2. Orthogonality.

Throughout this chapter the symbol U will stand for a fixed Euclidean vector space.

1 Orthogonal vectors.

Definition 1.1 *Two vectors $\bar{x}, \bar{y} \in U$ are said to be* **orthogonal** *if*

$$\bar{x} \cdot \bar{y} = 0.$$

Let us see some properties that follow from the previous definition:

1. *The only vector which is orthogonal to itself is $\bar{0}$.*
   Proof: Just take into account that the scalar product is associated to a positive definite quadratic form. Therefore:
   $$\bar{x} \cdot \bar{x} = 0 \iff \bar{x} = \bar{0}.$$
2. *Two nonzero vectors are orthogonal if and only if they form an angle of $\frac{\pi}{2}$.*
   Proof: If $\bar{x}, \bar{y}$ are two nonzero vectors:
   $$\bar{x}, \bar{y} \text{ orthogonal } \iff \bar{x} \cdot \bar{y} = 0 \iff cos(\bar{x}, \bar{y}) = \frac{\bar{x} \cdot \bar{y}}{\|\bar{x}\|\|\bar{y}\|} = 0 \iff \angle(\bar{x}, \bar{y}) = \frac{\pi}{2}.$$
3. **Pythagorean Theorem.** *Two vectors $\bar{x}, \bar{y}$ are orthogonal if and only if*
   $$\|\bar{x} + \bar{y}\|^2 = \|\bar{x}\|^2 + \|\bar{y}\|^2.$$
   Proof: Just take into account that:
   $$\|\bar{x} + \bar{y}\|^2 = (\bar{x} + \bar{y}) \cdot (\bar{x} + \bar{y}) = \|\bar{x}\|^2 + \|\bar{y}\|^2 + 2\bar{x} \cdot \bar{y}.$$

2 Orthogonal systems.

2.1 Definition

Definition 2.1 *A system of vectors $\{\bar{u}_1, \dots, \bar{u}_n\}$ is said to be* **orthogonal**, *if its vectors are pairwise orthogonal:*

$$\bar{u}_i \cdot \bar{u}_j = 0 \quad \text{for any } i \neq j,\ i, j \in \{1, \dots, n\}.$$

Definition 2.2 *A system of vectors $\{\bar{u}_1, \dots, \bar{u}_n\}$ is said to be* **orthonormal**, *if it is orthogonal and all its vectors are unitary:*

$$\bar{u}_i \cdot \bar{u}_j = \delta_i^j \quad \text{for any } i, j \in \{1, \dots, n\}.$$

Proposition 2.3 *Any orthogonal system which does not contain the vector $\bar{0}$ is linearly independent.*

Proof: Suppose $\{\bar{u}_1, \dots, \bar{u}_n\}$ is an orthogonal system none of whose vectors is zero. Suppose there are scalars $\alpha^1, \dots, \alpha^n$ satisfying:

$$\alpha^1\bar{u}_1 + \dots + \alpha^n\bar{u}_n = \bar{0}.$$

If we perform the scalar product by a vector $\bar{u}_j$ we get:

$$(\alpha^1\bar{u}_1 + \dots + \alpha^n\bar{u}_n) \cdot \bar{u}_j = \bar{0} \cdot \bar{u}_j = 0 \quad \Rightarrow \quad \alpha^1\bar{u}_1 \cdot \bar{u}_j + \dots + \alpha^n\bar{u}_n \cdot \bar{u}_j = \bar{0}$$

Taking into account that it is an orthogonal system, $\bar{u}_i \cdot \bar{u}_j = 0$ if $i \neq q$. Hence

$$\alpha^j\bar{u}_j \cdot \bar{u}_j = 0.$$

Since $\bar{u}_j \neq 0$, we deduce $\bar{u}_j \cdot \bar{u}_j \neq 0$ and we obtain that $\alpha^j = 0$ for every $j \in \{1, \dots, n\}$. ■

2.2 Orthogonal bases.

Definition 2.4 *An* **orthogonal basis** *is a basis which is also an orthogonal system.*

From the definition of an orthogonal system it is clear that:

B **is an orthogonal basis** $\iff$ **The Gram matrix** G_B **is diagonal**

Definition 2.5 *An* **orthonormal basis** *is a basis which is also an orthonormal system.*

From the definition of orthonormal system it follows that:

B **is an orthonormal basis** $\iff$ **The Gram matrix** G_B **is the identity**

In our study of symmetric quadratic forms we saw that all of them are diagonalizable by congruence. If they are also positive definite, then they are congruent to the identity matrix. Applying this fact to a Euclidean space we deduce:

Theorem 2.6 *Every Euclidean space has an orthonormal basis.*

2.3 Gram-Schmidt orthogonalization method.

This is a procedure for obtaining an orthogonal basis $\{\bar{u}_1, \ldots, \bar{u}_n\}$ starting from any basis $\{\bar{e}_1, \ldots, \bar{e}_n\}$.

