Part III

Affine geometry.

1. The affine space.

1 Definition and properties.

Definition 1.1 *Let E be a nonempty set. E is an* **affine space** *associated to a vector space V, if there is a map:*

$$\begin{array}{ccc} E \times E & \longrightarrow & V \\ (P,Q) & \longrightarrow & \bar{v} = \overline{PQ} \end{array}$$

satisfying the properties:

(a) $\forall P \in E$ and $\forall \bar{v} \in V$ there is an unique $Q \in E$ satisfying that $\bar{v} = \overline{PQ}$.

(b) $\forall P, Q, R \in E$ it is satisfied that $\overline{PQ} + \overline{QR} = \overline{PR}$ (Chasles relation).

We will call the elements of an affine space **points**. *If, in addition, the vector space V is Euclidean, then E is said to be a* **Euclidean affine space**.

By the property (a), given a point $P \in E$ and a vector $\bar{v} \in V$ we can define the point $Q := P + \bar{v}$, as the only point $Q \in E$ satisfying $\bar{v} = \overline{PQ}$. Geometrically, the point Q is obtained by moving the point P in the direction and distance indicated by the vector $\bar{v}$.

The following properties are immediate consequences of the definition of affine space. For any points $A, B, C, A', B' \in E$ the following holds:

1. $\overline{AB} = \bar{0} \iff A = B$.
   Proof: By condition (b) we have:
   $$\overline{AA} + \overline{AB} = \overline{AB} \quad \Rightarrow \quad \overline{AA} = \bar{0}.$$
   On the other hand, by condition (a) we know that there is a unique point B satisfying $\bar{0} = \overline{AB}$. ■
2. $\overline{AB} = -\overline{BA}$.
   Proof: By condition (b) and the above property:
   $$\overline{AB} + \overline{BA} = \overline{AA} = \bar{0} \quad \Rightarrow \quad \overline{AB} = -\overline{BA}.$$
   ■
3. $\overline{AB} = \overline{CB} - \overline{CA}$.
   Proof: It is a direct consequence of condition (b). ■
4. $\overline{AB} = \overline{A'B'} \quad \Rightarrow \quad \overline{AA'} = \overline{BB'}$.
   Proof: We use condition (b):
   $$\left.\begin{array}{ccccc} \overline{AA'} & = & \overline{AB} & + & \overline{BA'} \\ \overline{BB'} & = & \overline{BA'} & + & \overline{A'B'} \end{array}\right\} \quad \Rightarrow \quad \overline{AA'} - \overline{BB'} = \overline{AB} - \overline{A'B'} = \bar{0} \quad \Rightarrow$$
   $$\Rightarrow \quad \overline{AA'} = \overline{BB'}$$
   ■

Although the theory of affine spaces can be developed in any dimension, we will work mainly on two particular cases: the **Euclidean affine plane** E_2 defined over the vector space $\mathbb{R}^2$ and **the Euclidean affine space** E_3 defined over the vector space $\mathbb{R}^3$.

2 Affine coordinate system and affine coordinates.

We will work on a Euclidean affine space E associated to a real vector space V of dimension n.

Definition 2.1 *An* **affine coordinate system** *$\{O; \bar{e}_1, \ldots, \bar{e}_n\}$ of E is formed by a point $O \in E$ and a basis of V, $\{\bar{e}_1, \ldots, \bar{e}_n\}$.*

In particular, if the basis $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is orthonormal, then the affine coordinate system is said to be **rectangular.**

Definition 2.2 *Let $\{O; \bar{e}_1, \ldots, \bar{e}_n\}$ be an affine coordinate system in E. The* **affine coordinates of a point** *$P \in E$ are the contravariant coordinates of the vector $\overline{OP}$ respect to the basis $\{\bar{e}_1, \ldots, \bar{e}_n\}$.*

We notice that the affine coordinates of a point are well defined since:

- Given O and P, the vector $\overline{OP}$ is uniquely defined.

- The coordinates of a vector $(\overline{OP})$ in a basis of a vector space are unique.

2.1 Change of coordinate system.

Suppose we have two coordinate systems $\{O; \bar{e}_1, \ldots, \bar{e}_n\}$ and $\{P; \bar{e}'_1, \ldots, \bar{e}'_n\}$. We denote
$$B = \{\bar{e}_1, \ldots, \bar{e}_n\}; \qquad B' = \{\bar{e}'_1, \ldots, \bar{e}'_n\}.$$

We know how they are related to each other. Specifically

$(\bar{e}') = (\bar{e})M_{BB'}$
$(p) = (p^1, \ldots, p^n)$ coordinates of point P relative to the reference $\{O; \bar{e}_1, \ldots, \bar{e}_n\}$

Remember that this means that the coordinates of the vector $\overline{OP}$ relative to the basis B, are $(p^1, \ldots, p^n)$.

