3. Orthogonal transformations.

Throughout the chapter U will denote an Euclidean vector space.

1 Definition.

Definition 1.1 *A* **orthogonal transformation** *f of an Euclidean space U is an endomorphism that preserves the scalar product.*

$$f(\bar{x}) \cdot f(\bar{y}) = \bar{x} \cdot \bar{y} \quad \textit{for any } \bar{x}, \bar{y} \in U.$$

Let us see that preservation of the scalar product is actually sufficient, that is:

Proposition 1.2 *If a mapping $f : U \longrightarrow U$ preserves the scalar product then it is a linear mapping.*

Proof: Fix $\bar{x}, \bar{y} \in U$, $\alpha, \beta \in \mathbb{R}$. We want to prove that

$$f(\alpha\bar{x} + \beta\bar{y}) = \alpha f(\bar{x}) + \beta f(\bar{y}).$$

We have

$$(f(\alpha\bar{x} + \beta\bar{y}) - \alpha f(\bar{x}) - \beta f(\bar{y})) \cdot (f(\alpha\bar{x} + \beta\bar{y}) - \alpha f(\bar{x}) - \beta f(\bar{y})) =$$

$$= f(\alpha\bar{x} + \beta\bar{y}) \cdot f(\alpha\bar{x} + \beta\bar{y}) + \alpha^2 f(\bar{x}) \cdot f(\bar{x}) + \beta^2 f(\bar{y}) \cdot f(\bar{y}) - $$
$$-2\alpha f(\alpha\bar{x} + \beta\bar{y}) \cdot f(\bar{x}) - 2\beta f(\alpha\bar{x} + \beta\bar{y}) \cdot f(\bar{y}) + 2\alpha\beta f(\bar{x}) \cdot f(\bar{y}) =$$

$$= (\alpha\bar{x} + \beta\bar{y}) \cdot (\alpha\bar{x} + \beta\bar{y}) + \alpha^2 \bar{x} \cdot \bar{x} + \beta^2 \bar{y} \cdot \bar{y} -$$
$$-2\alpha(\alpha\bar{x} + \beta\bar{y}) \cdot \bar{x} - 2\beta(\alpha\bar{x} + \beta\bar{y}) \cdot \bar{y} + 2\alpha\beta\bar{x} \cdot \bar{y} =$$

$$= ((\alpha\bar{x} + \beta\bar{y}) - \alpha\bar{x} - \beta\bar{y}) \cdot ((\alpha\bar{x} + \beta\bar{y}) - \alpha\bar{x} - \beta\bar{y}) = 0$$

So $f(\alpha\bar{x} + \beta\bar{y}) - \alpha f(\bar{x}) - \beta f(\bar{y}) = 0$. ■

Using this proposition we can give a characterization of orthogonal transformations:

Theorem 1.3 *$f : U \longrightarrow U$ is an orthogonal transformation if and only if it transforms orthonormal bases into orthonormal bases.*

Proof: If f is an orthogonal transformation, it is clear that it takes orthonormal bases into orthonormal bases. Just take into account that f preserves the scalar product and that an orthogonal system is linearly independent.

Let us try the converse. Bearing in mind the previous proposition, it is enough to assume that f takes orthonormal bases into orthonormal bases and prove that then f preserves the scalar product.

Let $B = \{\bar{u}_1, \ldots, \bar{u}_n\}$ be an orthonormal basis of U and $f(B) = \{f(\bar{u}_1), \ldots, f(\bar{u}_n)\}$ the orthonormal basis image of B. Given $\bar{x}, \bar{y} \in U$ with coordinates $(x^i), (y^j)$ with respect to the basis B, we have:

$$f(\bar{x}) \cdot f(\bar{y}) = f(x^i \bar{u}_i) \cdot f(y^j \bar{u}_j) = x^i y^j f(\bar{u}_i) \cdot f(\bar{u}_j).$$

Taking into account that the bases B and $f(B)$ are orthonormal, we deduce the following:

$$f(\bar{x}) \cdot f(\bar{y}) = x^i y^j f(\bar{u}_i) \cdot f(\bar{u}_j) = x^i y^j \delta_{ij} = x^i y^j \bar{u}_i \cdot \bar{u}_j = (x^i \bar{u}_i) \cdot (y^j \bar{u}_j) = \bar{x} \cdot \bar{y}.$$

■

2 Properties.

Let $f : U \longrightarrow U$ be an orthogonal transformation. Let us see some of the main properties it satisfies.