The procedure is as follows:

1. We take $\bar{u}_1 = \bar{e}_1$.
2. We define $\bar{u}_2 = \bar{e}_2 + \alpha_2^1 \bar{u}_1$.
   To find the parameter α_2^1 we require that $\bar{u}_2$ be orthogonal with $\bar{u}_1$:
   $$\bar{u}_2 \cdot \bar{u}_1 = 0 \quad \Rightarrow \quad \bar{e}_2 \cdot \bar{u}_1 = -\alpha_2^1 \bar{u}_1 \cdot \bar{u}_1 \quad \Rightarrow \quad \alpha_2^1 = -\frac{\bar{e}_2 \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}.$$
   Now $\mathcal{L}\{\bar{e}_1, \bar{e}_2\} = \mathcal{L}\{\bar{u}_1, \bar{u}_2\}$.
3. We define $\bar{u}_3 = \bar{e}_3 + \alpha_3^1 \bar{u}_1 + \alpha_3^2 \bar{u}_2$.
   To find the parameters we require that $\bar{u}_3$ be orthogonal to both $\bar{u}_1$ and $\bar{u}_2$:
   $$\bar{u}_3 \cdot \bar{u}_1 = 0 \quad \Rightarrow \quad 0 = \bar{e}_3 \cdot \bar{u}_1 + \alpha_3^1 \bar{u}_1 \cdot \bar{u}_1 + \alpha_3^2 \bar{u}_2 \cdot \bar{u}_1 \quad \Rightarrow \quad \alpha_3^1 = -\frac{\bar{e}_3 \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}.$$
   $$\bar{u}_3 \cdot \bar{u}_2 = 0 \quad \Rightarrow \quad 0 = \bar{e}_3 \cdot \bar{u}_2 + \alpha_3^1 \bar{u}_1 \cdot \bar{u}_2 + \alpha_3^2 \bar{u}_2 \cdot \bar{u}_2 \quad \Rightarrow \quad \alpha_3^2 = -\frac{\bar{e}_3 \cdot \bar{u}_2}{\bar{u}_2 \cdot \bar{u}_2}.$$
   Now $\mathcal{L}\{\bar{e}_1, \bar{e}_2, \bar{e}_3\} = \mathcal{L}\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$.
4. We continue this process until completing the basis. **Specifically, the k-th step is as follows:**
   We define $\bar{u}_k = \bar{e}_k + \alpha_k^1 \bar{u}_1 + \ldots + \alpha_k^{k-1} \bar{u}_{k-1}$ with:
   $$\boxed{\alpha_k^i = -\frac{\bar{e}_k \cdot \bar{u}_i}{\bar{u}_i \cdot \bar{u}_i}} \quad , \qquad i = 1, \ldots, k-1,$$
   where $\mathcal{L}\{\bar{e}_1, \ldots, \bar{e}_k\} = \mathcal{L}\{\bar{u}_1, \ldots, \bar{u}_k\}$.
   It is interesting to note that all of these expressions make sense because the vectors $\bar{u}_i$ are independent. In particular they are not null and $\bar{u}_i \cdot \bar{u}_i \neq 0$.
   On the other hand we also see that if $\bar{e}_k$ is already orthogonal to $\bar{u}_1, \ldots, \bar{u}_{k-1}$, then $\bar{u}_k = \bar{e}_k$.

