Part IV

Conics and Quadrics.

1. Conics.

Throughout this chapter we will work on the Euclidean affine plane E_2 with respect to a rectangular affine coordinate system $\{O; \bar{e}_1, \bar{e}_2\}$. We will denote by (x, y) the coordinates with respect to this reference and by (x, y, t) the homogeneous coordinates.

1 Definition and equations.

Definition 1.1 *A* **conic section** *(or simply conic) is a plane curve determined, in coordinates, by a quadratic equation.*

In this way the **general equation of a conic will be**:

$$\boxed{a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + 2a_{13}x + 2a_{23}y + a_{33} = 0}$$

with $(a_{11}, a_{22}, a_{12}) \neq (0, 0, 0)$ (to ensure that the equation is quadratic).

Other equivalent expressions of the equation of a conic are:

1. In terms of the **matrix A associated to the conic** (every symmetric matrix determines a conic):

$$\begin{pmatrix} x & y & 1 \end{pmatrix} T \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0, \quad \text{with } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{pmatrix}.$$

2. Based on the **matrix T of quadratic terms**:

$$\begin{pmatrix} x & y \end{pmatrix} T \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} a_{13} & a_{23} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + a_{33} = 0, \quad \text{with } T = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \neq \Omega.$$

3. In homogeneous coordinates:

$$\begin{pmatrix} x & y & t \end{pmatrix} T \begin{pmatrix} x \\ y \\ t \end{pmatrix} = 0, \quad \text{with } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{pmatrix}.$$

From the last equation we deduce that, in homogeneous coordinates, the points of the conic are the self-conjugate vectors of the quadratic form that determines the associated matrix A.

Definition 1.2 *We will say that a conic is* **degenerate** *when it is formed by two lines (same or different, real or imaginary).*

We will see later that **a conic is degenerate when its associated matrix A has zero determinant**.

2 Intersection of a line and a conic.

We consider a conic given by a symmetric matrix A. Let $P = (p)$ and $Q = (q)$ be any two points. Let us express in homogeneous coordinates the intersection of the conic with the line joining these points:

$$\begin{aligned} \text{line } PQ &\equiv (x) = \alpha(p) + \beta(q). \\ \text{conic} &\equiv (x)^t A(x) = 0. \end{aligned}$$

Substituting the first equation into the second we obtain

$$(\alpha(p) + \beta(q))^t A(\alpha(p) + \beta(q)) = 0 \iff \alpha^2(p)^t A(p) + 2\alpha\beta(p)^t A(q) + \beta^2(q)^t A(q) = 0.$$

- If $(p)^t A(p) = (p)^t A(q) = (q)^t A(q) = 0$ the equation holds for any pair (α, β). In this case the line is contained in the conic.

- In other case, we obtain a quadratic equation with discriminant:

$$\frac{1}{4}\Delta = [(p)^t A(q)]^2 - [(p)^t A(p)][(q)^t A(q)].$$

There are three possibilities:

1. $\Delta > 0$**: Secant line.** There are two different real solutions, so the line intersects the conic in two different points.
2. $\Delta = 0$**: Tangent line.** There is a double solution, so the line intersects the conic in a double point.
3. $\Delta < 0$**: Exterior line.** There are no real solutions. The line does not intersect the conic.

We can apply this to two situations:

1. **Tangent line to the conic at a point P on the conic.** If P is on the conic then $(p)^t A(p) = 0$. Therefore the tangent line will have the equation:

$$\boxed{(p)^t A(x) = 0}$$

2. **Tangent lines to the conic from an exterior point P.** If P is a point exterior to the conic, the tangent lines to it will be obtained by the equation:

$$\boxed{[(p)^t A(x)]^2 - [(p)^t A(p)][(x)^t A(x)] = 0}$$

taking into account that this equation will be decomposed as a product of two linear equations.

In the next section we will see how **polarity** will provide another method to obtain tangent lines.

3 Polarity.

We work with a conic whose associated matrix is A.

Definition 3.1 *Given a point P with homogeneous coordinates (p^1, p^2, p^3) and a conic determined by a matrix A, we call* **polar line of P with respect to the conic** *the line of equation:*

$$(\,p^1 \quad p^2 \quad p^3\,)\, A \begin{pmatrix} x \\ y \\ t \end{pmatrix} = 0.$$

We denote this line by r_P. The point P is said to be the **pole** *of the line.*

Remark 3.2 *The concepts of pole and polar line are dual to each other. Suppose that the conic defined by A is non-degenerate. Given a pole P and its polar line r_P* **the family of lines passing through P** *corresponds to* **the polar lines of the points of r_P**. *To check this we simply consider the following. Let (p) be the coordinates of P. If B and C are two points of r_P, with coordinates (b) and (c) respectively, it follows that:*

$$\begin{array}{lcl} (p)^t A(b) = 0 & \Rightarrow & P \in \text{ polar line of } B \\ (p)^t A(c) = 0 & \Rightarrow & P \in \text{ polar line of } C \end{array}$$