1. *It preserves the scalar product.*
2. *It ransforms orthonormal bases into orthonormal bases.*
3.
4. *It preserves the norm:*

$$\|f(\bar{x})\|^2 = f(\bar{x}) \cdot f(\bar{x}) = \bar{x} \cdot \bar{x} = \|\bar{x}\|^2.$$

5. *It preserves angles:*

$$cos(f(\bar{x}), f(\bar{y})) = \frac{f(\bar{x}) \cdot f(\bar{y})}{\|f(\bar{x})\|\|f(\bar{y})\|} = \frac{\bar{x} \cdot \bar{y}}{\|\bar{x}\|\|\bar{y}\|} = cos(\bar{x}, \bar{y}).$$

6. *It is bijective.*
   Proof: It is onto because it takes orthonormal bases into orthonormal bases. By the dimensions formula, it is also injective.
   We can also see directly that it is injective by checking that the kernel is $\{0\}$.

$$f(\bar{x}) = \bar{0} \quad \Rightarrow \quad \bar{x} \cdot \bar{x} = f(\bar{x}) \cdot f(\bar{x}) = 0 \quad \Rightarrow \quad \bar{x} = \bar{0}.$$

7. *The composition of orthogonal transformations is an orthogonal transformation.*
   Proof: If f and g are orthogonal transformations, we have:

$$(g \circ f)(\bar{x}) \cdot (g \circ f)(\bar{y}) = g(f(\bar{x})) \cdot g(f(\bar{y})) = f(\bar{x}) \cdot f(\bar{y}) = \bar{x} \cdot \bar{y}.$$

8. *The inverse of an orthogonal transformation is an orthogonal transformation.*
   Proof: If f is orthogonal

$$f^{-1}(\bar{x}) \cdot f^{-1}(\bar{y}) = f(f^{-1}(\bar{x})) \cdot f(f^{-1}(\bar{y})) = \bar{x} \cdot \bar{y}.$$

9. *If f is an endomorphism and B has a basis of U, we have:*

$$\boxed{f \text{ is an orthogonal transformation} \iff {F_B}^t G_B F_B = G_B}$$

Proof: We have:

$$\begin{aligned} f \text{ orthogonal} &\iff f(\bar{x}) \cdot f(\bar{y}) = \bar{x} \cdot \bar{y}, \quad \forall \bar{x}, \bar{y} \in U \iff \\ &\iff (F_B(x))^t G_B F_B(y) = (x) G_B (y)^t \quad \forall \bar{x}, \bar{y} \in U \iff \\ &\iff (x)^t {F_B}^t G_B F_B (y) = (x) G_B (y)^t \quad \forall \bar{x}, \bar{y} \in U \iff \\ &\iff {F_B}^t G_B F_B = G_B. \end{aligned}$$

10. *If f is an endomorphism and B is an orthonormal basis of U, we have:*

$$\boxed{f \text{ orthogonal transformation} \iff {F_B}^t F_B = Id \iff F_B \text{ is orthogonal}}$$

Proof: It is a consequence of the previous property, taking into account that the Gram matrix relative to an orthonormal basis is the identity.

11. *If f is an orthogonal transformation and B a basis of U, then $|F_B| = \pm 1$.*

Proof: We have:

$$|G_B| = |{F_B}^t G_B F_B| = |F_B||G_B||F_B| \quad \Rightarrow \quad |F_B|^2 = 1 \quad \Rightarrow \quad |F_B| = \pm 1.$$

3 Eigenvalues and eigenvectors.

Proposition 3.1 *Any real* **eigenvalue** *of an orthogonal transformation is either* 1 *or* -1.