This method can also be used to build an **orthonormal** basis. To do this, we simply have to **normalize** the obtained basis. In general, if $\{\bar{u}_1, \ldots, \bar{u}_n\}$ is an *orthogonal* basis, then
$$\{\frac{\bar{u}_1}{\|\bar{u}_1\|}, \ldots, \frac{\bar{u}_n}{\|\bar{u}_n\|}\}$$
is an **orthonormal** basis. This process is called **normalization.**

2.4 Incomplete orthogonal basis theorem.

Theorem 2.7 *Let U be an n-dimensional Euclidean space. If $\{\bar{u}_1, \ldots, \bar{u}_p\}$ is an orthogonal system of p nonzero vectors, with $p < n$, then there exists a system of vectors $\{\bar{u}_{p+1,}, \ldots, \bar{u}_n\}$ whose union with the first:*
$$\{\bar{u}_1, \ldots, \bar{u}_p, \bar{u}_{p+1,}, \ldots, \bar{u}_n\}$$
is an orthogonal basis.

Proof: Given the system $\{\bar{u}_1, \ldots, \bar{u}_p\}$, we know that we can complete it up to a basis of U:
$$\{\bar{u}_1, \ldots, \bar{u}_p, \bar{e}_{p+1,}, \ldots, \bar{e}_n\}$$
We now apply Gram-Schmidt orthogonalization method to this basis. The first p vectors remain the same because they already form an orthogonal system. In this way we will obtain an orthogonal basis:
$$\{\bar{u}_1, \ldots, \bar{u}_p, \bar{u}_{p+1,}, \ldots, \bar{u}_n\}.$$

3 Some properties of orthonormal bases.

In this section we will see the advantages of choosing an orthonormal basis when working in the context of an Euclidean space.

3.1 Gram matrix in an orthonormal basis.

Theorem 3.1 *The Gram matrix of a scalar product with respect to an orthonormal basis is the identity.*

Proof: If $B = \{\bar{e}_1, \ldots, \bar{e}_n\}$ is an orthonormal basis, then:
$$(G_B)_{ij} = \bar{e}_i \cdot \bar{e}_j = \delta_{ij} \quad \Rightarrow \quad G_B = Id.$$

3.2 Expression of the scalar product in an orthonormal basis.

If $B = \{\bar{e}_1, \ldots, \bar{e}_n\}$ is an orthonormal basis and $\bar{x}, \bar{y}$ are vectors whose coordinates relative to this basis are $(x^i), (y^j)$ then
$$\boxed{\bar{x} \cdot \bar{y} = \begin{pmatrix} x^1 & \ldots & x^n \end{pmatrix} \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix}} \qquad \text{or} \qquad \boxed{\bar{x} \cdot \bar{y} = (x)^t (y)}$$

Therefore the norm of a vector can be obtained as follows:

$$\boxed{\|\bar{x}\| = \sqrt{(x^1)^2 + \ldots + (x^n)^2}}$$

3.3 Covariant coordinates with respect to an orthonormal basis.

Theorem 3.2 *The covariant coordinates of any vector with respect to an orthonormal basis are the same as the contravariant ones.*

Proof: Just take into account that the Gram matrix with respect to the orthonormal basis is the identity.

3.4 Relation between orthonormal bases.

Let B and B' be two orthonormal bases of U:

$$B = \{\bar{e}_1, \ldots, \bar{e}_n\}; \qquad B' = \{\bar{e}'_1, \ldots, \bar{e}'_n\}.$$

We know that:

$$G_{B'} = (M_{BB'})^t G_B M_{BB'}$$

In addition, since $G_B = G_{B'} = Id$ are orthonormal bases, we therefore deduce that:

Theorem 3.3 *The change-of-basis matrix between two orthonormal bases B, B' is orthogonal, that is:*

$$(M_{BB'})^t M_{BB'} = Id.$$

It is interesting to note that the fact that a matrix is orthogonal means that its inverse coincides with its transpose. This makes the change of basis between orthonormal bases especially convenient.

4 Orthogonal projection.

4.1 Orthogonal subspaces.

Definition 4.1 *Two subspaces S_1 and S_2 of a Euclidean vector space U are said to be* **orthogonal** *when every element of S_1 is orthogonal to all elements of S_2*

$$\bar{x} \cdot \bar{y} = 0 \quad \textit{for any} \quad \bar{x} \in S_1,\ \bar{y} \in S_2.$$

Proposition 4.2 *If S_1 and S_2 are two subspaces generated respectively by the vectors $\{\bar{u}_1, \ldots, \bar{u}_p\}$ and $\{\bar{v}_1, \ldots, \bar{v}_q\}$, then a necessary and sufficient condition for S_1 and S_2 to be orthogonal is:*

$$\bar{u}_i \cdot \bar{v}_j = 0 \quad \textit{for all } i \in \{1, \ldots, p\},\ j \in \{1, \ldots, q\}.$$

Proof: The necessity of the condition is clear. Let us see the sufficiency. Fix $\bar{x} \in S_1$, $\bar{y} \in S_2$. We need to see that if the condition of the statement is fulfilled, then $\bar{x}$ and $\bar{y}$ are orthogonal.