Therefore the pencil of straight lines that passes through P will be:

$$\begin{array}{rl} \textit{lines through } P & \iff \alpha(b)^t A(x) + \beta(c)^t A(x) = 0 \iff \\ & \iff (\alpha(b) + \beta(c))^t A(x) = 0 \iff \\ & \iff \textit{polar lines of the points } \alpha(b) + \beta(c) \iff \\ & \iff \textit{polar lines of the points of } r_P \end{array}$$

Let's see the geometric interpretation of the polar line. Let C be the (nondegenerate) conic defined by A, let P a pole and r_P its corresponding polar line:

1. *If P is not on the conic and the polar line intersects the conic, then the points of intersection of the polar line and the conic are the points of tangency of the lines tangent to the conic through P.*

   Proof: Let $X \in C \cap r_p$. Then the following equations hold:

   $$\begin{array}{rcl} X \in C & \iff & (x)^t A(x) = 0 \\ X \in r_p & \iff & (p)^t A(x) = 0 \\ \text{line joining } X \text{ and } P & \iff & \alpha(p) + \beta(x) = 0 \end{array}$$

   Let's see that the line joining P and X is tangent to C. We find the intersection of this line and the conic:

   $$\alpha^2(p)^t A(p) + 2\alpha\beta(p)^t A(x) + \beta^2(x)^t A(x) = 0 \quad \Rightarrow \quad \alpha^2(p)^t A(p) = 0$$

   that is to say, there is only one solution and therefore the line PX is tangent to the conic in X.

   Observation: As a consequence of this, to calculate the **tangents to a conic from an exterior point** P, it is enough to calculate the polar line of P and intersect it with the conic. The requested lines are those that join these points with P.

2. *If P is on the conic, then the polar line is the line* **tangent to the conic at point P.**

   Note: This is a particular case of the previous situation.

3. *If P is not on the conic and the polar line does not intersect the conic, then the polar line r_P is obtained as follows: take a line passing through P and find the points where this line intersects the conic. The corresponding tangents to the conic through these points intersect at a point on the polar line r_p.*

   Proof: It is a consequence of the previous observations.

4 Notable points and lines associated with a conic.

In what follows we will work on a conic whose associated matrix with respect to a certain coordinate system is A.

4.1 Singular points.

Definition 4.1 *A* **singular point** *of a curve is a point of non-differentiability of it.*

Equivalently, a **singular point** *of a curve is a point with two or more tangent lines.*

Equivalently, if every line passing through a point P of a curve intersects the curve with multiplicity > 1 at P, then P is a **singular point.**

Let us see when singular points appear in a conic. Let $P = (p)$ be a point on the conic. We take any line passing through P. To do this we choose any point $Q = (q)$ that is not on the conic and join it with P. Its equation in homogeneous coordinates is:

$$(x) = \alpha(p) + \beta(q)$$

If we intersect with the conic, we obtain the equation:

$$2\alpha\beta(p)^t A(q) + \beta^2(q)^t A(q) = 0$$

The point P is singular if the only solution of this equation is $\beta = 0$ with multiplicity 2, for any point (q), that is, if:

$$(p)^t A(q) = 0 \text{ for any } (q).$$

This is true when:

$$(p)^t A = \bar{0} \iff A(p) = 0$$

We conclude the following:

Theorem 4.2 *A conic given by a matrix A has singular points if and only if $det(A) = 0$. In this case, the conic is called* **degenerate** *and the singular points (affine) are those that satisfy the equation:*

$$\boxed{A\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \bar{0}}$$

4.2 Center.

Definition 4.3 *The* **center of a conic** *is an affine point which is the symmetry center of the conic.*

Let us denote by $(a, b, 1)$ the homogeneous coordinates of the center. Let us see how to find it:

- We consider the equation of a straight line that passes through the center and has a certain direction vector (p, q):

$$(x, y, t) = (a, b, 1) + \lambda(p, q, 0).$$

- We substitute in the conic equation and obtain:

$$\begin{pmatrix} p & q & 0 \end{pmatrix} A \begin{pmatrix} p \\ q \\ 0 \end{pmatrix} \lambda^2 + 2\begin{pmatrix} a & b & 1 \end{pmatrix} A \begin{pmatrix} p \\ q \\ 0 \end{pmatrix} \lambda + \begin{pmatrix} a & b & 1 \end{pmatrix} A \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = 0.$$

- For $(a, b, 1)$ to be the center, the solutions of λ must be opposite values for any direction $(p, q) \neq (0, 0)$. This means that:

$$\begin{pmatrix} a & b & 1 \end{pmatrix} A \begin{pmatrix} p \\ q \\ 0 \end{pmatrix} = 0 \text{ for any } (p, q) \neq (0, 0).$$

- We deduce that the **equation of the center** is:

$$\begin{pmatrix} a & b & 1 \end{pmatrix} A = (0, 0, h)$$

or equivalently:

$$\boxed{\begin{pmatrix} a \\ b \\ 1 \end{pmatrix} A = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}}$$

4.3 Asymptotes and asymptotic directions.

Definition 4.4 *The* **asymptotic directions** *are the points at infinity that belong to the conic.*

