2. The projective space.

1 Introduction.

The objective of this section is to describe some algebraic tools that will allow us to work with the points at infinity of affine spaces and subspaces. In this way the **projective space** or **extended affine space** is the affine space to which the **points at infinity** are added. We will define some new coordinates that will allow us to handle these points.

Although this construction may seem arbitrary at first, the main advantage will be that situations that were handled as particular cases in the affine space will now be inserted in a general context in the extended affine space. For example, while in the affine plane two different lines either intersect at a point or are parallel, in **the projective plane two different lines always intersect at a point.**

Although the construction can be done on spaces of arbitrary dimension, we will work on E_2 and E_3. Specifically, and unless otherwise indicated, we will develop the theory in E_3 (all constructions can be easily carried over to E_2 by eliminating a coordinate).

Finally, we point out that here we will only give a not too rigorous outline of these ideas, just as a basis for the concept of points at infinity.

1.1 Homogeneous coordinates and projective space.

Suppose we have a reference system $\{O; \bar{e}_1, \bar{e}_2, \bar{e}_3\}$ in E_3.

Definition 1.1 *If (x^1, x^2, x^3) are the affine coordinates of a point $X \in E$, the* **homogeneous coordinates** *of X are (tx^1, tx^2, tx^3, t) where t is a nonzero real number.*

According to this definition, the correspondence between affine and homogeneous coordinates is the following:

Affine	$\longrightarrow$	Homogeneous	Homogeneous	$\longrightarrow$	Affine
(x^1, x^2, x^3)		(y^1, y^2, y^3, t)	(y^1, y^2, y^3, t)		(x^1, x^2, x^3)
		with			with
		$y^1 = tx^1$			$x^1 = y^1/t$
		$y^2 = tx^2$			$x^2 = y^2/t$
		$y^3 = tx^3$			$x^3 = y^3/t$

It is important to observe the following:

Homogeneous coordinates of a point are NOT unique.

In fact, (y^1, y^2, y^3, t) and (z^1, z^2, z^3, t') are the homogeneous coordinates of the same point X when they are proportional, that is, when there exists a $\lambda \neq 0$ such that:

$$(y^1, y^2, y^3, t) = \lambda(z^1, z^2, z^3, t').$$

Taking this into account we can define the projective space as:

Definition 1.2 *If $\{O; \bar{e}_1, \bar{e}_2, \bar{e}_3\}$ is an affine coordinate system of E_3 we define* **the projective space** *as the set determined by the points of homogeneous coordinates $(y^1, y^2, y^3, t) \neq (0, 0, 0, 0)$, so that proportional coordinates define the same point.*

The extended affine space of an affine space E will be denoted by $\overline{E}$.

We have seen that when $t \neq 0$ these points correspond to points of the usual affine space. However, in the extended affine space we also include those with coordinate $t = 0$. In the next section we will see how they are interpreted.

1.2 Improper points or points at infinity.

1.2.1 Motivation: why the points with coordinate $t = 0$ represent points at infinity.

At the moment we have seen how to represent a point in affine space with new coordinates that we have called homogeneous. Let us see how these new coordinates will allow us to work with the points at infinity. To motivate what we will do we think of the following.

Suppose we have a straight line that passes through the point $P = (p^1, p^2, p^3)$ and with direction vector $\bar{u} = (u^1, u^2, u^3)$. Its parametric equation, in affine coordinates is:

$$(x^1, x^2, x^3) = (p^1, p^2, p^3) + \alpha(u^1, u^2, u^3) = (p^1 + \alpha u^1, p^2 + \alpha u^2, p^3 + \alpha u^3)$$

Intuitively, the point at infinity of the line will appear when α tends to ∞. However, with these coordinates we have no way to formalize this.

Let's write this equation in homogeneous coordinates (y^1, y^2, y^3, t)

$$(y^1, y^2, y^3, t) = (t(p^1 + \alpha u^1), t(p^2 + \alpha u^2), t(p^3 + \alpha u^3), t)$$

Since proportional homogeneous coordinates still refer to the same points, when $\alpha \neq 0$ we can write the equation as:

$$(y^1, y^2, y^3, t) = \frac{1}{\alpha}((p^1 + \alpha u^1), (p^2 + \alpha u^2), (p^3 + \alpha u^3), 1)$$

Now if we do $\alpha = \infty$ (or more strictly, we take limits as $\alpha \to \infty$) the above expression becomes:

$$(y^1, y^2, y^3, t) = (u^1, u^2, u^3, 0).$$

In other words, the point at infinity of the initial straight line is the one with homogeneous coordinates $(u^1, u^2, u^3, 0)$. We note that these coordinates only depend on the directing vector of the line, not on the point (p^1, p^2, p^3). Therefore it is clear that another line parallel to the first would have the same point at infinity.

