of the quadric are the self-conjugate vectors of the quadratic form determined by the associated matrix A.

Definition 1.2 *We will say that a quadric is* **degenerate** *when its associated matrix has a null determinant.*

2 Intersection of a line and a quadric.

We consider a quadric given by a symmetric matrix A. Let $P = (p)$ and $Q = (q)$ be any two points. Let us calculate the intersection of the line joining them and the quadric, in homogeneous coordinates:

$$\begin{aligned} \text{line } PQ &\equiv (x) = \alpha(p) + \beta(q). \\ \text{quadric} &\equiv (x)^t A(x) = 0. \end{aligned}$$

Substituting the first equation in the second one:

$$(\alpha(p) + \beta(q))^t A(\alpha(p) + \beta(q)) = 0 \iff \alpha^2 (p)^t A(p) + 2\alpha\beta(p)^t A(q) + \beta^2 (q)^t A(q) = 0.$$

- If $(p)^t A(p) = (p)^t A(q) = (q)^t A(q) = 0$ the equation holds for any pair (α, β). **The line is contained in the quadric**.

- In other case, we obtain a quadratic equation whose discriminant is

$$\frac{1}{4}\Delta = [(p)^t A(q)]^2 - [(p)^t A(p)][(q)^t A(q)].$$

We have three possibilities:

1. $\Delta > 0$**: Secant line.** There are two different real solutions, so the line intersects the quadric at two different points.
2. $\Delta = 0$**: Tangent line.** There is a double solution, so the line intersects the quadric at a double point.
3. $\Delta < 0$**: Exterior line.** There are no real solutions. The line does not intersect the quadric.

We can apply this to the following situations:

1. **Tangent plane to the quadric at a point P on the quadric.** The tangent plane to the quadric at a point P is formed by all the tangent lines at that point. Since P is in the quadric we have $(p)^t A(p) = 0$. Therefore the tangent plane has the following equation:

$$\boxed{(p)A(x)^t = 0}$$

2. **Cone of lines tangent to the quadric from an exterior point P.** If P is a point exterior to the quadric, the tangent lines to it will be obtained by the equation:

$$\boxed{[(p)^t A(x)]^2 - [(p)^t A(p)][(x)^t A(x)] = 0}$$

In general, this equation corresponds to the cone of center P formed by all tangent lines to the quadric.

Again **polarity** will give us another method to calculate these lines.

3 Polarity.

We work with a quadric whose associated matrix is A.

Definition 3.1 *Given any point P with homogeneous coordinates (p^1, p^2, p^3, t_p) and a quadric determined by a matrix A, the* **polar plane of P with respect to the quadric** *is the plane with equation*

$$\begin{pmatrix} p^1 & p^2 & p^3 & t_p \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = 0.$$

P is said to be the **pole** *of the plane.*

Remark 3.2 *Analogously to what happened in the case of conics, the concepts of pole and polar plane are dual to each other. Suppose that the quadric defined by A is non-degenerate. Given a pole P and its polar plane π_P* **the family of planes passing through** *P corresponds to* **the polar planes of the points of** *π_p. Indeed, let (p) be the coordinates of P. If B, C and D are three points of π_P, with coordinates $(b), (c)$ and (d) respectively, it follows that:*

$$\begin{array}{lcl} (p)^t A(b) = 0 & \Rightarrow & P \in \text{ polar plane of } B \\ (p)^t A(c) = 0 & \Rightarrow & P \in \text{ polar plane of } C \\ (p)^t A(d) = 0 & \Rightarrow & P \in \text{ polar plane of } D \end{array}$$

Therefore the pencil of planes that passes through P will be:

$$\begin{array}{ll} \textit{planes through } P & \Longleftrightarrow \alpha(b)^t A(x) + \beta(c)^t A(x) + \gamma(d)^t A(x) = 0 \Longleftrightarrow \\ & \Longleftrightarrow (\alpha(b) + \beta(c) + \gamma(d))^t A(x) = 0 \Longleftrightarrow \\ & \Longleftrightarrow \textit{polar planes of points } \alpha(b) + \beta(c) + \gamma(d) \Longleftrightarrow \\ & \Longleftrightarrow \textit{polar planes of the points of } \pi_P. \end{array}$$

Let us see the geometric interpretation of the polar plane. Let C be the (non-degenerate) quadric defined by A, P the pole and π_P the corresponding polar plane:

1. *If P is not on the quadric and the polar plane intersects the quadric, then the points of intersection of the polar plane and the quadric are the points of tangency of the tangent lines through P to the quadric.*

   Proof: Let $X \in C \cap \pi_P$. Then:

$$\begin{array}{rcl} X \in C & \Longleftrightarrow & (x)^t A(x) = 0. \\ X \in \pi_p & \Longleftrightarrow & (p)^t A(x) = 0. \\ \text{line joining } X \text{ and } P & \Longleftrightarrow & \alpha(p) + \beta(x) = 0. \end{array}$$

   Let us see that the line joining P and X is tangent to C. We intersect this line with the quadric and obtain:

$$\alpha^2 (p)^t A(p) + 2\alpha\beta (p)^t A(x) + \beta^2 (x)^t A(x) = 0 \quad \Rightarrow \quad \alpha^2 (p)^t A(p) = 0$$

   that is to say, there is only one solution and therefore the line PX is tangent to the quadric line in X.

2. *If P is on the quadric, then the polar plane is the plane* **tangent to the quadric at the point P.**

   Note: This is a particular case of the previous situation.