These vectors can be written as:

$$\bar{x} = \alpha^1 \bar{u}_1 + \ldots + \alpha^p \bar{u}_p; \quad \bar{y} = \beta^1 \bar{v}_1 + \ldots + \beta^q \bar{v}_q.$$

By the properties of the scalar product:

$$\bar{x} \cdot \bar{y} = \sum_{i=1}^{p} \sum_{j=1}^{q} \alpha^i \beta^j \bar{u}_i \cdot \bar{v}_j.$$

By hypothesis $\bar{u}_i \cdot \bar{v}_j = 0$, hence $\bar{x} \cdot \bar{y} = 0$.

4.2 Supplementary orthogonal subspace.

Definition 4.3 *Given a Euclidean vector space U and a subset $V \subset U$, the set*

$$V^{\perp} = \{\bar{x} \in U \,|\, \bar{x} \cdot \bar{v} = 0, \ \textit{for all } \bar{v} \in V\}.$$

is called the subspace **orthogonal to** *V.*

We note that this definition is equivalent to that of the conjugate space of U, which we have seen in the chapter on quadratic forms. Therefore, we immediately obtain the following properties:

1. *$V^{\perp}$ is a vector subspace.*
2. $V \subset W \quad \Rightarrow \quad W^{\perp} \subset V^{\perp}$.
3. *If $V = \mathcal{L}\{\bar{v}_1, \ldots, \bar{v}_p\}$ then $V^{\perp} = \{\bar{v}_1, \ldots, \bar{v}_p\}^{\perp}$.*
4. *If V is a vector subspace, V and $V^{\perp}$ are complementary subspaces.*

   Proof: First we obtain that $V \cap V^{\perp} = \{0\}$:

   $$\bar{x} \in V \cap V^{\perp} \quad \Rightarrow \quad \bar{x} \cdot \bar{x} = 0 \quad \Rightarrow \quad \bar{x} = \bar{0}.$$

   Now let us see that $V + V^{\perp} = U$. Let $\{\bar{v}_1, \ldots, \bar{v}_p\}$ be an orthogonal basis of V. By the incomplete orthogonal basis theorem, we can complete it up to an orthogonal basis of the whole space U:

   $$\{\bar{v}_1, \ldots, \bar{v}_p, \bar{v}_{p+1}, \ldots, \bar{v}_n\}$$

Since $\bar{v}_i \cdot \bar{v}_j = 0$ for $i \in \{1, \ldots, p\}$, $j \in \{p+1, \ldots, n\}$ we deduce

$$\bar{v}_j \in V^{\perp}, \quad \text{for } j \in \{p+1, \ldots, n\}$$

and therefore

$$\mathcal{L}\{\bar{v}_{p+1}, \ldots, \bar{v}_n\} \subset V^{\perp} \quad \Rightarrow \quad dim(V^{\perp}) \geq n - p$$

So:

$$n \geq dim(V + V^{\perp}) = dim(V) + dim(V^{\perp}) - dim(V \cap V^{\perp}) \geq p + n - p = n$$

and therefore

$$dim(V + V^{\perp}) = n \quad \Rightarrow \quad V + V^{\perp} = U.$$

4.3 Orthogonal projection

Definition 4.4 *Given a subspace V of a Euclidean space U we define the* **orthogonal projection onto** V *as the mapping*

$$p_V : \begin{array}{ccc} U & \longrightarrow & U \\ \bar{x} & \longrightarrow & \bar{x}_1 \end{array} \quad \textit{where} \quad \bar{x} = \bar{x}_1 + \bar{x}_2 \quad \textit{with} \quad \bar{x}_1 \in V, \bar{x}_2 \in V^{\perp}.$$

The above definition makes sense because as we have seen, V and $V^{\perp}$ are complementary vector subspaces.