1.2.2 Definition of improper points or points at infinity.

After the above discussion, it is reasonable to make the following definition:

Definition 1.3 *Fixed a reference $\{O; \bar{e}_1, \bar{e}_2, \bar{e}_3\}$. If X has homogeneous coordinates (y^1, y^2, y^3, t), X is said to be a* **improper point** *or a* **point at infinity** *if $t = 0$. In particular, if $\bar{u} = (u^1, u^2, u^3)$ are the coordinates of a vector of V, the point at infinity in the direction of $\bar{u}$ has homogeneous coordinates*

$$(\lambda u^1, \lambda u^2, \lambda u^3, 0), \text{ for any } \lambda \neq 0.$$

Therefore, when we work in homogeneous coordinates and we want to find the improper points (or infinity) of a certain geometric object, we will have to intersect the equations of said object with the equation $t = 0$.

In particular:

- In the projective plane $\overline{E_2}$, $t = 0$ is the equation of the **line of points from infinity.**

- In the projective space $\overline{E_3}$, $t = 0$ is the equation of the **plane of points at infinity.**

1.3 Change of coordinate system in homogeneous coordinates.

Suppose we have two affine coordinate systems $\{O; \bar{e}_1, \bar{e}_2, \bar{e}_3\}$ and $\{P; \bar{e}'_1, \bar{e}'_2, \bar{e}'_3\}$. We fix the following notation for affine and homogeneous coordinates with respect to each of the references:

Reference.	Affine coordinates.	Homogeneous coordinates.
$\{O; \underbrace{\bar{e}_1, \bar{e}_2, \bar{e}_3}_{B}\}$	(x^1, x^2, x^3)	(y^1, y^2, y^3, t)
$\{P; \underbrace{\bar{e}'_1, \bar{e}'_2, \bar{e}'_3}_{B'}\}$	(x'^1, x'^2, x'^3)	(y'^1, y'^2, y'^3, t')

Further suppose that:

$(e') = (e)M_{BB'}$ and $P = (p^1, p^2, p^3)$ in affine coordinates with respect to $\{O; \bar{e}_1, \bar{e}_2, \bar{e}_3\}$.

Then we know what the relationship is between the affine coordinates with respect to each reference:

$$(x) = (p) + M_{BB'}(x') \iff \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix} + M_{BB'} \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}$$

Let us see what is the relationship between the homogeneous coordinates. We note that the previous formula can also be written in matrix terms as:

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ 1 \end{pmatrix} = \left(\begin{array}{ccc|c} & & & p^1 \\ & M_{BB'} & & p^2 \\ & & & p^3 \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \\ 1 \end{pmatrix}$$

But since proportional homogeneous coordinates refer to the same point, $(x^1, x^2, x^3, 1)$ and (y^1, y^2, y^3, t) denote the same point; and the same happens with $(x'^1, x'^2, x'^3, 1)$ and (y'^1, y'^2, y'^3, t'). Therefore the formula is:

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ t \end{pmatrix} = \left(\begin{array}{ccc|c} & & & p^1 \\ & M_{BB'} & & p^2 \\ & & & p^3 \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} y'^1 \\ y'^2 \\ y'^3 \\ t' \end{pmatrix}$$

1.4 Cartesian equations of affine subspaces in homogeneous coordinates.

By using the formulas relating the affine and homogeneous coordinates of a point, we will see how to go from the cartesian equations of an affine subspace in affine coordinates to homogeneous coordinates. Remember that once a reference is fixed, the homogeneous coordinates (y^1, y^2, y^3, t) correspond to the affine ones

$$(x^1, x^2, x^3) = (y^1/t, y^2/t, y^3/t).$$

1.4.1 Lines in $\bar{E}_2$.

Suppose we have the Cartesian equation of a line:

$$a_1x^1 + a_2x^2 + a_3 = 0$$

Changing these coordinates to homogeneous we have:

$$a_1\frac{y^1}{t} + a_2\frac{y_2}{t} + a_3 = 0 \iff \boxed{a_1y^1 + a_2y^2 + a_3t = 0}$$

If we now want to find the infinity points of this line, we only have to intersect it with the line at infinity $t = 0$:

$$\left.\begin{array}{r} a_1y^1 + a_2y^2 + a_3t = 0 \\ t = 0 \end{array}\right\} \Rightarrow a_1y^1 + a_2y^2 = 0 \Rightarrow (y^1, y^2) = (a_2, -a_1)$$

so that we obtain that the point at infinity corresponds to the direction marked by the directing vector of the line.