As we will see later, there are quadrics formed by two families of real lines, that is, quadrics such that two real lines pass through each of their points. The tangent plane can be used used to calculate these lines. It is enough to take into account the following result:

Theorem 3.3 *The tangent plane to a quadric (if it is not contained in it) cuts it into two lines (real or imaginary), or a double line.*

Proof: Since the quadric is defined by an equation of degree 2, the intersection of this surface with a plane not contained in it is also determined by an equation of degree 2. Therefore it will be a conic.

Let a quadric be determined by the matrix A and $P = (p)$ be a point of the quadric. The equation of the tangent plane in P is:

$$(p)A(x)^t = 0$$

Let $Q = (q)$ be any point at the intersection of the quadric and the tangent plane. Let's see that the line that joins P and Q is contained in the quadric. We have:

$$\begin{array}{rcl} P \in \text{quadric} & \Rightarrow & (p)^t A(p) = 0 \\ Q \in \text{quadric} & \Rightarrow & (q)^t A(q) = 0 \\ Q \in \text{tangent plane} & \Rightarrow & (p)^t A(q) = 0 \end{array}$$

In other words, the points (p) and (q) satisfy exactly the conditions we found in the previous section which ensure that the line joining them is contained in the quadric. ■

4 Change of coordinate system.

Let $R_1 = \{O; \bar{e}_1, \bar{e}_2, \bar{e}_3\}$ and $R_2 = \{Q; \bar{e}'_1, \bar{e}'_2, \bar{e}'_3\}$. We denote by (x, y, z) and (x', y', z') respectively the affine coordinates in each of the coordinate systems. Let us suppose that

- The point Q has coordinates (q^1, q^2, q^3) with respect to the first coordinate system.

- $(\bar{e}') = (\bar{e})C$, where $C = M_{BB'}$.

Then we know that the change-of-coordinates formula is

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \left(\begin{array}{ccc|c} & & & q^1 \\ & C & & q^2 \\ & & & q^3 \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \iff \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = B \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix}.$$

Arguing exactly as in the case of conics we obtain:

Theorem 4.1 *Two matrices A, A' of the same quadric with respect to two different coordinate systems R_1, R_2, are congruent*

$$A' = B^t A B$$

where B is the change-of-coordinates matrix from R_2 to R_1, in homogeneous coordinates.

Theorem 4.2 *The matrices of quadratic terms T, T' of a quadric with respect to two different coordinate systems R_1, R_2 are congruent*

$$T' = C^t T C$$

where C is the change-of-coordinates matrix from the base of R_2 to that of R_1.

5 Classification of quadrics and reduced equation.

Given a quadric defined by a symmetric matrix A, we find its reduced equation. This consists of performing a reference change such that the equation of the quadric with respect to that new reference is as simple as possible. Once more we will need to apply

1. **A rotation.** It allows us to place the axis or axes of the quadric parallel to the coordinate axes of the new reference. The matrix of quadratic terms in the new reference will be diagonal.

2. **A translation.** This allows us to place the center(s) (if any) of the quadric at the origin of coordinates (otherwise we will take a vertex to the origin of coordinates).

As in the case of conics, we make the following **important remark**:

We will assume that at least one term of the diagonal of matrix T of quadratic terms is nonnegative.

If this property is not fulfilled, it is enough to work with the matrix $-A$ instead of with A. In this way we ensure that T always has at least one positive eigenvalue.

5.1 Step I: Reduction of quadratic terms (the rotation).

Since the matrix T of quadratic terms is symmetric and not zero, it has three real eigenvalues λ_1, λ_2, and λ_3, with $\lambda_1 \neq 0$. We will assume that **the eigenvalues are ordered according to the positive-negative-null criterion**, and that the number of positive eigenvalues is always greater than the number of negative ones (this can always be achieved by changing the sign of A if necessary). Furthermore, we know that there exists an orthonormal basis of eigenvectors $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ such that:

$$T' = C^t T C \text{ with } (\bar{u}) = (\bar{e})C \text{ and } T' = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

The change-of-coordinates equation is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

so that in the new basis the equation of the quadric is:

$$\begin{pmatrix} x' & y' & z' \end{pmatrix} C^t T C \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + 2 \begin{pmatrix} a_{14} & a_{24} & a_{34} \end{pmatrix} C \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + a_{44} = 0$$

Operating, we obtain:

$$\boxed{\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 + 2b_{14}x' + 2b_{24}y' + 2b_{34}z' + b_{44} = 0}$$

5.2 Step II: Reduction of linear terms (the translation[3]).

Now from the previous equation we complete the expressions of x', y' and z' to the square of a binomial (if possible), adding and subtracting the appropriate terms. Specifically:

[3] If $\lambda_2 = \lambda_3 = 0$ and $b_{24}^2 + b_{34}^2 \neq 0$ we will still have to do a new rotation.