Let $X \in E$ be any point in the affine space. We will denote:

$(x) = (x^1, \ldots, x^n)$ coordinates of the point X with respect to the system $\{O; \bar{e}_1, \ldots, \bar{e}_n\}$
$(x') = (x'^1, \ldots, x'^n)$ coordinates of the point X with respect to the system $\{P; \bar{e}'_1, \ldots, \bar{e}'_n\}$

The coordinates of X with respect to the system $\{O; \bar{e}_1, \ldots, \bar{e}_n\}$ are the coordinates of the vector $\overline{OX}$ relative to the basis B. The coordinates of X relative to the system$\{P; \bar{u}_1, \ldots, \bar{u}_n\}$ are the coordinates of the vector $\overline{PX}$ relative to the basis B'. Let us see how these sets of coordinates are related. We have:

$$\overline{PX} = \overline{OX} - \overline{OP} = (\bar{e})(x) - (\bar{e})(p) = (\bar{e})((x) - (p))$$

Therefore it is clear that the coordinates of $\overline{PX}$ in the basis B are $(x) - (p)$. Now we simply translate the coordinates of this vector relative to basis B into coordinates relative to basis B':

$$\boxed{(y) = M_{B'B}[(x) - (p)]} \iff \boxed{(y) = {M_{BB'}}^{-1}[(x) - (p)]}$$

3 Affine subspaces.

Definition 3.1 *Let E be a Euclidean affine space associated to a vector space V. Given a point $P \in E$ and a vector subspace $U \subset V$, the* **affine subspace** *which passes through P and has direction U is defined as the set of points:*

$$P + U = \{Q \in E |\, Q = P + \bar{u},\, \bar{u} \in U\} = \{Q \in E |\, \overline{PQ} \in U\}$$

The **dimension** *of the affine subspace is the dimension of the vector subspace U defining its direction. If this dimension is* 0**,** 1 **or** 2 *the affine variety will be called respectively a* **point***, a* **line** *or a* **plane**.

Let us see the different types of equations of the affine subspaces in the affine plane E_2 and in the affine space E_3.

3.1 Affine subspaces in the affine plane E_2.

We will work on the Euclidean affine plane E_2 with respect to a reference system $\{O; \bar{e}_1, \bar{e}_2\}$.

0. Points: A point in E_2 is determined by giving its coordinates.

1. Lines: A **line** is **determined** by giving **a point** $P = (p^1, p^2)$ through which it passes and **a vector** $\bar{u} = (u^1, u^2)$ that generates the subspace defining its direction. We can describe the points $X = (x^1, x^2)$ of the line with any of the following equations[1]:

$X = P + \alpha\bar{u}$ **Vector equation.**

$\left.\begin{array}{l} x^1 = p^1 + \alpha u^1 \\ x^2 = p^2 + \alpha u^2 \end{array}\right\}$ **Parametric equations.**

$\frac{x^1 - p^1}{u^1} = \frac{x^2 - p^2}{u^2}$ **Continuous equation.**(1)

$a_1x^1 + a_2x^2 + b = 0$ **Cartesian equation.**

where the continuous equation can be obtained from the parametric equations by eliminating the parameter, and the Cartesian equation from the continuous one by eliminating denominators.

The Cartesian equation satisfies the following interesting property:

Theorem 3.2 *If $a_1x^1 + a_2x^2 + a_3 = 0$ is the Cartesian equation of a line r, then (a_1, a_2) are the (covariant) coordinates of a vector orthogonal to any vector in the direction of the line.*

Proof: A vector $\bar{v}$ in the direction of the line is determined by two points X and Y contained in the line, such that $\bar{v} = \overline{XY}$. We have

$$\left.\begin{array}{lll} X \in r & \Rightarrow & a_1x^1 + a_2x^2 + a_3 = 0 \\ Y \in r & \Rightarrow & a_1y^1 + a_2y^2 + a_3 = 0 \end{array}\right\} \Rightarrow (a_1)(y^1 - x^1) + (a_2)(y^2 - x^2) = 0$$

We therefore deduce that the vector v is orthogonal to the vector with (covariant) coordinates (a_1, a_2). ∎

The vector of covariant coordinates (a_1, a_2) is called a **normal vector to the line**.