5 Symmetric endomorphisms.

5.1 Definition.

Definition 5.1 *Let U be an Euclidean space. An endomorphism $f : U \longrightarrow U$ is said to be* **symmetric** *if:*

$$\bar{x} \cdot f(\bar{y}) = \bar{y} \cdot f(\bar{x}) \quad \forall \bar{x}, \bar{y} \in U.$$

If $B = \{\bar{e}_1, \ldots, \bar{e}_n\}$ is a basis of U, G_B is the Gram matrix with respect to B and F_B is the matrix of an endomorphism with respect to the same basis B, then the symmetry condition reads as follows:

$$\begin{aligned} \bar{x} \cdot f(\bar{y}) = \bar{y} \cdot f(\bar{x}) \quad &\Longleftrightarrow \bar{x} \cdot f(\bar{y}) = f(\bar{x}) \cdot \bar{y} \Longleftrightarrow \\ &\Longleftrightarrow (x)^t G_B F_B(y) = (F_b(x))^t G_B(y) \Longleftrightarrow \\ &\Longleftrightarrow (x)^t G_B F_B(y) = (x)^t (F_B)^t G_B(y) \end{aligned}$$

for any $\bar{x}, \bar{y} \in U$ with coordinates $(x^i), (y^i)$ with respect to the basis B, respectively.

Therefore:

$$\boxed{f \text{ symmetric endomorphism} \iff G_B F_B = F_B^t G_B}$$

In particular if B is an orthonormal basis:

$$\boxed{\begin{array}{c} f \text{ symmetric endomorphism} \iff {F_B}^t = F_B \iff F_B \text{ is symmetric} \\ \\ \text{(where } B \text{ is an } \textbf{orthonormal} \text{ basis)} \end{array}}$$

5.2 Eigenvalues and eigenvectors of a symmetric endomorphism.

Proposition 5.2 *Let $f : U \longrightarrow U$ be a symmetric endomorphism. If λ and μ are two different eigenvalues of f, then the characteristic subspaces S_λ and S_μ are orthogonal.*

Proof: Let $\bar{x} \in S_\lambda$, $\bar{y} \in S_\mu$ be nonzero vectors. Since f is symmetric, we have:

$$\bar{x} \cdot f(\bar{y}) = \bar{y} \cdot f(\bar{x}).$$

Since $\bar{x}$, $\bar{y}$ are eigenvectors associated respectively to λ and μ we obtain

$$\bar{x} \cdot (\mu \bar{y}) = \bar{y} \cdot (\lambda \bar{x}) \quad \Rightarrow \quad (\lambda - \mu)\bar{x} \cdot \bar{y} = 0.$$

Taking into account that $\lambda \neq \mu$ we see that $\bar{x} \cdot \bar{y} = 0$ and therefore $\bar{x}$ and $\bar{y}$ are orthogonal.

Theorem 5.3 *Any symmetric matrix $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ has n real eigenvalues (counted with multiplicity).*

Proof: We know that there are always exactly n complex eigenvalues, corresponding to the n roots of the characteristic polynomial of A. We have to prove that all of them are real. Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue and let $\bar{x} = (x^1, \ldots, x^n) \in \mathbb{C}^n$ be an associated nonzero eigenvector. We denote by $c(\bar{x})$ the conjugate of the vector $\bar{x}$ whose components are the conjugates of each component of $\bar{x}$.

Remember that the conjugate of a complex number $a + b\imath$ is $a - b\imath$. We will use the following properties:

- A complex number is real if and only if it is equal to its conjugate.

- The conjugation "behaves well" with respect to the sum and product of complex numbers.

- The product of a nonzero complex number by its conjugate is a positive real number:

$$(a + b\imath)(a - b\imath) = a^2 + b^2 > 0.$$

In our case we have:

$$A(x) = \lambda(x) \quad \Rightarrow \quad c(x)^t A(x) = \lambda c(x)^t (x) \qquad (1)$$

On the other hand, by applying conjugation on both sides of the identity $A(x) = \lambda(x)$ and taking into account that since A is real $c(A) = A$, we deduce

$$c(A)c(x) = c(\lambda)c(x) \quad \Rightarrow \quad Ac(x) = c(\lambda)c(x) \quad \Rightarrow \quad (x)^t Ac(x) = c(\lambda)(x)^t c(x).$$