- For x':

$$\lambda_1 x'^2 + 2b_{14}x' = \lambda_1(x'^2 + 2\frac{b_{14}}{\lambda_1}x' + \frac{b_{14}^2}{\lambda_1^2}) - \frac{b_{14}^2}{\lambda_1} = \lambda_1(x' + \frac{b_{14}}{\lambda_1})^2 - \frac{b_{14}^2}{\lambda_1}$$

- For y', if $\lambda_2 \neq 0$:

$$\lambda_2 y'^2 + 2b_{24}y' = \lambda_2(y'^2 + 2\frac{b_{24}}{\lambda_2}y' + \frac{b_{24}^2}{\lambda_2^2}) - \frac{b_{24}^2}{\lambda_2} = \lambda_2(y' + \frac{b_{24}}{\lambda_2})^2 - \frac{b_{24}^2}{\lambda_2}$$

- For z':

- If $\lambda_3 \neq 0$:

$$\lambda_3 z'^2 + 2b_{34}z' = \lambda_3(z'^2 + 2\frac{b_{34}}{\lambda_3}z' + \frac{b_{34}^2}{\lambda_3^2}) - \frac{b_{34}^2}{\lambda_3} = \lambda_3(z' + \frac{b_{34}}{\lambda_3})^2 - \frac{b_{34}^2}{\lambda_3}$$

- If $\lambda_3 = 0$, $\lambda_2 \neq 0$ and $b_{34} \neq 0$:

$$2b_{34}z' + b_{44} - \frac{b_{24}^2}{\lambda_2} - \frac{b_{14}^2}{\lambda_1} = 2b_{34}(z' + \frac{1}{2b_{34}}(b_{44} - \frac{b_{24}^2}{\lambda_2} - \frac{b_{14}^2}{\lambda_1}))$$

- If $\lambda_2 = \lambda_3 = 0$ and $b_{24}^2 + b_{34}^2 \neq 0$. In this case, in addition to the translation, there is still a rotation to be made:

$$2b_{24}y' + 2b_{34}z' + b_{44} - \frac{b_{14}^2}{\lambda_1} = 2c_{24}\frac{b_{24}y' + b_{34}z' + \frac{1}{2}(b_{44} - \frac{b_{14}^2}{\lambda_1})}{c_{24}}$$

with $c_{24} = \sqrt{b_{24}^2 + b_{34}^2}$.

We make the corresponding change of coordinates in each case and obtain the following reduced forms:

$$\boxed{\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 + 2b_{14}x' + 2b_{24}y' + 2b_{34}z' + b_{44} = 0}$$

Eigenvalues and coefficients	Change of coordinates and reduced equation.
$\lambda_1, \lambda_2, \lambda_3 \neq 0$	$x'' = x' + \frac{b_{14}}{\lambda_1}$ $y'' = y' + \frac{b_{24}}{\lambda_2}$ $z'' = z' + \frac{b_{34}}{\lambda_3}$ $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 + \lambda_3 z''^2 + c_{44} = 0}$
$\lambda_1, \lambda_2 \neq 0$ $\lambda_3 = 0$ $b_{34} \neq 0$	$x'' = x' + \frac{b_{14}}{\lambda_1}$ $y'' = y' + \frac{b_{24}}{\lambda_2}$ $z'' = z' + \frac{1}{2b_{34}}(b_{44} - \frac{b_{24}^2}{\lambda_2} - \frac{b_{14}^2}{\lambda_1})$ $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 + 2c_{34}z'' = 0}$
$\lambda_1, \lambda_2 \neq 0$ $\lambda_3 = 0$ $b_{34} = 0$	$x'' = x' + \frac{b_{14}}{\lambda_1}$ $y'' = y' + \frac{b_{24}}{\lambda_2}$ $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 + c_{44} = 0}$
$\lambda_1 \neq 0$ $\lambda_2 = \lambda_3 = 0$ $b_{24}^2 + b_{34}^2 \neq 0$	$x'' = x' + \frac{b_{14}}{\lambda_1}$ $y'' = \frac{b_{24}y' + b_{34}z' + \frac{1}{2}(b_{44} - \frac{b_{14}^2}{\lambda_1})}{c_{24}}$ $z'' = \frac{b_{34}y' - b_{24}z'}{c_{24}}$ $\boxed{\lambda_1 x''^2 + 2c_{24}y'' = 0}$, $c_{24} = \sqrt{b_{24}^2 + b_{34}^2}$
$\lambda_1 \neq 0$ $\lambda_2 = \lambda_3 = 0$ $b_{24}^2 + b_{34}^2 = 0$	$x'' = x' + \frac{b_{14}}{\lambda_1}$ $\boxed{\lambda_1 x''^2 + c_{44} = 0}$

In other words, we are left with a reduced equation of the form:

$$\lambda_1 x''^2 + \lambda_2 y''^2 + \lambda_3 z''^2 + 2c_{24}y'' + 2c_{34}z'' + c_{44} = 0$$

with the following possibilities for the values of λ_2, λ_3, c_{24}, c_{34}, and c_{44}:

1. If $\lambda_2 > 0$ and $\lambda_3 > 0$, then $c_{24} = c_{34} = 0$ and if:
   (a) $c_{44} > 0$, then the reduced equation is:
   $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 + \lambda_3 z''^2 + |c_{44}| = 0}$ Imaginary ellipsoid.
   (b) $c_{44} = 0$, then the reduced equation is:
   $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 + \lambda_3 z''^2 = 0}$ Imaginary cone.
   (c) $c_{44} < 0$, then the reduced equation is:
   $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 + \lambda_3 z''^2 - |c_{44}| = 0}$ Real ellipsoid.
2. If $\lambda_2 > 0$, $\lambda_3 < 0$, then $c_{24} = c_{34} = 0$ and if:
   (a) $c_{44} > 0$, then the reduced equation is:
   $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 - |\lambda_3| z''^2 + c_{44} = 0}$ Hyperboloid of two sheets.
   (b) $c_{44} = 0$, then the reduced equation is:
   $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 - |\lambda_3| z''^2 = 0}$ Real cone.
   (c) $c_{44} < 0$, then the reduced equation is:
   $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 - |\lambda_3| z''^2 - |c_{44}| = 0}$ Hyperboloid of one sheet.
3. If $\lambda_2 > 0$ and $\lambda_3 = 0$, then $c_{24} = 0$ and if:
   (a) $c_{34} \neq 0$, then $c_{44} = 0$ and the reduced equation is:
   $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 + 2c_{34} z'' = 0}$ Elliptic paraboloid.
   (b) $c_{34} = 0$:
      i. if $c_{44} > 0$, then the reduced equation is:
      $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 + c_{44} = 0}$ Imaginary elliptical cylinder.
      ii. if $c_{44} = 0$, then the reduced equation is:
      $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 = 0}$ Imaginary intersecting planes.
      iii. if $c_{44} < 0$, then the reduced equation is:
      $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 - |c_{44}| = 0}$ Real elliptical cylinder.
4. If $\lambda_2 < 0$ and $\lambda_3 = 0$, then $c_{24} = 0$ and if:
   (a) $c_{34} \neq 0$, then $c_{44} = 0$ and the reduced equation is:
   $\boxed{\lambda_1 x''^2 - |\lambda_2| y''^2 + 2c_{34} z'' = 0}$ Hyperbolic paraboloid.
   (b) $c_{34} = 0$, then $c_{34} = 0$ and,:
      i. if $c_{44} \neq 0$, then the reduced equation is:
      $\boxed{\lambda_1 x''^2 - |\lambda_2| y''^2 + c_{44} = 0}$ Hyperbolic cylinder.
      ii. if $c_{44} = 0$, then the reduced equation is:
      $\boxed{\lambda_1 x''^2 - |\lambda_2| y''^2 = 0}$ Real intersecting planes.
5. If $\lambda_2 = \lambda_3 = 0$, then $c_{34} = 0$ and if:
   (a) $b_{24}^2 + b_{34}^2 > 0$, then $c_{44} = 0$ and the reduced equation is:
   $\boxed{\lambda_1 x''^2 + 2c_{24} y'' = 0}$ Parabolic cylinder.
   (b) $b_{24}^2 + b_{34}^2 = 0$, then $c_{24} = c_{34} = 0$ and:
      i. if $c_{44} > 0$, then the reduced equation is:
      $\boxed{\lambda_1 x''^2 + c_{44} = 0}$ Imaginary parallel planes.

ii. if $c_{44} = 0$, then the reduced equation is:

$$\boxed{\lambda_1 x''^2 = 0} \qquad \text{Double plane.}$$

iii. if $c_{44} < 0$, then the reduced equation is:

$$\boxed{\lambda_1 x''^2 - |c_{44}| = 0} \qquad \text{Real parallel planes.}$$

5.3 Classification based on the signatures of T and A.

Since the change of coordinates is equivalent to applying a transformation of the form BAB^t to the matrix A, to classify the quadric we can diagonalize (as far as possible) the matrix A by congruence, but with the following restriction:

The last row can neither be added to the others, nor multiplied by a scalar, nor changed position.

If the matrix A is diagonalizable, we can completely classify the quadric based on the signature of A. Except for a change of sign, we will be at one of the following cases.

Signature A	A **diagonalizes.**
$+,+,+;+$	Imaginary ellipsoid
$+,+,+;-$	Real ellipsoid
$+,+,-;-$	Hyperboloid of 1 sheet
$+,+,-;+$	Hyperboloid of 2 sheets
$+,+,-;0$	Real cone
$+,+,+;0$	Imaginary cone
$+,+,0;+$	Imaginary elliptical cylinder
$+,+,0;-$	Real elliptical cylinder
$+,-,0;+$	Hyperbolic cylinder
$+,-,0;0$	Real intersecting planes
$+,+,0;0$	Imaginary intersecting planes
$+,0,0;-$	Real parallel planes
$+,0,0;+$	Imaginary parallel planes
$+,0,0;0$	Double plane

If A does not diagonalize, the quadric is parabolic. It can be classified in terms of the signature of T.

Signature T	A **DOES NOT diagonalize.**
$+,+,0$	Elliptical Paraboloid
$+,-,0$	Hyperbolic paraboloid
$+,0,0$	Parabolic cylinder

It must be taken into account that the diagonalization of A by congruence does not have to coincide with the reduced form of the quadric. That is, it can be used to classify the quadric but not to calculate its reduced equation.

We can also directly calculate the signature of T, finding its eigenvalues or by diagonalization and use the rank and determinant of A to specify the classification. In this way we do not distinguish between the real or imaginary character of the elliptical cylinder and the intersecting planes.

	rank(A) = 4	
Signature T	$\|A\| > 0$	$\|A\| < 0$
$+,+,+$	Imaginary ellipsoid	Real ellipsoid
$+,+,-$	Hyperboloid of 1 sheets	Hyperboloid of 2 sheets
$+,+,0$	Elliptical Paraboloid	
$+,-,0$	Hyperbolic paraboloid	
Signature T	**rank(A) = 3**	
$+,+,+$	Imaginary cone	
$+,+,-$	Real cone	
$+,+,0$	Real or imaginary elliptical cylinder	
$+,-,0$	Hyperbolic cylinder	
$+,0,0$	Parabolic cylinder	
Signature T	**rank(A) = 2**	
$+,+,0$	Imaginary intersecting planes	
$+,-,0$	Real intersecting planes	
$+,0,0$	Real or imaginary parallel planes	
Signature T	**rank(A) = 1**	
$+,0,0$	Double plane	

In addition **when the quadric is non-degenerate** we can calculate the reduced equation, from the eigenvalues $\lambda_1 > 0, \lambda_2$ and λ_3 of T and $|A|$:

1. If $|T| \neq 0$, then:

$$\lambda_1 x''^2 + \lambda_2 y''^2 + \lambda_3 z''^2 + c = 0, \quad \text{with} \quad c = \frac{|A|}{|T|}.$$

2. If $|T| = 0$, then:

$$\lambda_1 x''^2 + \lambda_2 y''^2 - 2cz'' = 0, \quad \text{with} \quad c = \sqrt{\frac{|A|}{-\lambda_1\lambda_2}}.$$

We can even give the reference in which these reduced forms are obtained:

1. In the case of $|T| \neq 0$ (of elliptic or hyperbolic type), the new coordinate system contains an orthonormal basis of eigenvectors of T and the new origin is the center of the quadric. One just has to be careful to order the eigenvectors in a way that is consistent with how the eigenvalues are ordered.
2. In the case of $|T| = 0$ (of parabolic type), the new coordinate system contains an orthonormal basis of eigenvectors of T and the new origin is the vertex. Now, in addition to correctly ordering the eigenvectors, we must check whether the sign of the eigenvector associated with the null eigenvalue has been chosen correctly.

5.4 Obtaining the planes that form the quadrics of range 1 or 2.

Once the quadric has been classified, if it is of range 1 or 2, the most efficient way to obtain the planes forming it is the following:

1. If the quadric correspond to parallel planes (real or imaginary) or a double plane, we obtain the plane of centers. If it is a double plane we are done. In other case we intersect the quadric with any line (as simple as possible) and we obtain two points (real or imaginary). The planes that form the quadric are parallel to the plane of centers and pass through said points.
2. If they are intersecting planes (real or imaginary), we obtain the line of centers. Then we intersect the quadric with any line (as simple as possible) which does not intersect the line of centers and we obtain two points (real or imaginary). The planes that form the quadric are those generated by the line of centers and said points.

6 Notable points, lines and planes associated with a quadric.

In what follows we will work on a quadric whose associated matrix with respect to a certain coordinate system is A.

6.1 Singular points.

Definition 6.1 *A* **singular point** *of a surface is a point of non-differentiability of the surface.*

Equivalently, a **singular point** *of a surface is a point with more than one plane of tangency.*

Equivalently, if every line passing through a point P of a surface intersects it with multiplicity > 1 in P, then P is a **singular point.**

Let us see when singular points appear in a quadric. Let $P = (p)$ be a point of it. We take any line passing through P. To do this we choose any point $Q = (q)$ that is not in the quadric and join it with P. Its equation in homogeneous coordinates is:

$$(x) = \alpha(p) + \beta(q).$$

If we intersect it with the quadric, we obtain the equation:

$$2\alpha\beta(p)^t A(q) + \beta^2(q)^t A(q) = 0.$$

The point P is singular if the only solution of this equation is $\beta = 0$ with multiplicity 2, for any point (q), that is, if:

$$(p)^t A(q) = 0 \text{ for any } (q).$$

This is true when:

$$(p)A = \bar{0} \iff A(p)^t = 0.$$

We conclude the following:

Theorem 6.2 *A quadric given by a matrix A has singular points (proper or improper) if and only if $det(A) = 0$. In this case, the quadric is called* **degenerate** *and the singular points (affine) are those that satisfy the equation:*

$$\boxed{A\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}$$

6.2 Center.

Definition 6.3 *A* **center of a quadric** *is an affine point which is a center of symmetry of the quadric.*

We denote the homogenous coordinates of the center by $(a, b, c, 1)$. Let us see how to calculate it:

- We consider the equation of a line that passes through the center and has a certain direction vector (p, q, r):

$$(x, y, z, 1) = (a, b, c, 1) + \lambda(p, q, r, 0).$$

- We substitute in the equation of the quadric and obtain:

$$(p \quad q \quad r \quad 0)\,A\begin{pmatrix} p \\ q \\ r \\ 0 \end{pmatrix}\lambda^2 + 2\,(a \quad b \quad c \quad 1)\,A\begin{pmatrix} p \\ q \\ r \\ 0 \end{pmatrix}\lambda + (a \quad b \quad c \quad 1)\,A\begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix} = 0$$

- For $(a, b, c, 1)$ to be a center, the solutions of λ must be opposite values for any direction $(p, q, r) \neq (0, 0, 0)$. This means that:

$$\begin{pmatrix} a & b & c & 1 \end{pmatrix} A \begin{pmatrix} p \\ q \\ r \\ 0 \end{pmatrix} = 0 \text{ for any } (p, q, r) \neq (0, 0, 0)$$

- We deduce that the **equation of the center** is:

$$\boxed{A \begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ h \end{pmatrix}}$$

6.3 Asymptotic directions.

Definition 6.4 *The* **asymptotic directions** *are those points at infinity contained in the quadric.*

From the definition it is clear that the asymptotic directions $(p, q, r, 0)$ are obtained by solving the equation:

$$\boxed{\begin{pmatrix} p & q & r & 0 \end{pmatrix} A \begin{pmatrix} p \\ q \\ r \\ 0 \end{pmatrix} = 0, \qquad (p, q, r) \neq (0, 0, 0)}$$

6.4 Diametral planes and diameters.

Definition 6.5 *A* **diametral plane** *of a quadric is the polar (affine) plane of a point at infinity. The point at infinity is called* **conjugate direction with the diametral plane**.