A **line** is also **determined by two different points** P and Q. In this case, we can obtain the previous equations taking as the direction vector the one joining these points $\bar{u} = \overline{PQ}$. For instance the continuous equation would be in this case

$$\frac{x^1 - p^1}{q^1 - p^1} = \frac{x^2 - p^2}{q^2 - p^2}$$

Again, for this equation to be valid, the denominators must be nonzero.

Finally, it is also sometimes useful to write directly the equation of a line whose intersections with the axes are known. The line passing through the points $(a, 0)$ and $(0, b)$ has the equation:

$$\frac{x^1}{a} + \frac{x^2}{b} = 1$$

[1] For the continuous equation to make sense, the denominators u^1 and u^2 must be nonzero.

This equation is sometimes called the **canonical equation** of the line.

2. Planes: The only plane in E_2 is the affine plane E_2 itself, so it doesn't make practical sense to talk about its equations.

3.2 Affine subspaces in the affine space E_3.

We will work in the Euclidean affine space E_3 with respect to a reference system $\{O; \bar{e}_1, \bar{e}_2, \bar{e}_3\}$.

0. Points: A point at E_3 is determined by giving its coordinates.
1. Lines: A **line** is **determined** by giving **a point** $P = (p^1, p^2, p^3)$ through which it passes and **a vector** $\bar{u} = (u^1, u^2, u^3)$ generating its direction subspace. We can describe the points $X = (x^1, x^2, x^3)$ on the line by means of any of the following equations[2]:

$$X = P + \alpha\bar{u}$$ **Vector equation.**

$$\left.\begin{array}{l} x^1 = p^1 + \alpha u^1 \\ x^2 = p^2 + \alpha u^2 \\ x^3 = p^3 + \alpha u^3 \end{array}\right\}$$ **Parametric Equations.**

$$\frac{x^1-p^1}{u^1} = \frac{x^2-p^2}{u^2} = \frac{x^3-p^3}{u^3}$$ **Continuous Equation.**

$$\left.\begin{array}{l} a_1x^1 + a_2x^2 + a_3x^3 + a_4 = 0 \\ b_1x^1 + b_2x^2 + b_3x^3 + b_4 = 0 \end{array}\right\} \text{ with } rank\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = 2$$ **Intersection of two planes.**

The equation of the line expressed as the intersection of two planes, can be obtained (for example) from the parametric equations by eliminating the parameter or from the continuous one by eliminating denominators.

A **line** is also **determined by two different points** P and Q. Again we can obtain the preceding equations taking as the direction vector the one that joins the points $\bar{u} = \overline{PQ}$.

2. Planes: A **plane** is **determined by a point** $P = (p^1, p^2, p^3)$ through which it passes and a basis $\{\bar{u}, \bar{v}\}$ of the vector subspace that marks **its direction**. Let us see the different equations that characterize the points $X = (x^1, x^2, x^3)$ of the plane:

$$X = P + \alpha\bar{u} + \beta\bar{v}$$ Vector equation.

$$\left.\begin{array}{l} x^1 = p^1 + \alpha u^1 + \beta v^1 \\ x^2 = p^2 + \alpha u^2 + \beta v^2 \\ x^3 = p^3 + \alpha u^3 + \beta v^3 \end{array}\right\}$$ Parametric equations.

$$\begin{vmatrix} x^1-p^1 & x^2-p^2 & x^3-p^3 \\ u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \end{vmatrix} = 0$$
$$\Updownarrow$$
$$a_1x^1 + a_2x^2 + a_3x^3 + a_4 = 0$$ Cartesian equation.

Again, the following interesting property holds:

Theorem 3.3 *If $a_1x^1 + a_2x^2 + a_3x^3 + a_4 = 0$ is the Cartesian equation of a plane, then (a_1, a_2, a_3) are the (covariant) coordinates of a vector orthogonal to any vector in the direction of the plane.*

The vector of (covariant) coordinates (a_1, a_2, a_3) is said to be a **normal vector to the plane**.

A **plane** is also **determined by three not collinear points** P, Q, R. In this case we can write the previous equations taking $\overline{PQ}$ and $\overline{PR}$ as direction vectors.

Finally, if only the intersection points of the plane with the axes are known: $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ we can directly write the **canonical equation**:

$$\frac{x^1}{a} + \frac{x^2}{b} + \frac{x^3}{c} = 1.$$

4 Pencil of affine subspaces.

A pencil of affine subspaces consists of a family of affine subspaces that depend "linearly" on some parameters. Here we will see two particular cases.