Proof: If $t : U \longrightarrow U$ is an orthogonal transformation, λ is a real eigenvalue and $\bar{x}$ is a nonzero eigenvector associated to λ, we have:

$$t(\bar{x}) \cdot t(\bar{x}) = \bar{x} \cdot \bar{x} \quad \Rightarrow \quad \lambda^2 \bar{x} \cdot \bar{x} = \bar{x} \cdot \bar{x} \quad \Rightarrow \quad \lambda^2 = 1.$$

■

Proposition 3.2 *If $t : U \longrightarrow U$ is an orthogonal transformation, the* **characteristic subspaces** *associated with dfferent eigenvalues are orthogonal.*

Proof: Recall that t can only have 1 and -1 as its real eigenvalues. Let $\bar{x} \in S_1$, $\bar{y} \in S_{-1}$. We have

$$\bar{x} \cdot \bar{y} = t(\bar{x}) \cdot t(\bar{y}) = -\bar{x} \cdot \bar{y} \quad \Rightarrow \quad 2\bar{x} \cdot \bar{y} = 0 \quad \Rightarrow \quad \bar{x} \cdot \bar{y} = 0.$$

Therefore S_1 and S_{-1} are orthogonal. ■

Theorem 3.3 *If an orthogonal transformation t is diagonalizable, then it admits an orthonormal basis of eigenvectors.*

Proof: By the previous theorem, the characteristic subspaces corresponding to different eigenvalues are orthogonal. Therefore, if the transformation is diagonalizable, we can build a basis of orthogonal eigenvectors by joining bases of each one of the two characteristic subspaces formed by orthogonal eigenvectors. ■

4 Relative orientation of the bases.

Definition 4.1 *Let U be a finite-dimensional real vector space. Two bases B and B' are said to have the same* **orientation** *if the corresponding change-of-basis matrix has a positive determinant:*

$$|M_{BB'}| > 0$$

Proposition 4.2 *The relation "to have the same orientation" is an equivalence relation. Furthermore, its quotient set has two elements.*

Proof: Let us see that it satisfies the three properties that characterize equivalence relations.

1. Reflexive: $M_{BB} = Id \Rightarrow |M_{BB}| = 1 \Rightarrow$ every basis B has the same orientation as itself.
2. Symmetric: if B, B' have the same orientation, then $|M_{B'B}| > 0$, and:

$$|M_{B'B}| = |{M_{BB'}}^{-1}| = \frac{1}{|M_{BB'}|} > 0.$$

3. Transitive:

$$\left.\begin{array}{l} B, B' \text{ have the same orientation} \quad \Rightarrow \quad |M_{BB'}| > 0 \\ B', B'' \text{ have the same orientation} \quad \Rightarrow \quad |M_{B'B''}| > 0 \end{array}\right\} \Rightarrow$$
$$\Rightarrow \quad |M_{BB''}| = |M_{BB'} M_{B'B''}| = |M_{BB'}||M_{B'B''}| > 0.$$

Therefore B and B'' have the same orientation.

Now let us prove that there are only two equivalence classes. Let $B = \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n\}$ be a basis of U. It is clear that the basis:

$$B' = \{-\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n\}$$

has a different orientation than B', because the change-of-basis matrix is:

$$M_{BB'} = \begin{pmatrix} -1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \quad \Rightarrow \quad |M_{B'B}| = -1.$$

Let us see that any other basis B'' has the same orientation as one of these two. If B'' has a different orientation than B, then $|M_{B''B}| < 0$ and:

$$|M_{B'B''}| = |M_{B'B}||M_{BB''}| > 0.$$

and therefore B'' has the same orientation as B'.

5 Direct and inverse orthogonal transformations.

Definition 5.1 *Let $t : U \longrightarrow U$ be an orthogonal transformation, B a basis of U and let T_B the matrix associated to t with respect to the basis B. We say that t is*

- *a* **direct** *transformation if $|T_B| = 1$.*
- *an* **inverse** *transformation if $|T_B| = -1$.*

Bearing in mind the change-of-basis formula for the matrix associated with t

$$T'_B = (M_{BB'})^{-1} T_B M_{BB'},$$

we see that the definition does not depend on the chosen basis, since similar matrices have the same determinant.

On the other hand, we can give the following characterization of direct and inverse transformations:

Proposition 5.2 *Let $t : U \longrightarrow U$ be an orthogonal transformation. Then t is*

- **direct** *if and only if it preserves the orientation of the bases.*
- **inverse** *if and only if it changes the orientation of the bases.*