The **asymptotes** *are the lines that pass through the center and have an asymptotic direction.*

From the definition it is clear that the asymptotic directions $(p, q, 0)$ are obtained by solving the equation:

$$\boxed{\begin{pmatrix} p & q & 0 \end{pmatrix} A \begin{pmatrix} p \\ q \\ 0 \end{pmatrix} = 0, \qquad (p, q) \neq (0, 0)}$$

If we expand this equation, we get:

$$a_{11}p^2 + 2a_{12}pq + a_{22}q^2 = 0$$

We see that it is a quadratic equation whose discriminant is $-|T|$. Therefore:

- If $|T| > 0$, then there are no asymptotic directions (conic **elliptic** type).
- If $|T| = 0$, there is an asymptotic direction (conic of **parabolic** type).
- If $|T| < 0$, there are two asymptotic directions (conic of **hyperbolic** type).

4.4 Diameters.

Definition 4.5 *A* **diameter** *of a conic is the (affine) polar line of a point at infinity. The point at infinity is called the* **conjugate direction** *to the diameter.*

Remark 4.6 *Any diameter passes through the center (or centers) of the conic.*

Proof: Suppose A is the matrix of the conic and $(a, b, 1)$ is a center. A diameter has the equation:

$$\begin{pmatrix} u^1 & u^2 & 0 \end{pmatrix} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

where (u^1, u^2) is the conjugate direction. On the other hand we saw that if $(a, b, 1)$ is the center it satisfies

$$\begin{pmatrix} a & b & 1 \end{pmatrix} A = \begin{pmatrix} 0 & 0 & h \end{pmatrix} \iff A \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$$

We deduce that:

$$\begin{pmatrix} u^1 & u^2 & 0 \end{pmatrix} A \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = 0$$

and therefore the diameter contains the center. ■

4.5 Axes.

Definition 4.7 *The diameters perpendicular to their conjugate direction are called* **axes**. *Geometrically, they are symmetry axes.*

Remark 4.8 *The conjugate directions of the axes are the eigenvectors of T associated with nonzero eigenvalues.*

Proof: Let $(u^1, u^2, 0)$ be a point at infinity and

$$\begin{pmatrix} u^1 & u^2 & 0 \end{pmatrix} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

the equation of the corresponding diameter. Operating, we obtain that the normal vector of the line is:

$$\begin{pmatrix} u^1 & u^2 \end{pmatrix} T$$

Since this line must be perpendicular to the conjugate direction, this normal vector must be parallel to (u^1, u^2) and therefore:

$$\begin{pmatrix} u^1 & u^2 \end{pmatrix} T = \lambda \begin{pmatrix} u^1 & u^2 \end{pmatrix}$$

We deduce that (u^1, u^2) is an eigenvector of T, associated to the eigenvalue λ. Finally we take into account that if $\lambda = 0$, then the previous line would be the line at infinity, and therefore it is not an axis. ■

4.6 Vertices.

Definition 4.9 *The points of intersection of the axes with the conic are called* **vertices**.

4.7 Foci, directrices and eccentricity.

We will define these three concepts for non-degenerate conics, that is, those whose associated matrix A is nonsingular.

Definition 4.10 *A* **focus** *of a conic is a point F which satisfies the following condition: the quotient between the distances from any point of the conic to F and to its polar line d is constant:*

$$\frac{d(X,F)}{d(X,d)} = e, \quad \textit{for any point } X \textit{ on the conic.}$$

The polar line of a focus is called **directrix**.

The constant e is called **eccentricity**.

5 Description of non-degenerate conics.

5.1 The real ellipse.

The reduced equation of an ellipse is:

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a, b \neq 0}$$

(we will assume $a \geq b$, so that the largest radius of the ellipse is placed on the OX axis).

When $a = b$ it is a **circumference** of radius a. Its notable points and lines are:

1. **Center:** $(0, 0)$.
2. **Asymptotic directions:** Do not exist.
3. **Asymptotes:** Do not exist.
4. **Diameters:** Any line passing through the center.
5. **Axes:** If $a \neq b$ the axes are $x = 0$ and $y = 0$. If $a = b$ any diameter is an axis.
6. **Vertices:** If $a \neq b$ the vertices are $(-a, 0), (a, 0), (0, -b)$ and $(0, b)$. If $a = b$ any point on the conic is a vertex.
7. **Foci:** $(-c, 0)$ and $(c, 0)$ with $\boxed{a^2 = b^2 + c^2}$.
8. **Directrices:** $x = \frac{a^2}{c}$ and $x = -\frac{a^2}{c}$.
9. **Eccentricity:** $e = \frac{c}{a} < 1$ (when $e = 0$ the conic is a circle.)

All these values are obtained by directly using the associated matrix to the canonical equation of the conic:

$$A = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

and applying the definitions seen for each of the previous concepts. In particular let us compute foci, directrices, and eccentricity.