4.1 Pencil of lines in E_2.

Let r and s be two lines of E_2 with cartesian equations:

$$\begin{array}{lcl} r & \equiv & a_1x^1 + a_2x^2 + a_3 = 0 \\ s & \equiv & b_1x^1 + b_2x^2 + b_3 = 0 \end{array}$$

The **pencil of lines formed by** r **and** s is the set of all lines defined by the following equations:

$$\{\alpha(a_1x^1 + a_2x^2 + a_3) + \beta(b_1x^1 + b_2x^2 + b_3) = 0 |\ \alpha, \beta \in \mathbb{R},\ (\alpha, \beta) \neq (0, 0)\}$$

[2] For the continuous equation to make sense the denominators u^1, u^2 and u^3 must be non-null.

or also

$$\{(a_1x^1 + a_2x^2 + a_3) + \mu(b_1x^1 + b_2x^2 + b_3) = 0|\ \mu \in \mathbb{R}\} \cup \{s\}.$$

If r and s intersect at a point P, the pencil of lines is formed by all the lines that pass through the point P.

If r and s are parallel, then the pencil of lines consists of all the lines parallel to them.

4.2 Pencil of planes in E_3.

Given two planes π_1 and π_2 in E_3 with Cartesian equations

$$\begin{array}{rcl} \pi_1 & \equiv & a_1x^1 + a_2x^2 + a_3x^3 + a_4 = 0 \\ \pi_2 & \equiv & b_1x^1 + b_2x^2 + b_3x^3 + b_4 = 0 \end{array}$$

the **pencil of planes formed by π_1 and π_2** is the set of all the planes defined by the following equations:

$$\{\alpha(a_1x^1+a_2x^2+a_3x^3+a_4)+\beta(b_1x^1+b_2x^2+b_3x^3+b_4) = 0|\ \alpha,\beta \in \mathbb{R},\ (\alpha,\beta) \neq (0,0)\}$$

or also

$$\{(a_1x^1 + a_2x^2 + a_3x^3 + a_4) + \mu(b_1x^1 + b_2x^2 + b_3x^3 + b_4) = 0|\ \mu \in \mathbb{R}\} \cup \{s\}.$$

If π_1 and π_2 have a nonempty intesection, the pencil of planes is formed by all the planes that pass through the intersection $\pi_1 \cap \pi_2$.

If π_1 and π_2 are parallel, then the pencil of planes consists of all the planes parallel to them.

5 Angles.

5.1 Angle between two lines.

Both in the affine plane E_2 and in the affine space E_3 **the angle formed by two lines is the one formed by their direction vectors**. We will always adopt the smaller of the two angles they form.

If r and s are two lines with direction vectors $\bar{u}$ and $\bar{v}$ respectively, then the angle they form is given by:

$$cos(r,s) = |cos(\bar{u},\bar{v})| = \frac{|\bar{u}\cdot\bar{v}|}{\|\bar{u}\|\|\bar{v}\|}.$$

In this way, two lines are parallel if their direction vectors are proportional, or equivalently, if they form an angle of zero degrees.

Two lines are perpendicular if their direction vectors are orthogonal, or equivalently if they form an angle of $\frac{\pi}{2}$.

5.2 Angle between two planes.

The **angle formed by two planes** in the affine space E_3 is **the smallest angle formed by their corresponding normal vectors**.

Specifically, if π_1 and π_2 are two planes with normal vectors $\bar{n}_1$ and $\bar{n}_2$ respectively, then the angle they form is given by:

$$cos(\pi_1,\pi_2) = |cos(\bar{n}_1,\bar{n}_2)| = \frac{|\bar{n}_1\cdot\bar{n}_2|}{\|\bar{n}_1\|\|\bar{n}_2\|}$$

Now two planes are parallel if they form an angle of zero degrees or equivalently if their normal vectors are proportional.

Two planes are perpendicular if their normal vectors are perpendicular or equivalently if they form an angle of $\frac{\pi}{2}$.

5.3 Angle between a line and a plane.

The angle between a line and a plane in the Euclidean space is the complement of the one formed by the direction vector of the line and the normal vector of the plane.