A **diameter** *is a straight line which is the intersection of two Diametral planes.*

Remark 6.6 *Any diameter passes through the center (or centers) of the quadric.*

Proof: Suppose A is the matrix of the quadric and $(a, b, c, 1)$ is a center. A diametral plane has the equation:

$$\begin{pmatrix} u^1 & u^2 & u^3 & 0 \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0$$

where (u^1, u^2, u^3) is the conjugate direction. On the other hand we saw that if $(a, b, c, 1)$ is the center it satisfies:

$$A \begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ h \end{pmatrix}.$$

We deduce that

$$\begin{pmatrix} u^1 & u^2 & u^3 & 0 \end{pmatrix} A \begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix} = 0$$

and therefore the diametral plane contains the center. Since the diameters are intersections of diametral planes, they also contain the center. ■

6.5 Principal planes, axes and vertices.

Definition 6.7 *The* **principal planes** *are the diametral planes orthogonal to their conjugate direction.*

They **axes** *are the intersections of the principal planes.*

The **vertices** *are the intersection of the axes with the quadric.*

Remark 6.8 *The conjugate directions of the principal planes are the eigenvectors of T associated with nonzero eigenvalues.*

Proof: Let $(u^1, u^2, u^3, 0)$ be a point at infinity and

$$\begin{pmatrix} u^1 & u^2 & u^3 & 0 \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0$$

the equation of the corresponding diametral plane. Operating, we obtain that the normal vector of the plane is:

$$T \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}.$$

Since this line must be orthogonal to the conjugate direction, this normal vector must be parallel to (u^1, u^2, u^3) and therefore:

$$T \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} = \lambda \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}.$$

We deduce that (u^1, u^2, u^3) is an eigenvector of T, associated to the eigenvalue λ. Finally we take into account that if $\lambda = 0$, then the previous plane would be the plane at infinity, and therefore it is not an axis. ■

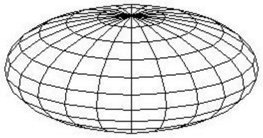
7 Description of the real quadrics of rank 3 and 4.

7.1 Real quadrics of rank 4.

7.1.1 Real ellipsoid.

The reduced equation of an ellipsoid is:

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \qquad a,b,c\neq 0}$$



Its notable points, lines and planes are:

1. **Center:** $(0,0,0)$.
2. **Asymptotic directions:** Does not have.
3. **Diametral planes and diameters:** Any plane and any line passing through the center.
4. **Principal planes, axes and vertices:**

   (a) $a\neq b\neq c$:

Principal Planes	Axes	Vertices
$x=0;$	$y=0;\quad z=0;$	$(-a,0,0),(a,0,0)$
$y=0;$	$x=0;\quad z=0;$	$(0,b,0),(0,-b,0)$
$z=0;$	$x=0;\quad y=0;$	$(0,0,c),(0,0,-c)$

   (b) $a=b$ and $b\neq c$ (**Ellipsoid of revolution**):

Principal Planes	Axes	Vertices
$\alpha x+\beta y=0;$	$\alpha x+\beta y=0;\quad z=0;$	$(p,q,0)$ $p^2+q^2=a^2$
$z=0;$	$x=0;\quad y=0;$ (OZ $\equiv$ Axis of revolution)	$(0,0,c),(0,0,-c)$

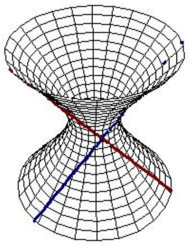
   (c) $a=b=c$ (**Sphere**):

   All planes and lines passing through the center are principal planes and axes. Therefore all points on the sphere are vertices. Any line passing through the center is an axis of revolution.

7.1.2 Hyperboloid of one sheet.

The reduced equation of a one-sheet hyperboloid is:

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \qquad a,b,c\neq 0}$$



Its notable points, lines and planes are:

1. **Center:** $(0,0,0)$.
2. **Asymptotic directions:** The cone formed by all the vectors (x,y,z) that satisfy the equation:
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$
3. **Diametral planes and diameters:** Any plane and any line passing through the center.
4. **Principal planes, axes and vertices:**

   (a) $a\neq b$:

Principal Planes	Axes	Vertices
$x=0;$	$y=0;\quad z=0;$	$(-a,0,0),(a,0,0)$
$y=0;$	$x=0;\quad z=0;$	$(0,b,0),(0,-b,0)$
$z=0;$	$x=0;\quad y=0;$	

   (b) $a=b$ (**Hyperboloid of one sheet of revolution**):

Principal Planes	Axes	Vertices
$\alpha x+\beta y=0;$	$\alpha x+\beta y=0;\quad z=0;$	$(p,q,0)$ $p^2+q^2=a^2$
$z=0;$	$x=0;\quad y=0;$ (OZ $\equiv$ Axis of revolution)	

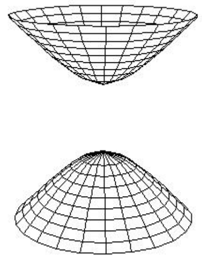
The hyperboloid of one sheet is a ruled surface. The two families of straight lines contained in the quadric are:

$$\left.\begin{array}{rcl} \left(\frac{x}{a}+\frac{z}{c}\right) & = & \alpha\left(1+\frac{y}{b}\right) \\ \alpha\left(\frac{x}{a}-\frac{z}{c}\right) & = & \left(1-\frac{y}{b}\right) \end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{rcl} \left(\frac{x}{a}+\frac{z}{c}\right) & = & \beta\left(1-\frac{y}{b}\right) \\ \beta\left(\frac{x}{a}-\frac{z}{c}\right) & = & \left(1+\frac{y}{b}\right) \end{array}\right. .$$