If we consider the point $(c, 0)$ with $a^2 = b^2 + c^2$, its polar line will be:

$$(c, 0, 1)A(x, y, 1)^t = 0 \iff \frac{c}{a^2}x - 1 = 0 \iff x = \frac{a^2}{c}.$$

Now given any point $X = (x, y)$ on the conic, let us find the quotient between the distances of that point from the focus and the directrix. First of all, taking into account that (x, y) satisfies the equation of the conic and that $a^2 = b^2 + c^2$, we have:

$$\begin{aligned} d(F, X) &= \sqrt{(xc)^2 + y^2} = \sqrt{x^2 + c^2 - 2cx + \frac{a^2b^2 - b^2x^2}{a^2}} = \\ &= \sqrt{\frac{c^2x^2}{a^2} - 2xc + a^2} = a - \frac{cx}{a} = \frac{a^2 - cx}{a}. \end{aligned}$$

On the other hand:

$$d(directrix, X) = \frac{a^2}{c} - x = \frac{a^2 - cx}{c}.$$

Therefore:

$$\frac{d(F, X)}{d(d, X)} = \frac{\frac{a^2 - cx}{a}}{\frac{a^2 - cx}{c}} = \frac{c}{a}$$

We deduce that indeed $(c, 0)$ is a focus and the eccentricity is $\frac{c}{a}$. Analogously it can be seen that $(-c, 0)$ is the other focus of the conic.

5.1.1 The ellipse as a locus.

Definition 5.1 *The* **ellipse** *can also be defined as the locus of the points in the plane whose sum of distances to the foci is constant.*

Let us see that this definition is coherent with the equation and foci given previously. Given a point (x, y) of the conic we saw that:

$$d(F_1, X) = \frac{a^2 - cx}{a}; \qquad d(F_2, X) = \frac{a^2 + cx}{a}.$$

Therefore:

$$d(F_1, X) + d(F_2, X) = \frac{a^2 - cx}{a} + \frac{a^2 + cx}{a} = 2a.$$

5.2 The hyperbola.

The reduced equation of a hyperbola is:

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad a, b \neq 0}.$$

Its notable points and lines are:

1. **Center:** $(0, 0)$.
2. **Asymptotic directions:** $(a, b, 0)$ and $(a, -b, 0)$.
3. **Asymptotes:** $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.
4. **Diameters:** Any line passing through the center.
5. **Axes:** $x = 0$ and $y = 0$.
6. **Vertices:** $(-a, 0)$ and $(a, 0)$.
7. **Foci:** $(-c, 0)$ and $(c, 0)$ with $\boxed{c^2 = a^2 + b^2}$.
8. **Directrices:** $x = \frac{a^2}{c}$ and $x = -\frac{a^2}{c}$.
9. **Eccentricity:** $e = \frac{c}{a} > 1$.

All these values are obtained directly by using the associated matrix to the canonical equation of the conic:

$$A = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & -\frac{1}{b^2} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We next compute explicitly the foci, directrices and eccentricity.

If we consider the point $(c, 0)$ with $c^2 = a^2 + b^2$, its polar line will be:

$$(c, 0, 1)A(x, y, 1)^t = 0 \iff \frac{c}{a^2}x - 1 = 0 \iff x = \frac{a^2}{c}$$

Now given a point $X = (x, y)$ of the conic let us find the quotient between the distances from the point to the focus and the directrix. First of all, taking into account that (x, y) satisfies the equation of the conic and that $c^2 = a^2 + b^2$, we have:

$$\begin{aligned} d(F, X) &= \sqrt{(xc)^2 + y^2} = \sqrt{x^2 + c^2 - 2cx + \frac{b^2x^2 - a^2b^2}{a^2}} = \\ &= \sqrt{\frac{c^2x^2}{a^2} - 2xc + a^2} = \left|\frac{cx}{a} - a\right| = \frac{|cx - a^2|}{a}. \end{aligned}$$

On the other hand:

$$d(d, X) = x - \frac{a^2}{c} = \frac{|cx - a^2|}{c}.$$

Therefore:

$$\frac{d(F,X)}{d(d,X)} = \left|\frac{\frac{cx-a^2}{a}}{\frac{cx-a^2}{c}}\right| = \frac{c}{a}$$

We deduce that $(c,0)$ is a focus and the eccentricity is $\frac{c}{a}$. Analogously it can be seen that $(-c,0)$ is the other focus of the conic.

5.2.1 The hyperbola as a locus.

Definition 5.2 *The* **hyperbola** *can also be defined as the locus of the points in the plane the difference of whose distances to the foci in absolute value is constant.*

Let us see that this definition is coherent with the equation and foci given previously. Given a point (x,y) of the conic we saw that:

$$d(F_1,X) = \frac{|cx-a^2|}{a}; \qquad d(F_2,X) = \frac{|cx+a^2|}{a}.$$

Therefore, if $x \geq a$

$$d(F_1,X) - d(F_2,X) = \frac{cx-a^2}{a} - \frac{cx+a^2}{a} = -2a.$$

and if $x \leq -a$:

$$d(F_1,X) - d(F_2,X) = \frac{a^2-cx}{a} - \frac{-cx-a^2}{a} = 2a.$$

5.3 The parabola.

The reduced equation of a parabola is:

$$\boxed{x^2 = 2py, \qquad p \neq 0}.$$

(We will assume $p > 0$ so that the parabola is located in the positive half plane.)