Thus, if r is a line with direction vector $\bar{u}$ and π is a plane with normal vector $\bar{n}$, then the angle they form is given as:

$$sin(r,\pi) = |cos(\bar{u},\bar{n})| = \frac{|\bar{u}\cdot\bar{n}|}{\|\bar{u}\|\|\bar{n}\|}.$$

The line and the plane will be parallel, when the direction vector of the line and the normal vector of the plane are perpendicular; equivalently, when the line and the plane form an angle of zero degrees.

The line and the plane will be perpendicular, when the direction vector of the line and the normal vector of the plane are proportional; equivalently, when the line and the plane form an angle of $\frac{\pi}{2}$.

6 Distances.

6.1 Distance between two points.

Definition 6.1 *Given two points P and Q in an arbitrary affine Euclidean space E, we define the* **distance between** P **and** Q *as the norm of the vector they form:*

$$d(P,Q) = \|\overline{PQ}\|$$

As a consequence of the properties seen for the Euclidean norm, the distance between two points fulfills the following properties. For any points $P, Q, R \in E$ we have:

1. $d(P,Q) \geq 0$.
2. $d(P,Q) = 0 \iff P = Q$.
3. $d(P,Q) = d(Q,P)$.
4. $d(A,C) \leq d(A,B) + d(B,C)$ (Triangle Inequality).

6.2 Distance from a point to an affine subspace.

Definition 6.2 *The* **distance between a point** P **and an affine subspace** F *is defined as the smallest of the distances between P and each of the points of F:*

$$d(P,F) = min\{d(P,X) |\, X \in F\}.$$

Intuitively we have a geometric idea of what is the smallest distance between a point P and a subspace. The shortest "path" between the two is the segment perpendicular to the affine subspace joining it to the point. Let us formalize this.

Definition 6.3 *Given a point P and an affine subspace F, such that $P \notin F$, we call* **orthogonal projection** *of P onto F to a point $Q \in F$ such that $\overline{PQ}$ is orthogonal to all vectors in F.*

Theorem 6.4 *Given a point P and an affine subspace F, such that $P \notin F$, we denote by Q the orthogonal projection of P onto F. Then*

$$d(P,F) = d(P,Q)$$

Proof: We will use the fact that, since $\overline{PQ}$ is orthogonal to F, if we take any point $X \in F$ then $\overline{QX}$ is orthogonal to $\overline{PQ}$ and the Pythagorean theorem can be applied. Therefore:

$$\|\overline{PX}\|^2 = \|\overline{PQ}\|^2 + \|\overline{QX}\|^2 \quad \Rightarrow \quad \|\overline{PX}\| = \sqrt{\|\overline{PQ}\|^2 + \|\overline{QX}\|^2}$$

Then

$$\begin{aligned} d(P,f) &= min\{d(P,X) |\, X \in F\} = min\{\|\overline{PX}\| |\, X \in F\} = \\ &= min\{\sqrt{\|\overline{PQ}\|^2 + \|\overline{QX}\|^2} |\, X \in F\} = \|\overline{PQ}\| = d(P,Q). \end{aligned}$$

■

We will apply this to several particular cases in E_2 and in E_3.

6.2.1 Distance from a point to a line in E_2.

We consider in E_2 the line r of equation:

$$r \equiv ax + by + c = 0$$

and the point $P = (x_0, y_0)$. We denote by $\bar{n} = (a,b)$ the (covariant) coordinates of the vector normal to r.

Let Q be the projection of P onto r and let $A = (x_1, y_1)$ be any point of r. You have:

$$\overline{AP}\cdot\bar{n} = (\overline{AQ}+\overline{QP})\cdot\bar{n} = \overline{AQ}\cdot\bar{n}+\overline{QP}\cdot\bar{n} = \overline{QP}\cdot\bar{n} = \|QP\|\|\bar{n}\|cos(\overline{QP},\bar{n}) = \pm\|QP\|\|\bar{n}\|$$

Therefore:

$$\boxed{d(P,r) = \frac{|\overline{AP}\cdot\bar{n}|}{\|\bar{n}\|}}$$

If we introduce coordinates we have:

$$\overline{AP} = (x_0 - x_1, y_0 - y_1) \quad \Rightarrow \quad \overline{AP}\cdot\bar{n} = ax_0 - ax_1 + by_0 - by_1 = ax_0 + bx_0 + c$$

and we get:

$$\boxed{d(P,r) = \frac{|ax_0 + by_0 + c|}{\|\bar{n}\|}}$$

6.2.2 Distance from a point to a line in E_3.

We consider the line r in E_3, defined by a point A and a direction vector $\bar{u}$. On the other hand, we consider a point $P \in E_3$. We call Q the projection of P on r. Let's see two ways to calculate the distance from P to r:

Method I: Using the cross product.