Proof: Let $B = \{\bar{u}_1, \dots, \bar{u}_n\}$ be a basis of U. Let $t(B) = \{t(\bar{u}_1), \dots, t(\bar{u}_n)\}$ be the image of this basis. If T_B is the matrix of the application t with respect to the basis B, we have:

$$(t(\bar{u}_j) = (\bar{u}_i)T_B.$$

Therefore, the change-of-basis matrix between B and $t(B)$ is:

$$M_{Bt(B)} = T.$$

So:

$$\begin{array}{ccccc} t \text{ direct} & \Longleftrightarrow & |M_{Bt(B)}| = |T| = 1 & \Longleftrightarrow & B \text{ and } t(B) \text{ have the same orientation.} \\ t \text{ inverse} & \Longleftrightarrow & |M_{Bt(B)}| = |T| = -1 & \Longleftrightarrow & B \text{ and } t(B) \text{ have different orientations.} \end{array}$$

■

The following properties hold:

1. *The composition of two direct orthogonal transformations is direct.*
2. *The composition of two inverse orthogonal transformations is direct.*
3. *The composition of two orthogonal transformations, of which one is direct and the other one is inverse, is inverse.*
4. *The inverse of a direct orthogonal transformation is direct.*
5. *The inverse of an inverse orthogonal transformation is inverse.*

6 Orthogonal transformations on $\mathbb{R}^2$.

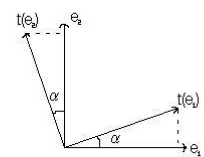
We will start by describing all orthogonal transformations of $\mathbb{R}^2$

6.1 Rotations on $\mathbb{R}^2$.

Theorem 6.1 *Let $B = \{\bar{e}_1, \bar{e}_2\}$ be an* **orthonormal** *basis of $\mathbb{R}^2$ and α any angle. The associated matrix of the* **rotation of angle α with respect to the orientation given by the basis** *B is*

$$T_B = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Proof: If we consider the following drawing:



$$\begin{array}{rcl} t(\bar{e}_1) & = & \cos(\alpha)\bar{e}_1 + \sin(\alpha)\bar{e}_2 \\ t(\bar{e}_2) & = & -\sin(\alpha)\bar{e}_1 + \cos(\alpha)\bar{e}_2 \end{array}$$

Therefore we obtain the matrix T_B described in the statement. ■

6.1.1 Procedure to find the matrix of a rotation of a given angle with respect to an arbitrary basis.

Suppose we are given a **not necessarily orthonormal** basis $B = \{\bar{e}_1, \bar{e}_2\}$ and we are asked to calculate the matrix of a rotation of angle α relative to this basis and with respect to the orientation it defines.

The steps to obtain the matrix are the following:

1. Find an orthonormal basis $B' = \{\bar{u}'_1, \bar{u}'_2\}$.

2. Check that this basis has the same orientation as the initial one, that is, $|M_{B'B}| > 0$. If it does not, we change the vector $\bar{u}_1'$ by $-\bar{u}_1'$.
3. The rotation matrix with respect to the basis B' is:

$$T_{B'} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

4. We perform a change of basis on the matrix $T_{B'}$:

$$T_B = M_{BB'} T_{B'} (M_{BB'})^{-1}.$$

It must be taken into account that, in all the steps where the scalar product (or norms) appears, we need to use the Gram matrix corresponding to the basis in which one is working. If this basis is orthonormal, the Gram matrix is the identity and the scalar product is the usual one.

6.2 Symmetries in $\mathbb{R}^2$.

Theorem 6.2 *Let $B = \{\bar{e}_1, \bar{e}_2\}$ be a* **orthogonal** *basis of $\mathbb{R}^2$. The matrix associated to the* **symmetry with respect to the axis generated by** $\bar{e}_1$ *is*

$$T_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proof: Just take into account that the axis of symmetry is invariant, while the vectors orthogonal to it have their sign changed:

$$t(\bar{e}_1) = \bar{e}_1; \quad t(\bar{e}_2) = -\bar{e}_2.$$

Remark 6.3 *We can also define* **a symmetry about the origin**, *but it is the same transformation as a rotation of angle π. The matrix associated with this symmetry with respect to any basis is $-Id$.*

6.2.1 Procedure to find the matrix of a symmetry in any basis.

Suppose we are given a **not necessarily orthonormal** basis $B = \{\bar{e}_1, \bar{e}_2\}$ and we are asked to calculate the matrix relative to B of a symmetry about the axis generated by a vector $\bar{u}_1$.