Its notable points and lines are:

1. **Center:** Does not exist (it has an "improper" center at $(0,1,0)$).
2. **Asymptotic directions:** $(0,1)$.
3. **Asymptotes:** Do not exist.
4. **Diameters:** Lines parallel to the OY axis.
5. **Axes:** $x = 0$.
6. **Vertices:** $(0,0)$.
7. **Focus:** $(0,\frac{p}{2})$.
8. **Directrices:** $y = -\frac{p}{2}$.
9. **Eccentricity:** $e = 1$.

Again, all these values are obtained directly the associated matrix associated to the conic:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -p \\ 0 & -p & 0 \end{pmatrix}$$

We compute again the foci, directrices and eccentricity.

If we consider the point $F = (0,\frac{p}{2})$ its polar line will be:

$$(0,\frac{p}{2},1)A(x,y,1)^t = 0 \iff -py - \frac{p^2}{2} = 0 \iff y + \frac{p}{2} = 0$$

Now given any point $X = (x,y)$ of the conic let us find the quotient between the distances from the point to the focus and the directrix. First of all considering that (x,y) satisfies the equation of the conic:

$$d(F,X) = \sqrt{x^2 + (y-\frac{p}{2})^2} = \sqrt{2py + y^2 + \frac{p^2}{4} - py} = \sqrt{y^2 + py + \frac{p^2}{4}} = y + \frac{p}{2}$$

Besides:

$$d(d,X) = y + \frac{p}{2}$$

We deduce that indeed $(0,\frac{p}{2})$ is a focus and the eccentricity is 1.

5.3.1 The parabola as a locus.

Definition 5.3 *The* **parabola** *can also be defined as the locus of points in the plane whose distance from the focus is the same as the distance from the directrix.*

This is an immediate consequence of the fact that the eccentricity is 1.

6 Change of coordinate system.

Let $R_1 = \{O; \bar{e}_1, \bar{e}_2\}$ and $R_2 = \{Q; \bar{e}'_1, \bar{e}'_2\}$. We denote by (x,y) and (x',y') respectively the coordinates relative to each of the coordinate systemes. Let us suppose

- The point Q has coordinates (q^1,q^2) with respect to the first system.
- $(e') = (e)C$, where $C = M_{BB'}$.

Then we know that the change of coordinate formula is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} + C \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x \\ y \\ t \end{pmatrix} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{C} & \begin{matrix} q^1 \\ q^2 \end{matrix} \\ 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x' \\ y' \\ t' \end{pmatrix}.$$

Suppose we have the equation of the conic given by a matrix A (and the corresponding T of quadratic terms) with respect to the system R_1:

$$\begin{pmatrix} x & y & t \end{pmatrix} A \begin{pmatrix} x \\ y \\ t \end{pmatrix} = 0 \iff \begin{pmatrix} x & y \end{pmatrix} T \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} a_{13} & a_{23} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + a_{33} = 0.$$

If we perform the change of coordinates in homogeneous coordinates we obtain:

$$\begin{pmatrix} x' & y' & t' \end{pmatrix} B^t A B \begin{pmatrix} x' \\ y' \\ t' \end{pmatrix} = 0 \text{ with } B = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{C} & \begin{matrix} q^1 \\ q^2 \end{matrix} \\ 0 & 0 & 1 \end{array} \right).$$

We deduce that the matrix of the conic in the new coordinate system is:

$$A' = B^t A B$$

and therefore:

Theorem 6.1 *The matrices A, A' of a conic with respect to two different coordinate systems R_1, R_2, are congruent matrices*

$$A' = B^t A B$$

where B is the change of coordinates matrix from R_2 to R_1, in homogeneous coordinates.

If we make the change of coordinates in affine coordinates we obtain:

$$\begin{pmatrix} x' & y' \end{pmatrix} C^t T C \begin{pmatrix} x' \\ y' \end{pmatrix} + \{\text{terms of degree} \leq 1\} = 0$$

Therefore we deduce that:

Theorem 6.2 *The matrices of quadratic terms T, T' of a conic with respect to two different affine coordinate systems R_1, R_2 are congruent matrices*

$$T' = CTC^t$$

where C is the change of coordinates matrix from the base of R_2 to that of R_1.

7 Classification of conics and reduced equation.

Given a conic defined by a symmetric matrix A, finding its reduced equation consists of making a change of coordinates so that the equation of the conic with respect to that new reference is as simple as possible. Specifically we carry out:

1. **A rotation.** Allows us to place the axis or axes of the conic parallel to the coordinate axes of the new reference. The matrix of quadratic terms in the new reference will be diagonal.
2. **A translation.** That allows us to place the center(s) (if any) of the conic at the origin of coordinates (otherwise we will take a vertex to the coordinate axis).

Before we start, we point out the following **important note**:

We will assume that at least one term of the diagonal of the matrix T of quadratic terms is nonnegative.