We have:

$$\|\overline{AP}\wedge\bar{u}\| = \|(\overline{AQ}+\overline{QP})\wedge\bar{u}\| = \|\overline{AQ}\wedge\bar{u} + \overline{QP}\wedge\bar{u}\| = \|\overline{QP}\wedge\bar{u}\| = \|QP\|\|\bar{u}\|$$

Therefore:

$$\boxed{d(P,r) = \frac{\|\overline{AP}\wedge\bar{u}\|}{\|\bar{u}\|}}$$

Method II: Calculating the projection Q of P onto r.

To calculate Q we use the following. Since Q is a point on the line r, it satisfies the parametric equation:

$$Q = A + \alpha\bar{u} \quad \Rightarrow \quad \overline{AQ} = \alpha\bar{u} \quad \Rightarrow \quad \overline{PQ} = \overline{PA} + \alpha\bar{u}$$

We require that $\overline{PQ}$ must be perpendicular to $\bar{u}$ and therefore:

$$\overline{PQ} \cdot \bar{u} = 0 \quad \Rightarrow \quad \overline{PA} \cdot \bar{u} = -\alpha\bar{u} \cdot \bar{u} \quad \Rightarrow \quad \alpha = \frac{-\overline{PA} \cdot \bar{u}}{\|\bar{u}\|^2}$$

Once we know α we also know Q. Now the desired distance is $\|PQ\|$.

6.2.3 Distance from a point to a plane in E_3.

We consider in E_3 the plane π with equation

$$\pi \equiv ax + by + cz + d = 0$$

and the point $P = (x_0, y_0, z_0)$. We denote by $\bar{n} = (a, b, c)$ the covariant coordinates of the vector normal to π.

Let Q be the projection of P onto π and let $A = (x_1, y_1, z_1)$ be any point on π. The argument is exactly the same as in the case of the distance from a point to a line in E_2. We have:

$$\overline{AP}\cdot\bar{n} = (\overline{AQ}+\overline{QP})\cdot\bar{n} = \overline{AQ}\cdot\bar{n}+\overline{QP}\cdot\bar{n} = \overline{QP}\cdot\bar{n} = \|QP\|\|\bar{n}\|cos(\overline{QP}, \bar{n}) = \pm\|QP\|\|\bar{n}\|$$

Therefore:

$$\boxed{d(P, \pi) = \frac{|\overline{AP} \cdot \bar{n}|}{\|\bar{n}\|}}$$

If we introduce coordinates and operate analogously to the case of the line we obtain:

$$\boxed{d(P, r) = \frac{|ax_0 + by_0 + cz_0 + d|}{\|\bar{n}\|}}$$

6.3 Distance between two affine subspaces.

Definition 6.5 *The* **distance between two affine subspaces** *F_1 and F_2 is defined as the smallest of the distances between any point in F_1 and any point in F_2:*

$$d(F_1, F_2) = min\{d(X, Y)|\, X \in F_1,\ Y \in F_2\}$$

Again, intuitively, the smallest distance between F_1 and F_2 is obtained as the length of any segment joining them and which is orthogonal to both subspaces. Specifically, the following result holds:

Theorem 6.6 *Given two affine subspaces F_1 and F_2, there exist two points $Q_1 \in F_1$ and $Q_2 \in F_2$ such that $\overline{Q_1Q_2}$ is perpendicular to F_1 and F_2 and for any pair of such points one has*

$$d(F_1, F_2) = d(Q_1, Q_2).$$

We will apply this result in several particular cases:

6.3.1 Distance between two parallel lines.

The procedure is the same for both E_2 and E_3. The distance between two parallel lines r and s is the distance from any point of s to r:

$$d(r, s) = d(B, r) \text{ with } B \in s.$$

6.3.2 Distance between two parallel planes of E_3.

The distance between two parallel planes π_1 and π_2 of E_3, is the distance from any point from π_2 to π_1:

$$d(\pi_1, \pi_2) = d(B, \pi_1) \text{ with } B \in \pi_2.$$

6.3.3 Distance in E_3 between a plane and a line parallel to it.

The distance between a plane π and a line r of E_3 parallel to it, is the distance from any point of the line r to π:

$$d(\pi, r) = d(B, \pi) \text{ with } B \in r.$$

6.3.4 Distance between two skew lines that intersect at E_3.

Let r and s be two skew lines in E_3 (lines that neither intersect nor are parallel). Suppose that their vector equations are respectively:

$$r \equiv X = A + \alpha\bar{u}_1$$
$$s \equiv X = B + \beta\bar{u}_2$$

We denote by $Q_1 \in r$ and $Q_2 \in s$ the points on the lines, satisfying that $\overline{Q_1Q_2}$ is perpendicular to r and s, and also $d(r, s) = d(Q_1, Q_2)$.