The steps to find this matrix are the following:

1. We complete $\bar{u}_1$ to an orthogonal basis: $B' = \{\bar{u}_1, \bar{u}_2\}$.
2. The symmetry matrix with respect to this basis will be

$$TB_{B'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3. We perform the change of basis on the matrix $T_{B'}$:

$$T_B = M_{BB'} T_{B'} (M_{BB'})^{-1}.$$

6.3 Classification of orthogonal transformations on $\mathbb{R}^2$.

Theorem 6.4 *Every orthogonal transformation t on $\mathbb{R}^2$, other than the identity, is a rotation or a symmetry about an axis.*

Proof: Let T be the matrix of t with respect to an orthonormal basis B. We know that

$$T^t T = Id. \qquad |T| = \pm 1. \qquad \text{The real eigenvalues of } T \text{ are } 1 \text{ or } -1.$$

Only the following possibilities exist:

1. T has a unique real eigenvalue $\lambda = 1$ with multiplicity 2. Then T is similar to the identity and therefore actually $T = Id$.
2. T has a unique real eigenvalue $\lambda = -1$ with multiplicity 2. So T is similar to $-Id$ and indeed $T = -Id$. This is a rotation of angle π or equivalently a symmetry with respect to the origin.
3. T has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. Then T is similar to the matrix

$$TB_{B'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In particular, there is an orthonormal basis B' with respect to which the matrix of t is the previous one. We deduce that it is a symmetry with respect to an axis. Bearing in mind that this axis is formed by vectors which remain invariant under this symmetry, we deduce that the axis is the characteristic subspace S_1.

4. T has no real eigenvalues. Suppose T has the form

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Since $TT^t = Id$

$$a^2 + c^2 = 1; \; ab + cd = 0; \; b^2 + d^2 = 1;$$

Furthermore, the characteristic polynomial is:

$$|T - \lambda Id| = \lambda^2 - (a+d)\lambda + ad - bc = 0$$

Taking into account that the discriminant is $(a+d)^2 - 4(ad - bc)$, since it does not have any real eigenvalue we deduce

$$ad - bc > 0.$$

We can put

$$a = \cos(\alpha), \quad c = \sin(\alpha); \qquad b = \sin(\beta), \quad d = \cos(\beta).$$

for certain values $\alpha, \beta \in \mathbb{R}$.

Now:

$$\begin{array}{lcl} ab + cd = 0 & \Rightarrow & \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta) = 0 \Rightarrow \sin(\alpha+\beta) = 0. \\ ad - bc > 0 & \Rightarrow & \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) > 0 \Rightarrow \cos(\alpha+\beta) > 0. \end{array}$$

We deduce that $\alpha + \beta = 0$ and therefore:

$$\begin{array}{l} b = \sin(\beta) = \sin(-\alpha) = -\sin(\alpha) = -a. \\ d = \cos(\beta) = \cos(-\alpha) = \cos(\alpha) = c. \end{array}$$

We see that T can be written as:

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

and therefore t is a rotation of angle α with respect to the orientation given by the basis B. ■

We summarize the classification in the following table:

Orthogonal transformations in $\mathbb{R}^2$		
Eigenvalues	**Classification**	**Type**
$\{1,1\}$	*Identity*	Direct
$\{1,-1\}$	*Symmetry with respect to* S_1.	Inverse
$\{-1,-1\}$	*Symmetry about the origin.* ⇕ Rotation of angle π.	Direct
not real	*Angle rotation* α: $T_B = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$ B **ORTHONORMAL** basis	Direct

7 Orthogonal transformations on $\mathbb{R}^3$.

We will start by describing the orthogonal transformations of $\mathbb{R}^3$.

7.1 Rotations in $\mathbb{R}^3$.

Theorem 7.1 *Let $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ be an* **orthonormal** *basis of $\mathbb{R}^3$ and α any angle. The* **rotation of angle α about the semi-axis generated by $\bar{e}_1$ and with respect to the orientation given by the basis** *B has the associated matrix:*

$$T_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Proof: It is enough to take into account that the semi-axis of rotation $\bar{e}_1$ remains invariant due to the rotation. Besides, we apply the rotation matrix on $\mathbb{R}^2$ we have seen in the previous section to the perpendicular plane generated by the orthonormal basis $\{\bar{e}_2, \bar{e}_3\}$.

Remark 7.2 *It must be taken into account that in order for a rotation in $\mathbb{R}^3$ to be well defined the following information is essential:*

1. *the semi-axis of rotation (not just the axis).*
2. *the orientation that serves as a reference.*
3. *the angle of rotation.*

The first two data allow us to identify in which direction the positive angles are taken.

