4. Vector product and mixed product.

In this chapter U will denote an Euclidean space of dimension 3.

1 Vector product.

1.1 Definition.

Definition 1.1 *Let $\bar{x}, \bar{y}$ be two vectors in U and suppose we fix the basis $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ as a reference. We define the* **vector product** *or* **cross product** $\bar{x} \wedge \bar{y}$ *of the vectors $\bar{x}$ and $\bar{y}$ as follows:*

1. *If $\bar{x}, \bar{y}$ are dependent, then $\bar{x} \wedge \bar{y} = \bar{0}$.*
2. *If $\bar{x}, \bar{y}$ are independent, then $\bar{x} \wedge \bar{y}$ is the only vector satisfying*
   (a) *$\bar{x} \wedge \bar{y}$ is orthogonal to both $\bar{x}$ and $\bar{y}$.*
   (b) *$\|\bar{x} \wedge \bar{y}\| = \|\bar{x}\| \cdot \|\bar{y}\| \cdot |\sin(\bar{x}, \bar{y})|$.*
   (c) *The basis $\{\bar{x}, \bar{y}, \bar{x} \wedge \bar{y}\}$ has the same orientation as B.*

This definition is consistent because given two independent vectors $\bar{x}, \bar{y}$ in a 3-dimensional Euclidean space, the space of vectors perpendicular to both has dimension 1. Therefore there is a unique vector in this subspace with a given norm and a given direction.

1.2 Analytical expression.

Suppose we fix an **orthonormal** basis $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ which we choose as a positive orientation reference. Then

Theorem 1.2 *The vector product of the vectors $\vec{x} = (x^1, x^2, x^3)_B$ and $\vec{y} = (y^1, y^2, y^3)_B$ can be obtained from the following determinant-like expression:*

$$\boxed{\bar{x} \wedge \bar{y} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix}}$$

Proof: First, let us note that even though the first row of this determinant is formed by vectors, the expression itself makes sense. It is enough to take into account that in each term of the expansion of the determinant there appears only one element from the first row; the corresponding term must be interpreted as a vector multiplied by a scalar, which is an operation that makes sense in a vector space.

If we put $A = \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{pmatrix}$ the theorem states that:

$$\bar{x} \wedge \bar{y} = \bar{z} \text{ with } \bar{z} = (A_{11}, A_{12}, A_{13})_B,$$

where A_{ij} is the cofactor corresponding to the position ij of A. Let us see that $\bar{z}$ meets the definition of vector product:

1. If $\bar{x}, \bar{y}$ are dependent, then $\bar{z} = 0$ because in this case the last two rows of A are linearly dependent and therefore all cofactors are null.

2. (a) We have:

$$\begin{aligned} \bar{x} \cdot \bar{z} &= x^1 A_{11} + x^2 A_{22} + x^3 A_{33} = \begin{vmatrix} x^1 & x^2 & x^3 \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix} = 0 \\ \bar{y} \cdot \bar{z} &= y^1 A_{11} + y^2 A_{22} + y^3 A_{33} = \begin{vmatrix} y^1 & y^2 & y^3 \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix} = 0 \end{aligned}$$

2. (b) We want to check that $\|\bar{x} \wedge \bar{y}\| = \|\bar{x}\|\|\bar{y}\||\sin(\bar{x}, \bar{y})|$. This is clearly equivalent to

$$\begin{aligned} \|\bar{x} \wedge \bar{y}\|^2 = \|\bar{x}\|\|\bar{y}\|^2(1 - \cos^2(\bar{x}, \bar{y})) &= \|\bar{x}\|^2\|\bar{y}\|^2 - \|\bar{x}\|^2\|\bar{y}\|^2 \cos^2(\bar{x}, \bar{y}) = \\ &= \|\bar{x}\|^2\|\bar{y}\|^2 - (\bar{x} \cdot \bar{y})^2 \end{aligned}$$

Note that from the proposed formula it follows

$$\|\bar{x} \wedge \bar{y}\|^2 = (x_1 y_2 - x_2 y_1)^2 + (x_1 y_3 - x_3 y_1)^2 + (x_2 y_3 - x_3 y_2)^2$$

and on the other hand

$$\|\bar{x}\|^2 = x_1^2 + x_2^2 + x_3^2, \quad \|\bar{y}\|^2 = y_1^2 + y_2^2 + y_3^2, \quad \bar{x} \cdot \bar{y} = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

From here it is easy to deduce the required identity.