Let us see several methods to calculate the distance between these lines:

Method I:

The steps are:

1. Find the plane π_1 that contains both r and the direction vector $\bar{u}_1 \wedge \bar{u}_2$.

2. $Q_2 = \pi_1 \cap s$.
3. Finally $d(r, s) = d(Q_2, r)$.

Method II:

The steps are:

1. Find the plane π_1 that contains r and the direction vector $\bar{u}_1 \wedge \bar{u}_2$.
2. Then $Q_2 = \pi_1 \cap s$.
3. Calculate the plane π_2 that contains s and the direction vector $\bar{u}_1 \wedge \bar{u}_2$.
4. Then $Q_1 = \pi_2 \cap r$.
5. Finally $d(r, s) = d(Q_1, Q_2)$.

Method III:

The steps are:

1. The point Q_1, because it is in r, has the form $Q_1 = A + \alpha\bar{u}_1$.
2. The point Q_2, because it is in s, has the form $Q_2 = B + \beta\bar{u}_2$.
3. The vector $\overline{Q_1Q_2}$ will be $B - A + \beta\bar{u}_2 - \alpha\bar{u}_1$.
4. We plug in the condition that $\overline{Q_1Q_2}$ must be perpendicular to $\bar{u}_1$ and $\bar{u}_2$. This gives us two equations which we will solve for α and β:
$$\begin{array}{l}(B - A + \beta\bar{u}_2 - \alpha\bar{u}_1) \cdot \bar{u}_1 = 0 \\ (B - A + \beta\bar{u}_2 - \alpha\bar{u}_1) \cdot \bar{u}_2 = 0\end{array}$$
5. Once α and β are known, we also know Q_1 and Q_2 and therefore $d(r, s) = d(Q_1, Q_2)$.

Method IV: This method is similar to those seen for the distance from a point to a line or a plane. By using the cross product, we have:

$$[\overline{AB}, \bar{u}_1, \bar{u}_2] = \overline{AB} \cdot (\bar{u}_1 \wedge \bar{u}_2) = (\overline{AQ_1} + \overline{Q_1Q_2} + \overline{Q_2B}) \cdot (\bar{u}_1 \wedge \bar{u}_2) = \overline{Q_1Q_2} \cdot (\bar{u}_1 \wedge \bar{u}_2)$$

Since the distance between r and s is $\|\overline{Q_1Q_2}\|$ we derive the following formula:

$$\boxed{d(r, s) = \frac{|[\overline{AB}, \bar{u}_1, \bar{u}_2]|}{\|\bar{u}_1 \wedge \bar{u}_2\|}}$$

7 Affine transformations.

Definition 7.1 *Given an affine space E associated to a vector space V and an automorphism $t : V \longrightarrow V$, an* **affine transformation** *associated to t is an application $A : E \longrightarrow E$ satisfying:*

$$\forall P, Q \in E, \text{ if } A(P) = P' \text{ and } A(Q) = Q' \text{ then } t(\overline{PQ}) = \overline{P'Q'}.$$

With this definition, if A is an affine transformation, the image by A of any point $X \in E$ is calculated as follows. Once fixed a point $P \in X$, if we call $X' = A(X)$ and $P' = A(P)$:

$$t(\overline{PX}) = \overline{P'X'} = X' - P' \quad \Rightarrow \quad X' = P' + t(\overline{PX}) \quad \Rightarrow \quad \boxed{A(X) = A(P) + t(\overline{PX})}$$

Points that are transformed into themselves by an affine transformation are called **double points** or **fixed points**.

Let us look at some important types of affine transformations.

7.1 Translations.

Definition 7.2 *A* **translation** *by the vector $\bar{v}$ is an affine transformation that takes each point X to the point $X + \bar{v}$.*

What a translation does is moving each point of E in the direction defined by the vector $\bar{v}$. Since it is an affine transformation it will have an associated automorphism $t : V \longrightarrow V$. Let's see what this one is. Given two points $P, Q \in E$, we have:

$$\begin{array}{l}P' = P + \bar{v} \quad \Rightarrow \quad \bar{v} = \overline{PP'} \\ Q' = Q + \bar{v} \quad \Rightarrow \quad \bar{v} = \overline{QQ'}\end{array}$$

Now
$$t(\overline{PQ}) = \overline{P'Q'} = \overline{P'P} + \overline{PQ} + \overline{QQ'} = -\bar{v} + \overline{PQ} + \bar{v} = \overline{PQ}$$

We deduce that the automorphism t is the **identity** mapping.