7.1.1 Procedure to find the matrix of a rotation in $\mathbb{R}^3$ relative to an arbitrary basis.

Suppose we are given a **not necessarily orthonormal** basis $B = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ and we are asked to calculate the matrix with respect to the basis B of a rotation of angle α, semiaxis $\bar{v}$ and with the orientation given by the basis B.

The steps to find this matrix are the following.

1. Find an **orthonormal** basis $B' = \{\bar{v}_1', \bar{v}_2', \bar{v}_3'\}$, such that

$$\bar{v}_1' = \frac{\bar{v}}{\|\bar{v}\|}$$

2. Check that this basis has the same orientation as the starting one, that is, $|M_{BB'}| > 0$. If it did not have it, we change the vector $\bar{v}_2'$ to $-\bar{v}_2'$.
3. The rotation matrix with respect to the basis B' is:

$$T_{B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

4. We perform the change of basis on the matrix $T_{B'}$:

$$T_B = M'_{BB}T_{B'}M^{-1}_{BB'}.$$

It will be useful to note that **if the starting basis B was already orthonormal, then the step matrix $M_{BB'}$ is orthogonal and its inverse coincides with its transpose.**

As before, on all the steps where the scalar product (or norms) appears, we must use the corresponding Gram matrix with respect to the basis we have fixed. If this basis is orthonormal, then the Gram matrix is the identity and the scalar product is the usual one.

7.2 Symmetries on $\mathbb{R}^3$

7.2.1 Symmetry about the origin.

Theorem 7.3 *Let B be any basis of $\mathbb{R}^3$. The matrix associated to the* **symmetry about the origin** *is $T_B = -Id$.*

Proof: Just take into account that the symmetry about the origin takes every vector to its opposite. ■

7.2.2 Symmetry about a line.

Theorem 7.4 *Let $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ be a* **orthogonal** *basis of $\mathbb{R}^3$. The matrix associated to the* **symmetry about the axis generated by** *$\bar{e}_1$ is*

$$T_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Proof: Just take into account that the axis of symmetry is invariant, while all vectors orthogonal to it are transformed into their opposites:

$$t(\bar{e}_1) = \bar{e}_1; \quad t(\bar{e}_2) = -\bar{e}_2; \quad t(\bar{e}_3) = -\bar{e}_3.$$

7.2.3 Symmetry about a plane.

Theorem 7.5 *Let $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ be a* **orthogonal** *basis of $\mathbb{R}^3$. The matrix associated to the* **symmetry about the plane generated by** *$\bar{e}_1, \bar{e}_2$ is*

$$T_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Proof: Just take into account that the plane of symmetry is invariant, while all vectors orthogonal to it are transformed into their opposites:

$$t(\bar{e}_1) = \bar{e}_1; \quad t(\bar{e}_2) = \bar{e}_2; \quad t(\bar{e}_3) = -\bar{e}_3.$$

7.2.4 Procedure to find the matrix of a symmetry relative to an arbitrary basis.

Suppose we are given a **not necessarily orthonormal** basis $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ and we are asked to calculate the matrix of a symmetry about either

1. the axis generated by a vector $\bar{u}_1$.
2. the plane generated by a vector $\bar{u}_1, \bar{u}_2$.

The steps to find this matrix are as follows.

1. **Symmetry about the axis generated by $\bar{u}_1$.**
   (a) We complete $\bar{u}_1$ to an orthogonal basis: $B' = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$. Actually we only need both $\bar{u}_2$ and $\bar{u}_3$ to be orthogonal to $\bar{u}_1$.
   (b) The matrix of the symmetry with respect to this basis will be:
   $$TB_{B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
   (c) We perform the change of basis on the matrix $T_{B'}$:
   $$T_B = M'_{BB}T_{B'}M^{-1}_{BB'}.$$
2. **Symmetry with respect to the plane generated by $\bar{u}_1, \bar{u}_2$.**
   (a) We complete $\bar{u}_1$ to a basis: $B' = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$, with $\bar{u}_3$ orthogonal to the plane $\mathcal{L}\{\bar{u}_1, \bar{u}_2\}$.
   (b) The matrix of the symmetry with respect to this basis will be:
   $$T_{B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
   (c) We perform the change of basis on the matrix $T_{B'}$:
   $$T_B = M'_{BB}T_{B'}M^{-1}_{BB'}.$$

7.3 Classification of orthogonal transformations in $\mathbb{R}^3$.

Theorem 7.6 *Every orthogonal transformation t in $\mathbb{R}^3$ other than the identity is either*

- a rotation, or

- a symmetry with respect to a point, a line or a plane, or

- the composition of a rotation and a symmetry about the plane orthogonal to the axis of rotation.