1.4.2 Planes in $\bar{E}_3$.

Suppose we have the cartesian equation of a plane:

$$a_1x^1 + a_2x^2 + a_3x^3 + a_4 = 0$$

Changing these coordinates to homogeneous we have:

$$a_1\frac{y^1}{t} + a_2\frac{y_2}{t} + a_3\frac{y^3}{t} + a_4 = 0 \iff \boxed{a_1y^1 + a_2y^2 + a_3y^3 + a_4t = 0}$$

If we now want to find the infinity points of this line, we only have to intersect it with the plane at infinity $t = 0$:

$$\left.\begin{array}{r} a_1y^1 + a_2y^2 + a_3y^3 + a_4t = 0 \\ t = 0 \end{array}\right\} \Rightarrow a_1y^1 + a_2y^2 + a_3y^3 = 0$$

Now this intersection corresponds to the points at infinity whose direction is determined by any vector contained in the plane.

1.4.3 Lines in $\bar{E}_3$.

Suppose we have the Cartesian equations of a line (as the intersection of two planes):

$$\left\{\begin{array}{l} a_1x^1 + a_2x^2 + a_3x^3 + a_4 = 0 \\ b_1x^1 + b_2x^2 + b_3x^3 + b_4 = 0 \end{array}\right.$$

Reasoning as before, in homogeneous a coordinates it remains:

$$\boxed{\begin{array}{l} a_1y^1 + a_2y^2 + a_3y^3 + a_4t = 0 \\ b_1y^1 + b_2y^2 + b_3y^3 + b_4t = 0 \end{array}}$$

The infinity point of said line is obtained by solving the system:

$$\begin{array}{r} a_1x^1 + a_2x^2 + a_3x^3 + a_4t = 0 \\ b_1x^1 + b_2x^2 + b_3x^3 + b_4t = 0 \\ t = 0 \end{array}$$

1.5 Parametric equations of affine subspaces in homogeneous coordinates.

Let us now see how the parametric equations of the affine subspaces are written in homogeneous coordinates.

1.5.1 Lines in $\bar{E}_2$.

If $P = (a^1, a^2, t_P)$ and $Q = (b^1, b^2, t_Q)$ are the **homogeneous coordinates** of two distinct points in the extended affine plane, then the parametric equation of the line that joins them is:

$$\boxed{(y^1, y^2, t) = \alpha(a^1, a^2, t_P) + \beta(b^1, b^2, t_Q)}$$

Observations on this equation:

- Although there are two parameters α and β, taking into account that proportional homogeneous coordinates define the same point, the object defined by these equations has dimension 1.

- Once again, the improper points of the line are obtained by intersecting with the equation $t = 0$.

- The advantage of this equation is that it is used to find both the equation of a line passing through two points, and the equation of a line known to a point and the directing vector. Simply keep in mind that the fact that a straight line has (u^1, u^2) as its direction vector means that it passes through the point at infinity $(u^1, u^2, 0)$.

1.5.2 Lines in $\bar{E}_3$.

It is totally analogous to the previous case. If $P = (a^1, a^2, a^3, t_P)$ and $Q = (b^1, b^2, b^3, t_Q)$ are the **homogeneous coordinates** of two different points in the extended affine space, then the parametric equation of the line that joins them is:

$$\boxed{(y^1, y^2, y^3, t) = \alpha(a^1, a^2, a^3, t_P) + \beta(b^1, b^2, b^3, t_Q)}$$

1.5.3 Planes in $\bar{E}_3$.

Now if we have three non-aligned points in the extended affine space $\bar{E}_3$ with homogeneous coordinates, $P = (a^1, a^2, a^3, t_P)$, $Q = (b^1, b^2, b^3, t_Q)$ and $R = (c^1, c^2, c^3, t_R)$, the parametric equation of the plane they define will be:

$$\boxed{(y^1, y^2, y^3, t) = \alpha(a^1, a^2, a^3, t_P) + \beta(b^1, b^2, b^3, t_Q) + \gamma(c^1, c^2, c^3, t_R)}$$

We note that this equation also allows us to calculate the equation of a plane from two points and a vector or a point and two vectors. It is only necessary to take into account that a vector $\bar{u} = (u^1, u^2, u^3)$ corresponds to the point at infinity $(u^1, u^2, u^3, 0)$.