It is clear that if $\bar{v} \neq 0$ then the translation defined by $\bar{v}$ has no fixed points.

Finally if we introduce a reference $\{O; \bar{e}_1, \ldots, \bar{e}_n\}$, the equation of the translation in coordinates is:

$$(x'^1, \ldots, x'^n) = (x^1, \ldots, x^n) + (v^1, \ldots, v^n)$$

where $(v^1, \ldots, v^n)$ are the coordinates of the vector $\bar{v}$ defining the translation.

7.2 Homotheties.

Definition 7.3 *Given a point $C \in E$ and a real number $k \neq 0, 1$, the* **homothety** *with center C and ratio k is defined as the transformation which takes each point X to*

$$X' = C + k\overline{CX} \quad \text{or equivalently} \quad \overline{CX'} = k\overline{CX}.$$

What a homothety does is increasing ($|k| > 1$) or decreasing ($|k| < 1$) the distance of all points in the Euclidean space with respect to a fixed center. In this way the distances of the images are multiplied by $|k|$, the areas by k^2 and the volumes by $|k|^3$.

It is clear that with this definition **the center C is a double point of the homothety**. Let us see that actually it is **the only one**. If X is a double point then

$$X = C + k\overline{CX} \;\Rightarrow\; \overline{CX} = k\overline{CX} \;\Rightarrow\; (1-k)\overline{CX} = 0 \;\Rightarrow\; \begin{cases} k = 1 \\ \text{or} \\ \overline{CX} = \bar{0} \end{cases}$$

As we assume $k \neq 1$ we deduce that $X = C$.

A homothety, being an affine transformation, will have an associated automorphism $t : V \longrightarrow V$. Let's see what it is. Given any $X \in E$ we have:

$$t(\overline{CX}) = \overline{C'X'} = \overline{CX'} = k\overline{CX}$$

We deduce that $t = k \cdot Id$.

Finally let us see what the equations of the homothety with respect to a given reference $\{O; \bar{e}_1, \ldots, \bar{e}_n\}$ look like. We have

$$\overline{OX'} = \overline{OC} + \overline{CX'} = \overline{OC} + k\overline{CX} = \overline{OC} + k(\overline{OX} - \overline{OC})$$

Hence

$$(x'^1, \ldots, x'^n) = (c^1, \ldots, c^n) + k(x^1 - c^1, \ldots, x^n c^n)$$

where $(c^1, \ldots, c^n)$ are the coordinates of the center C of the homothety.

7.3 Affine transformations associated with orthogonal maps: isometries.

Definition 7.4 *An* **isometry** *of a Euclidean affine space E is a distance-preserving affine transformation $A : E \longrightarrow E$. That is, it satisfies:*

$$d(A(P), A(Q)) = d(P, Q) \text{ for any pair of points } P, Q \in E.$$

It can be proved that an isometry of E_2 or E_3 is a composition of translations, rotations and/or symmetries. Specifically, the following rotations or symmetries may appear:

1. Rotations in E_2: Once fixed a center C and angle α the rotation of center C and angle α is constructed as:

$$X' = C + t_\alpha(\overline{CX})$$

where t_α is the rotation in $\mathbb{R}^2$ of angle α, with the corresponding orientation.
The only double point in this case is the center of rotation C.

2. Rotations in E_3: Once fixed a semi-axis r and an angle α the rotation of axis r and angle α is constructed as as:

$$X' = P + t_\alpha(\overline{PX})$$

where P is any point of r and t_α is the rotation in $\mathbb{R}^3$ of angle α with respect to the direction vector of r.
The double points are all the points on the line axis r.
It must be taken into account that the rotation is defined with respect to a semi-axis, so that the direction vector of r is chosen in a certain direction. This allows us to set the orientation of the rotation.

3. Orthogonal symmetries: Fixed an affine subspace $F = P + U$, the orthogonal symmetry with respect to f is constructed as:

$$X' = P + t_U(\overline{PX})$$

where t_U is the orthogonal symmetry in V with respect to the subspace U.