Proof: Let T be the matrix of t with respect to an orthonormal basis B. We know that

$$TT^t = Id. \qquad |T| = \pm 1. \qquad \text{The real eigenvalues of } T \text{ are 1 or } -1.$$

One of the following must be true:

1. T has a unique real eigenvalue $\lambda = 1$ with multiplicity 3. Then T is similar to the identity and therefore actually $T = Id$.
2. T has a unique real eigenvalue $\lambda = -1$ with multiplicity 3. So T is similar to $-Id$ and actually $T = -Id$. It is a symmetry about the origin.
3. T has eigenvalues $\lambda_1 = 1$, with multiplicity 2, and $\lambda_2 = -1$, with multiplicity 1. Then T is similar to the matrix
$$T_{B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
In particular, there is an orthonormal basis B' with respect to which the matrix of t is the previous one. We deduce that it is a symmetry with respect to a plane. Taking into account that this plane is formed by vectors which remain invariant under the symmetry, the plane of symmetry is the characteristic subspace S_1.
4. T has eigenvalues $\lambda_1 = 1$, with multiplicity 1, and $\lambda_2 = -1$, with multiplicity 2. Then T is similar to the matrix
$$T_{B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
Arguing as before, we see that t is a symmetry with respect to an axis corresponding to the characteristic subspace S_1.
5. T has a unique real eigenvalue $\lambda_1 = 1$ with multiplicity 1. So $\mathbb{R}^3$ can be decomposed as:
$$\mathbb{R}^3 = S_1 \oplus S_1^\perp$$
where $S_1^\perp$ is a 2-dimensional subspace invariant under t. The restriction of t to $S_1^\perp$ is an orthogonal transformation with no real eigenvalues. From the classification of orthogonal transformations in $\mathbb{R}^2$ we deduce that it is a rotation. Therefore there exists an orthonormal basis $B' = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ with
$$S_1 = \{\bar{u}_1\} \quad \text{and} \quad S_1^\perp = \{\bar{u}_2, \bar{u}_3\}$$
relative to which the matrix associated to t is:
$$T_{B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$
and therefore it is a rotation of angle α with respect to the orientation given by the basis B' and the semi-axis $\bar{u}_1$.
6. T has a unique real eigenvalue $\lambda_1 = -1$ with multiplicity 1. Arguing as before, we deduce that there exists an orthonormal basis $B' = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ with
$$S_{-1} = \{\bar{u}_1\} \quad \text{and} \quad S_{-1}^\perp = \{\bar{u}_2, \bar{u}_3\}$$
relative to which the matrix associated to t is:
$$T_{B'} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix},$$
and therefore t is the composition of a rotation of angle α about the semi-axis $\bar{u}_1$ and with respect to the orientation given by the basis B', and a symmetry about the plane $S_{-1}^\perp$. ■

We summarize the classification in the following table:

Orthogonal transformations on $\mathbb{R}^3$		
Eigenvalues	**Classification**	**Type**
$\{1,1,1\}$	*Identity*	Direct
$\{1,1,-1\}$	*Symmetry about the plane S_1.*	Inverse
$\{1,-1,-1\}$	*Symmetry about the S_1 axis.*	Direct
$\{-1,-1,-1\}$	*Symmetry about the origin.*	Inverse
1, mult=1.	*Rotation of angle α with respect to the semi-axis $\bar{u}_1$:* $T_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}$ $B = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ **ORTHONORMAL** basis $S_1 = \mathcal{L}\{\bar{u}_1\}$ $S_1^{\perp} = \mathcal{L}\{\bar{u}_2, \bar{u}_3\}$ Orientation given by basis B.	Direct
-1, mult=1.	*Composition of:* - *Rotation of angle α with respect to the semi-axis $\bar{u}_1$:* $T_B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(alpha) \end{pmatrix}$ $B = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ **ORTHONORMAL** basis $S_{-1} = \mathcal{L}\{\bar{u}_1\}$ $S_{-1}^{\perp} = \mathcal{L}\{\bar{u}_2, \bar{u}_3\}$ Orientation given by basis B. - *Symmetry about the plane $S_{-1}^{\perp}$.*	Inverse

7.4 An alternative method for classifying orthogonal transformations.

We will expose an alternative method for classifying orthogonal transformations in dimensions 2 and 3. This method is based on the following known result:

Proposition 7.7 *Neither the determinant nor the trace of the matrix associated with an endomorphism depend on the chosen basis.*

Given that we have already obtained the associated matrices to each type of orthogonal transformation relative to convenient orthonormal bases let us see how to classify them.