If this property is not fulfilled, it is enough to work with the matrix $-A$ instead of with A. In this way we ensure that T always has at least one positive eigenvalue.

7.1 Step I: Reduction of quadratic terms (the rotation).

Since the matrix T of quadratic terms is symmetric, it has two real eigenvalues λ_1 and λ_2, with $\lambda_1 \neq 0$. We will assume $\lambda_1 > 0$. Furthermore, we know that there exists an orthonormal basis of eigenvectors $\{\bar{u}_1, \bar{u}_2\}$ such that:

$$T' = C^t T C \text{ with } (\bar{u}) = (\bar{e})C \text{ and } T' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

The change of coordinates equation is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C \begin{pmatrix} x' \\ y' \end{pmatrix}$$

so that in the new base the equation of the conic is:

$$\begin{pmatrix} x' & y' \end{pmatrix} C^t T C \begin{pmatrix} x' \\ y' \end{pmatrix} + 2 \begin{pmatrix} a_{13} & a_{23} \end{pmatrix} C \begin{pmatrix} x' \\ y' \end{pmatrix} + a_{33} = 0$$

Operating, we obtain:

$$\boxed{\lambda_1 x'^2 + \lambda_2 y'^2 + 2b_{13}x' + 2b_{23}y' + b_{33} = 0}$$

7.2 Step II: Reduction of linear terms (the translation).

Now from the previous equation we complete the terms involving x' and y' to the square of a binomial, adding and subtracting the appropriate terms. Specifically:

- For x':

$$\lambda_1 x'^2 + 2b_{13}x' = \lambda_1(x'^2 + 2\frac{b_{13}}{\lambda_1}x' + \frac{b_{13}^2}{\lambda_1^2}) - \frac{b_{13}^2}{\lambda_1} = \lambda_1(x' + \frac{b_{13}}{\lambda_1})^2 - \frac{b_{13}^2}{\lambda_1}$$

- For y':

- If $\lambda_2 \neq 0$:

$$\lambda_2 y'^2 + 2b_{23}y' = \lambda_2(y'^2 + 2\frac{b_{23}}{\lambda_2}y' + \frac{b_{23}^2}{\lambda_2^2}) - \frac{b_{23}^2}{\lambda_2} = \lambda_2(y' + \frac{b_{23}}{\lambda_2})^2 - \frac{b_{23}^2}{\lambda_2}$$

- If $\lambda_2 = 0$ and $b_{23} \neq 0$:

$$2b_{23}y' + b_{33} = 2b_{23}(y' + \frac{b_{33}}{2b_{23}})$$

We make the corresponding translation in each case and obtain the following reduced forms:

If $\lambda_2 \neq 0$	If $\lambda_2 = 0$ and $b_{23} \neq 0$	If $\lambda_2 = b_{23} = 0$
Change of Coordinates: $x'' = x' + \frac{b_{13}}{\lambda_1}$ $y'' = y' + \frac{b_{23}}{\lambda_2}$	Change of Coordinates: $x'' = x' + \frac{b_{13}}{\lambda_1}$ $y'' = y' + \frac{b_{33}}{2b_{23}}$	Change of Coordinates: $x'' = x' + \frac{b_{13}}{\lambda_1}$
Reduced Equation: $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 + c_{33} = 0}$	Reduced Equation: $\boxed{\lambda_1 x''^2 + 2c_{23}y'' = 0}$	Reduced Equation: $\boxed{\lambda_1 x''^2 + c_{33} = 0}$

In other words, we are left with a reduced equation of the form:

$$\lambda_1 x''^2 + \lambda_2 y''^2 + 2c_{23}y'' + c_{33} = 0$$

with the following possibilities for the values of λ_2, c_{23}, and c_{33}:

1. If $\lambda_2 > 0$, then $c_{23} = 0$ and if:
   (a) $c_{33} > 0$, then the reduced equation is:

   $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 + c_{33} = 0}$ Imaginary ellipse.

   (b) $c_{33} = 0$, then the reduced equation is:

   $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 = 0}$ Imaginary lines intersecting at a point.

   (c) $c_{33} < 0$, then the reduced equation is:

   $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 + c_{33} = 0}$ Real ellipse.

2. if $\lambda_2 = 0$ and
   (a) $c_{23} \neq 0$, then $c_{33} = 0$ and the reduced equation is:

   $\boxed{\lambda_1 x''^2 + 2c_{23}y'' = 0}$ Parabola.

   (b) $c_{23} = 0$ and $c_{33} > 0$

   $\boxed{\lambda_1 x''^2 + c_{33} = 0}$ Imaginary parallel lines.

   (c) $c_{23} = 0$ and $c_{33} = 0$

   $\boxed{\lambda_1 x''^2 = 0}$ Real double line.

   (d) $c_{23} = 0$ and $c_{33} < 0$

   $\boxed{\lambda_1 x''^2 + c_{33} = 0}$ Real parallel lines.

3. If $\lambda_2 < 0$, then $c_{23} = 0$ and if:
   (a) $c_{33} \neq 0$, then the reduced equation is:

   $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 + c_{33} = 0}$ Hyperbola.