7.1.3 Hyperboloid with two sheets.

The reduced equation of a hyperboloid with two sheets is:

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0, \qquad a,b,c\neq 0}$$



Its notable points, lines and planes are:

1. **Center:** $(0,0,0)$.
2. **Asymptotic directions:** The cone formed by all the vectors (x,y,z) that satisfy the equation:
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$
3. **Diametral planes and diameters:** Any plane and any line passing through the center.
4. **Principal planes, axes and vertices:**

   (a) $a\neq b$:

Principal Planes	Axes	Vertices
$x=0;$	$y=0;\quad z=0;$	
$y=0;$	$x=0;\quad z=0;$	
$z=0;$	$x=0;\quad y=0;$	$(0,0,-c),(0,0,c)$

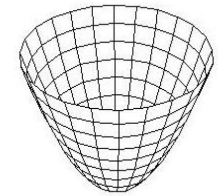
   (b) $a=b$ (**Hyperboloid of two sheets of revolution**):

Principal Planes	Axes	Vertices
$\alpha x+\beta y=0;$	$\alpha x+\beta y=0;\quad z=0;$	
$z=0;$	$x=0;\quad y=0;$ (OZ $\equiv$ Axis of revolution)	$(0,0,-c),(0,0,c)$

7.1.4 Elliptic paraboloid.

The reduced equation of an elliptic paraboloid is:

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0, \qquad a,b\neq 0}$$



Its notable points, lines and planes are:

1. **Center:** Does not have ($(0,0,1,0)$ could be considered an improper center).
2. **Asymptotic directions:** $(0,0,1)$.
3. **Diametral planes and diameters:** Planes and lines parallel to the OZ axis.
4. **Principal planes, axes and vertices:**

   (a) $a\neq b$:

Principal Planes	Axes	Vertices
$x=0;$ $y=0;$	$x=0;\quad y=0;$	$(0,0,0)$

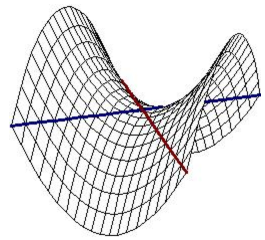
   (b) $a=b$ (**Elliptic paraboloid of revolution**):

Principal Planes	Axes	Vertices
$\alpha x+\beta y=0;$	$x=0;\quad y=0;$ (OZ $\equiv$ Axis of revolution)	$(0,0,0)$

7.1.5 Hyperbolic paraboloid.

The reduced equation of a hyperbolic paraboloid is:

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0, \qquad a, b \neq 0}$$



Its notable points, lines and planes are:

1. **Center:** Does not have ($(0, 0, 1, 0)$ could be considered an improper center).
2. **Asymptotic directions:** (a, b, z) and $(a, -b, z)$, for any $z \in \mathbb{R}$.
3. **Diametric planes and diameters:** Planes and lines parallel to the OZ axis.
4. **Principal planes, axes and vertices:**

Principal Planes	Axes	Vertices
$x = 0$; $y = 0$;	$x = 0; \quad y = 0;$	$(0, 0, 0)$

The hyperbolic paraboloid is a ruled surface. The two families of lines contained in the quadric are:

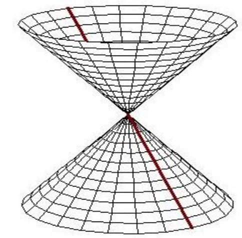
$$\left.\begin{array}{rcl} \alpha\left(\frac{x}{a} + \frac{y}{b}\right) & = & z \\ \left(\frac{x}{a} - \frac{y}{b}\right) & = & \alpha \end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{rcl} \left(\frac{x}{a} + \frac{y}{b}\right) & = & \beta \\ \beta\left(\frac{x}{a} - \frac{y}{b}\right) & = & z \end{array}\right. .$$

7.2 Real quadrics of rank 3.

7.2.1 Real cone.

The reduced equation of a real cone is:

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \qquad a, b, c \neq 0}$$



Its notable points, lines and planes are:

1. **Singular points:** $(0, 0, 0)$.
2. **Center:** $(0, 0, 0)$.
3. **Asymptotic directions:** Vectors (x, y, z) that satisfy the equation of the cone.
4. **Diametral planes and diameters:** Planes and lines passing through the center.
5. **Principal planes, axes and vertices:**

   (a) $a \neq b$:

Principal Planes	Axes	Vertices
$x = 0$;	$y = 0; \quad z = 0;$	$(0, 0, 0)$
$y = 0$;	$x = 0; \quad z = 0;$	
$z = 0$;	$x = 0; \quad y = 0;$	

   (b) $a = b$ (**Real cone of revolution**):

Principal Planes	Axes	Vertices
$\alpha x + \beta y = 0$; $z = 0$;	$\alpha x + \beta y = 0; \quad z = 0;$ $x = 0; \quad y = 0;$ (OZ $\equiv$ Axis of revolution)	$(0, 0, 0)$

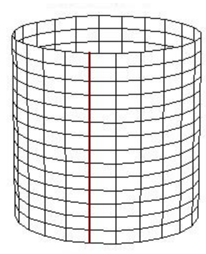
The cone is a ruled surface. The generatrices of the cone are given by the equations:

$$\begin{array}{rcl} \left(\frac{z}{c} + \frac{x}{a}\right) & = & \alpha\frac{y}{b} \\ \alpha\left(\frac{z}{c} - \frac{x}{a}\right) & = & \frac{y}{b} \end{array} .$$

7.2.2 Real elliptical cylinder.

The reduced equation of a real elliptical cylinder is:

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad a,b\neq 0}$$



Its notable points, lines and planes are:

1. **Singular points:** (improper) point $(0,0,1,0)$.
2. **Center:** The OZ axis is a line of centers ($x=0$; $y=0$).
3. **Asymptotic directions:** $(0,0,1)$.
4. **Diametral planes and diameters:** Planes containing the OZ axis and the OZ axis as the only diameter.
5. **Principal planes, axes and vertices:**

   (a) $a\neq b$:

Principal Planes	**Axes**	**Vertices**
$x=0$; $y=0$;	$x=0$; $\quad y=0$;	NO

   (b) $a=b$ (**Real cylinder of revolution**):

Principal Planes	**Axes**	**Vertices**
$\alpha x+\beta y=0$;	$x=0$; $\quad y=0$; (OZ $\equiv$ Axis of revolution)	NO

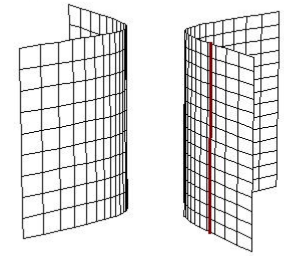
The elliptical cylinder is obviously a ruled surface. The family of lines is given by the equations:

$$\begin{aligned} \left(\frac{x}{a}+1\right) &= \alpha\frac{y}{b} \\ \alpha\left(\frac{x}{a}-1\right) &= \frac{y}{b} \end{aligned}.$$

7.2.3 Hyperbolic cylinder.

The reduced equation of a hyperbolic cylinder is:

$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \qquad a,b\neq 0}$$



Its notable points, lines and planes are:

1. **Singular points:** (improper) point $(0,0,1,0)$.
2. **Center:** The OZ axis is a line of centers ($x=0$; $y=0$).
3. **Asymptotic directions:** $(0,0,1)$, (a,b,z) and $(a,-b,z)$, for any $z\in\mathbb{R}$.
4. **Diametral planes and diameters:** Planes containing the OZ axis and the OZ axis as the only diameter.
5. **Principal planes, axes and vertices:**

Principal Planes	**Axes**	**Vertices**
$x=0$; $y=0$;	$x=0$; $\quad y=0$;	NO

The hyperbolic cylinder is obviously a ruled surface. The family of lines is given by the equations:

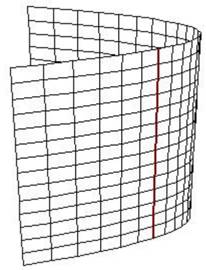
$$\begin{aligned} \left(\frac{x}{a}+\frac{y}{b}\right) &= \alpha \\ \alpha\left(\frac{x}{a}-\frac{y}{b}\right) &= 1 \end{aligned}$$

with $\alpha\neq 0$ and where the sign of α indicates in which branch of the surface the line is.

7.2.4 Parabolic cylinder.

The reduced equation of a parabolic cylinder is:

$$\boxed{x^2 - 2py = 0, \qquad p \neq 0}$$



Its notable points, lines and planes are:

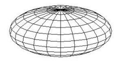
1. **Singular points:** (improper) point $(0, 0, 1, 0)$.
2. **Center:** It has no affine center, but a straight line with improper centers ($x = 0$; $t = 0$).
3. **Asymptotic directions:** $(0, y, z)$, for any $y, z \in \mathbb{R}$. That is, any direction in the plane $x = 0$.
4. **Diametric planes and diameters:** Planes parallel to the plane $x = 0$. It has no diameters.
5. **Principal planes, axes and vertices:**

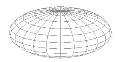
Principal Planes	**Axes**	**Vertices**
$x = 0$	NOT	NO

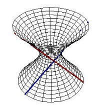
The parabolic cylinder is obviously a ruled surface. The family of lines is given by the equations:

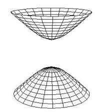
$$\begin{aligned} \alpha x &= 2py \\ x &= \alpha \end{aligned} \quad .$$

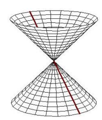
8 Reduced forms of quadrics.

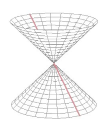
1. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ Ellipsoid 

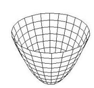
2. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+1=0$ Imaginary ellipsoid 

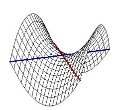
3. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0$ Hyperboloid of one sheet 

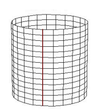
4. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0$ Hyperboloid of two sheets 

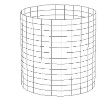
5. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ Cone 

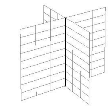
6. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$ Imaginary cone 

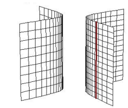
7. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0$ Elliptic paraboloid 

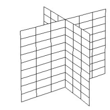
8. $\frac{x^2}{a^2}-\frac{y^2}{b^2}-z=0$ Hyperbolic paraboloid 

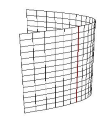
9. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ Elliptical cylinder 

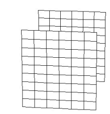
10. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+1=0$ Imaginary elliptical cylinder 

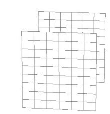
11. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$ Imaginary intersecting planes 

12. $\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$ Hyperbolic cylinder 

13. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ Pair of intersecting planes 

14. $y^2-2px=0$ Parabolic cylinder 

15. $x^2-a^2=0$ Pair of parallel planes 

16. $x^2+a^2=0$ Pair of imaginary parallel planes 

17. $x^2=0$ Double plane