2. (c) In order for the basis $B' = \{\bar{x}, \bar{y}, \bar{x} \wedge \bar{y}\}$ to have the same orientation as B, the change-of-basis matrix $M_{BB'}$ must have a positive determinant. But since $\bar{x} \wedge \bar{y} = (A_{11}, A_{12}, A_{13})_B$,

$$|M_{BB'}| = |M_{BB'}^t| = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ A_{11} & A_{12} & A_{13} \end{vmatrix}$$

and expanding through the last row we obtain

$$|M_{BB'}| = A_{11}^2 + A_{12}^2 + A_{13}^2 > 0.$$

1.3 Properties.

Taking into account the analytical expression of the vector product, it is easy to deduce the following properties:

1. $\bar{x} \wedge \bar{y} = \bar{0} \iff \bar{x}, \bar{y}$ *are dependent.*
2. *The vector product is bilinear, that is:*

$$\bar{x} \wedge (\alpha\bar{y} + \beta\bar{y}') = \alpha\bar{x} \wedge \bar{y} + \beta\bar{x} \wedge \bar{y}'.$$

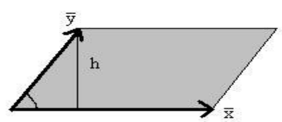
$$(\alpha\bar{x} + \beta\bar{x}') \wedge \bar{y} = \alpha\bar{x} \wedge \bar{y} + \beta\bar{x}' \wedge \bar{y}.$$

3. *The vector product is antisymmetric, that is:*

$$\bar{x} \wedge \bar{y} = -\bar{y} \wedge \bar{x}.$$

1.4 Geometric interpretation.

Theorem 1.3 *The norm of the vector product of two vectors $\bar{x}$, $\bar{y}$, coincides with the area of the parallelogram they determine:*



Proof: We have:

$$\text{Area of parallelogram} = \|\bar{x}\|h = \|\bar{x}\|\|\bar{y}\||\sin(\bar{x}, \bar{y})| = \|\bar{x} \wedge \bar{y}\|.$$

2 Mixed Product

Definition 2.1 *We define the* **mixed product** *of three vectors $\bar{x}, \bar{y}, \bar{z}$ of a 3-dimensional Euclidean space as:*

$$[\bar{x}, \bar{y}, \bar{z}] = (\bar{x} \wedge \bar{y}) \cdot \bar{z}.$$

As a consequence of the analytical expression we have obtained for the vector product, we deduce the following expression for the mixed product:

Theorem 2.2 *Let $\bar{x}, \bar{y}, \bar{z}$ be three vectors of U and B a basis. Then*

$$[\bar{x}, \bar{y}, \bar{z}] = \sqrt{|G_B|}\begin{vmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \\ z^1 & z^2 & z^3 \end{vmatrix} \qquad [\bar{x}, \bar{y}, \bar{z}] = \frac{1}{\sqrt{|G_B|}}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

The mixed product has the following properties:

1. $[\bar{x}, \bar{y}, \bar{z}] = 0 \iff \{\bar{x}, \bar{y}, \bar{z}\}$ *are dependent.*
2. *It is trilinear, that is:*

$$[\alpha\bar{x} + \beta\bar{x}', \bar{y}, \bar{z}] = \alpha[\bar{x}, \bar{y}, \bar{z}] + \beta[\bar{x}', \bar{y}, \bar{z}].$$

$$[\bar{x}, \alpha\bar{y} + \beta\bar{y}', \bar{z}] = \alpha[\bar{x}, \bar{y}, \bar{z}] + \beta[\bar{x}, \bar{y}', \bar{z}].$$

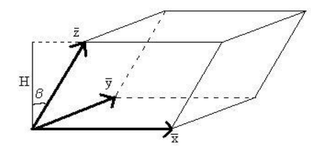
$$[\bar{x}, \bar{y}, \alpha\bar{z} + \beta\bar{z}'] = \alpha[\bar{x}, \bar{y}, \bar{z}] + \beta[\bar{x}, \bar{y}, \bar{z}'].$$

3. *It is antisymmetric, that is:*

$$[\bar{x}, \bar{y}, \bar{z}] = -[\bar{y}, \bar{x}, \bar{z}] = -[\bar{x}, \bar{z}, \bar{y}] = -[\bar{z}, \bar{y}, \bar{x}].$$

Finally, the mixed product of three vectors admits the following geometric interpretation.

Theorem 2.3 *The absolute value of the mixed product of three vectors $\bar{x}, \bar{y}, \bar{z}$ coincides with the volume of the parallelepiped they determine.*



Proof: We have:

Volume of the parallelepiped = Base area$\cdot H = \|\bar{x}\wedge\bar{y}\|\|z\|\cos(\theta) = |(\bar{x}\wedge\bar{y})\cdot\bar{z}| = |[\bar{x}, \bar{y}, \bar{z}$

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