   (b) $c_{33} = 0$, then the reduced equation is:

   $\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 = 0}$ Real lines intersecting at a point.

7.3 Classification and reduced equation as a function of $|T|$ and $|A|$.

Taking into account that the determinants of T and A are preserved by rotations and translations, we can rewrite the above classification in terms of $|T|$ and $|A|$. Again we assume that T **has at least one positive term on the diagonal**:

	$\lvert A\rvert > 0$	$\lvert A\rvert = 0$	$\lvert A\rvert < 0$
$\lvert T\rvert > 0$	Imaginary ellipse	Imaginary lines intersecting	Real ellipse
$\lvert T\rvert = 0$	Parabola	$rg(A) = 2$ { Imag. parallel lines / Real parallel lines }; $rg(A) = 1$ Double Line	Parabola
$\lvert T\rvert < 0$	Hyperbola	Intersecting real lines	Hyperbola

When the conic is non-degenerate we can calculate the reduced equation, from the eigenvalues (at least one positive) λ_1, λ_2 of T and $|A|$:

1. If $|T| \neq 0$, then we obtain

$$\lambda_1 x''^2 + \lambda_2 y''^2 + c = 0, \quad \text{with} \quad c = \frac{|A|}{|T|}$$

   If $|T| > 0$, that is, for an ellipse, to guarantee that the largest radius is on the new OX axis, the smallest of the two eigenvalues must be taken as λ_1.

   If $|T| < 0$, that is, for a hyperbola, to guarantee that the vertices are on the new OX axis, we must take as λ_1 the eigenvalue that has the same sign as $det(A)$.

2. If $|T| = 0$, then we obtain

$$\lambda_1 x''^2 - 2cy'' = 0, \quad \text{with} \quad c = \sqrt{-\frac{|A|}{\lambda_1}}$$

   with $\lambda_1 \neq 0$.

We can even give the reference in which these reduced forms are obtained:

1. In the case of $|T| \neq 0$ (ellipse or hyperbola), the base of the new reference is formed by the normalized eigenvectors of T and the new origin is located at the center of the conic. One just has to be careful to use for the eigenvectors the same ordering as the one chosen for the eigenvalues:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \text{center} \end{pmatrix} + \underbrace{\begin{pmatrix} eigenvec_1 & eigenvec_2 \end{pmatrix}}_{\substack{normalized \\ eigenvec\, of\, T}} \begin{pmatrix} x'' \\ y'' \end{pmatrix}$$

2. In the case of $|T| = 0$ (parabola), the base of the new reference is formed by the normalized eigenvectors of T and the new origin is placed at the vertex. Now, in addition to correctly sorting the eigenvectors, we must check whether the sign of the eigenvector associated with the null eigenvalue has been chosen correctly:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \text{vertex} \end{pmatrix} + \underbrace{\begin{pmatrix} eigenvec_1 & eigenvec_2 \end{pmatrix}}_{\substack{normalized \\ eigenvec\, of\, T}} \begin{pmatrix} x'' \\ y'' \end{pmatrix}$$

   In order for the choice of eigenvectors to be consistent with the reduced form of the parabola, the eigenvector associated with 0 must be oriented in such a way that:

$$\begin{pmatrix} a_{13} & a_{23} \end{pmatrix} \begin{pmatrix} eigenvec_2 \end{pmatrix} < 0.$$

7.4 Classification by diagonalization by congruence.

Another way to classify a conic given by a matrix A is to diagonalize this matrix by congruence, but with the following restriction:

The last row can neither be added to the others nor multiplied by a scalar nor changed position.

A "forbidden" operation with this row would mean that the transformation we make takes proper points into points at infinity and vice versa.

With this method we will arrive at a diagonal matrix (except if it is a parabola) that will allow us to easily classify the conic.

Remark 7.1 *It must be taken into account that the diagonal form that we obtain in this way, does* **NOT** *necessarily correspond to the* **reduced equation** *of the conic. That is, this method allows us to classify the conic, but NOT to give its reduced equation.*

7.5 Obtaining the lines that form the degenerate conics.

Once the conic has been classified and shown to be degenerate, the most effective way to find the lines that constitute it is the following:

1. If the conic consists of two parallel lines (real or imaginary) or a double line, we calculate the line of centers. If it is a double line we are done. Otherwise we intersect the conic with any line (as simple as possible) and we obtain two points (real or imaginary). The lines that form the conic are parallel to the line of centers passing through these points.
2. If the conic consists of two intersecting lines (real or imaginary), we calculate the center. Then we intersect the conic with any line (as simple as possible) that does not pass through the center and we obtain two points (real or imaginary). The lines that form the conic are those that join the center with these points.