Bearing in mind that A is symmetric, if we apply transposition on both sides of this identity we get:

$$c(x)^t A(x) = c(\lambda)c(x)^t(x).$$

We compare it with the equation (1) and obtain:

$$\lambda c(x)^t(x) = c(\lambda)c(x)^t(x) \quad \Rightarrow \quad (\lambda - c(\lambda))c(x)^t(x) = 0.$$

But since $\bar{x} \neq 0$,

$$c(x)^t(x) = x^1 c(x^1) + \ldots + x^n c(x^n) > 0.$$

We deduce that $\lambda - c(\lambda) = 0$, that is, λ is equal to its conjugate and is therefore a real number. ■

Theorem 5.4 *Every symmetric endomorphism of an n-dimensional Euclidean space has n real eigenvalues (counted with multiplicity).*

Proof: It is enough to take into account that the matrix of a symmetric endomorphism relative to any orthonormal basis is symmetric. Now the theorem is a consequence of the previous result. ■

5.3 Orthonormal bases of eigenvectors.

Theorem 5.5 *Let U be an n-dimensional Euclidean space. If f is a symmetric endomorphism then there exists an orthonormal basis formed by eigenvectors of f.*

Proof: We will prove that there is always an orthogonal basis of eigenvectors. The remaining step is immediate through the normalization process.

By the previous theorem we know that there are exactly n real eigenvalues (counted with multiplicity):

$$\left.\begin{matrix} \lambda_1 \\ \vdots \\ \lambda_k \end{matrix}\right\} \quad \text{with algebraic multiplicities} \quad \left\{\begin{matrix} m_1 \\ \vdots \\ m_k \end{matrix}\right.$$

so $m_1 + \ldots + m_k = n$. We know that the characteristic spaces give a decomposition of the space as a direct sum:

$$V = S_{\lambda_1} \oplus \ldots \oplus S_{\lambda_k}.$$

Additionally, each of them is orthogonal to the other ones. Choosing an orthogonal basis for each of them, we build an orthogonal basis of eigenvectors for the subspace V:

$$\{\bar{u}_1, \ldots, \bar{u}_p\}$$

It only remains to prove that V is actually the entire subspace U. We notice that **any eigenvector of f has to be contained in V.**

Suppose that $V \neq U$. Consider the space $V^\perp$. Fix $\bar{x} \in V^\perp$, let us see that $f(\bar{x}) \in V^\perp$. To do this, we must check that $f(\bar{x}) \cdot \bar{u}_i = 0$ for any $i = 1, \ldots, p$:

$$f(\bar{x}) \cdot \bar{u}_i \underset{f \text{ symmetric}}{=} \bar{x} \cdot f(\bar{u}_i) \underset{\bar{u}_i \text{ eigenvector}}{=} \bar{x} \cdot \lambda \bar{u}_i \underset{\bar{x} \in V^\perp}{=} 0.$$

Therefore the subspace $V^\perp$ is invariant by f. The restriction of f to $V^\perp$

$$g : V^\perp \longrightarrow V^\perp; \qquad g(\bar{x}) = f(\bar{x})$$

is again a symmetric endomorphism. All its eigenvalues are real, and therefore it contains at least one nonzero eigenvector. But this contradicts the fact that all eigenvectors are in V and $V \cap V^\perp = \{0\}$. ■

Corollary 5.6 *If f is a symmetric endomorphism of a Euclidean space, there exists an orthonormal basis with respect to which the associated matrix is diagonal.*

Proof: It is enough to apply the previous theorem. Because f is symmetric, there always exists an orthonormal basis formed by eigenvectors; but the matrix of any endomorphism with respect to a basis of eigenvectors is diagonal. ■

Corollary 5.7 *If $A \in M_{n\times n}(\mathbb{R})$ is a* **symmetric matrix:**

1. *All eigenvalues of A are real.*
2. *A is diagonalizable by similarity.*
3. *There exists an orthonormal basis of $\mathbb{R}^n$ with respect to the usual scalar product which is formed by eigenvectors of A .*
4. *There exists an orthogonal matrix P (that is, satisfying $P^t = P^{-1}$) such that:*

$$D = P^{-1}AP$$

where D is the diagonal matrix formed by the eigenvalues.