7.4.1 Orthogonal transformations on $\mathbb{R}^2$.

If the orthogonal transformation is a symmetry about a straight line, we saw that the associated matrix with respect to a suitable orthonormal basis (one of its vectors must generate the axis of symmetry) is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We get determinant -1 and trace 0.

If the orthogonal transformation is a rotation, the associated matrix relative to an orthonormal basis is:

$$\begin{pmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{pmatrix}$$

We have determinant 1 and trace $2cos(\alpha)$. This case includes the particular situations in which the angle is 0 and therefore the transformation is the identity or the angle is 180^o and the transformation is a symmetry about the origin.

We see that the trace allows us to calculate the cosine of the angle in such a way that it can be uniquely chosen in the interval $[0,\pi]$. The problem which remains to be solved is whether this angle should be taken with a positive or negative sign, depending on the orientation we are handling.

To do this, if we work with the orientation given by the basis $B = \{\bar{e}_1, \bar{e}_2\}$, we consider the basis $B' = \{\bar{e}_1, f(\bar{e}_1)\}$. If both bases have the same orientation, then the angle must be taken with a positive sign; if they are different, it is taken with a negative sign.

We summarize all this in the following table:

Orthogonal transformations on $\mathbb{R}^2$		
Orientation given by $B = \{\bar{e}_1, \bar{e}_2\}$. F associated matrix wrt any basis		
Det(F)	**Trace(F)**	**Classification**
1	2	*Identity*
1	-2	*Symmetry about the origin.* $\Updownarrow$ Rotation of angle π.
1	$\neq 2, -2$	*Rotation of angle α:* $B' = \{\bar{e}_1, \bar{f}(e_1)\}$ If $\lvert M_{BB'}\rvert > 0$, $\alpha = +arc\,cos(trace(F)/2)$. If $\lvert M_{BB'}\rvert < 0$, $\alpha = -arc\,cos(trace(F)/2)$.
-1	0	*Symmetry about S_1*

7.4.2 Orthogonal transformations on $\mathbb{R}^3$.

By analyzing trace and the determinant for each type of transformation, we obtain the following classification table.

Orthogonal transformations in $\mathbb{R}^3$		
Orientation given by basis B. F associated matrix wrt any basis		
Det(F)	**Trace(F)**	**Classification**
1	3	*Identity*
1	-1	*Symmetry about line* S_1
1	$\neq 3, -1$	*Rotation of angle* α *with respect to the semi-axis* $\bar{u}_1$*:* $\bar{u}_1 =$ eigenvector associated to 1 $\bar{v}$ linearly independent with $\bar{u}_1$ $B' = \{\bar{u}_1, \bar{v}, \bar{f}(v)\}$ If $\lvert M_{BB'}\rvert > 0$, $\alpha = +arc\,cos((trace(F) - 1)/2)$. If $\lvert M_{BB'}\rvert < 0$, $\alpha = -arc\,cos((trace(F) - 1)/2)$.
-1	-3	*Symmetry about the origin.*
-1	1	*Symmetry about the plane* S_1
-1	$\neq -3, 1$	Composition of one rotation and one symmetry - *Rotation of angle* α *with respect to the semi-axis* $\bar{u}_1$*:* $\bar{u}_1 =$ eigenvector associated to -1 $\bar{v}$ linearly independent with $\bar{u}_1$ $B' = \{\bar{u}_1, \bar{v}, \bar{f}(v)\}$ If $\lvert M_{B'B}\rvert > 0$, $\alpha = +arc\,cos((trace(F) + 1)/2)$. If $\lvert M_{B'B}\rvert < 0$, $\alpha = -arc\,cos((trace(F) + 1)/2)$. - *Symmetry with respect to the plane* $S_{-1}^{\perp}$.