8 Pencils of conics

Definition 8.1 *Given two conics C_1 and C_2 with equations:*

$$(x)^t A_1(x) = 0 \quad \text{and} \quad (x)^t A_2(x) = 0$$

the **pencil of conics** *generated by them corresponds to the family of conics with equations:*

$$\{\alpha[(x)^t A_1(x)] + \beta[(x)^t A_2(x)] = 0; \quad \alpha, \beta \in \mathbb{R}, \quad (\alpha, \beta) \neq (0,0)\}$$

or equivalently:

$$\{[(x)^t A_1(x)] + \mu[(x)^t A_2(x)] = 0; \quad \mu \in \mathbb{R})\} \cup \{(x)^t A_2(x) = 0\}$$

The conics in a pencil share some common properties with the conics generating the pencil. For example:

- If P is a **common point** of C_1 and C_2, then P belongs to all the conics of the pencil.

- If r is a **tangent** to C_1 and C_2 at a point P, then r is also tangent at P to each one of the conics in the pencil.

- If r is an **asymptote** common to C_1 and C_2, then r is also an asymptote to all the conics of the pencil.

- If P is a **singular point** of C_1 and C_2, then P is a singular point of all the conics of the pencil.

- If P is the **center** of C_1 and C_2, then P is a center of all the conics of the pencil.

Let us see how to build the families of conics that satisfy some of these conditions.

1. *Pencil of conics passing through four non-collinear points.*
   Suppose A, B, C, D are four non-collinear points. We consider the lines
   $$r_1 \equiv AB, \quad r_2 \equiv CD; \qquad s_1 \equiv AC; s_2 \equiv BD.$$
   The corresponding pencil is:
   $$\boxed{\alpha(r_1 \cdot r_2) + \beta(s_1 \cdot s_2) = 0}$$

2. *Pencil of conics through three non-aligned points and with the tangent line at one of them fixed.*
   Suppose A, B, C are three non-aligned points and tg_A is the tangent line at A. We consider the lines
   $$r_1 \equiv AB, \quad r_2 \equiv AC; \qquad s \equiv BC.$$
   The corresponding pencil is:
   $$\boxed{\alpha(r_1 \cdot r_2) + \beta(s \cdot tg_A) = 0}$$

2'. *Pencil of conics through two points and a fixed asymptote.*
   It is a particular case of the previous one, if we think that the asymptote is a tangent line at the point at infinity. Suppose B, C are the points and *asympt* is the asymptote. We consider the lines
   $$\begin{array}{l} r_1 \equiv \{\text{line through } B \text{ and parallel to } asympt\} \\ r_2 \equiv \{\text{line through } C \text{ and parallel to } asympt\} \end{array} \qquad s \equiv BC.$$
   The corresponding pencil is:
   $$\boxed{\alpha(r_1 \cdot r_2) + \beta(s \cdot asympt) = 0}$$

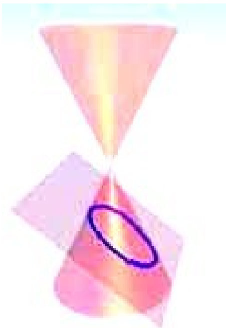
3. *Pencil of conics through two points and with the tangent lines to both fixed.*
   Suppose that A, B are two points and tg_A, tg_B the corresponding tangents. We consider the line
   $$r \equiv AB.$$
   The corresponding pencil is:
   $$\boxed{\alpha(r)^2 + \beta(tg_A \cdot tg_B) = 0}$$

3'. *Pencil of conics through a point, given the tangent line at it and also an asymptote.*
   Again it is a particular case of the previous one. Suppose A is the point and tg_A the corresponding tangent. Let *asympt* be the asymptote. We consider the line
   $$r \equiv \{\text{line through } A \text{ and parallel to } asympt\}$$
   The corresponding pencil is:
   $$\boxed{\alpha(r)^2 + \beta(tg_A \cdot asympt) = 0}$$

3". *Pencil of conics with two asymptotes known.*
   Again it is a particular case of the previous one. Suppose that $asympt_1$ and $asympt_2$ are the two known asymptotes. We consider the line:
   $$r \equiv \{\text{line at infinity}\}$$
   whose homogeneous equation is $t = 0$ and the affine equation is $1 = 0$ (!?). The corresponding pencil is:
   $$\boxed{\alpha(1)^2 + \beta(asympt_1 \cdot asympt_2) = 0}$$

9 Appendix: planar sections of a cone.

Non-degenerate conics.

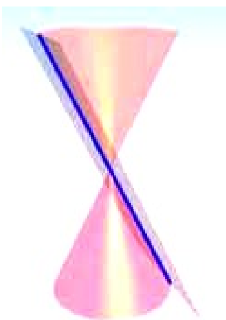


Ellipse.

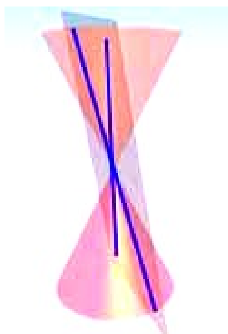
Hyperbola.

Parabola.

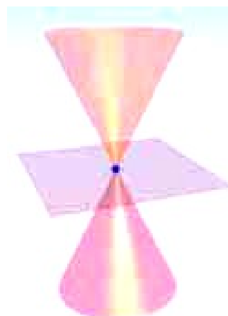
Degenerate conics.



Double line.



Two lines.



One point.